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Statements and Solutions

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ISBN 978-1-4471-5576-8 ISBN 978-1-4471-5577-5 (eBook)
DOI 10.1007/978-1-4471-5577-5
Springer London Heidelberg New York Dordrecht

Library of Congress Control Number: 2013949477

Translators: Helen Warbuton and Antonio Maravilla Burgos (HyA Translations)

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Operations Research Problems is an advanced textbook: designed primarily to meet the demands of a operations research and management science course taught at postgraduate M. Eng. or MBA level. This book includes an overview of each topic considered with worked examples in the text, problems, solutions to problems, end-of-chapter references and Index.

The objective of this book is to provide a valuable compendium of problems as a reference for undergraduate and postgraduate students, faculty, researchers and practitioners of operations research and management science. These problems can serve as a basis for the development or study of assignments and exams. Also, they can be useful as a guide for the first stage of the model formulation, i.e. the definition of a problem. The book is divided into 10 chapters that address the following topics: Linear programming, integer programming, nonlinear programming, network modelling, inventory theory, queueing theory, decision theory, games theory, dynamic programming and markov processes. Readers are going to find a considerable number of statements of operations research applications for management decision-making. The solutions of these problems are provided in a concise way although all topics start with a more developed overview. The proposed problems are based on the research experience of the authors in real-world companies so much as on the teaching experience of the authors in order to develop exam problems for industrial engineering and business administration studies.

The chapters have been arranged as follows:

Linear Programming

This book chapter provides some linear programming applications. After reading this chapter, the reader should be capable of: understanding the nature of resources optimisation problems and their formulation by linear programming; knowing the resolution graphic model for linear programming problems with two decision variables; comprehending the Simplex Method's logic nature and of applying it to solve linear programming models; formulating the dual problem of a linear programming model; economically interpreting the solutions of a dual problem; understanding the sensitivity analysis concept and of using it for decision making.

Integer Programming

Integer linear programming models are employed in a large number of problems with intrinsically integer variables. After reading this chapter, the reader should be capable of knowing and modelling different integer linear programming prototype problems and formulating these models with binary variables.

Non-Linear Programming

The objective of this book chapter is to help learn the formulation of nonlinear programming models and of presenting some of their applications in the industrial engineering and management domain. After reading this chapter, the reader should be capable of formulating different prototype nonlinear programming problems, and of modelling multivariant and multimodel functions with inequality constraints by the Kuhn-Tucker conditions.

Network Modelling

The purpose of this book chapter is to help learn the formulation of network modelling models and to show some of their applications in the industrial engineering and management area. Therefore, management problems are modelled using graphs, while models are solved with shortest path, maximal flow and minimal spanning tree problems. After reading this chapter, readers should be able to model and solve different prototype shortest path, maximal flow and minimal spanning tree problems and to model minimal cost flow problems.

Inventory Theory

The objective of this chapter is to help learn to formulate and solve deterministic models based on the Inventory Theory in the independent demand context and to show some of their applications in the industrial engineering and management area. Thus, management problems are modelled by analytical EOQ formulation along with some of its variants. After reading this chapter, readers should be able to model and solve different inventory problems by means of the basic EOQ model and its variants with discounts for volume, delivery times other than zero and backorders.

Queueing Theory

The objective of this book chapter is to help to learn how to formulate and solve analytical Queueing Theory models and to show some of their applications in the industrial engineering and management domain. Therefore, some management problems are modelled by the analytical formulation of various steady-state queueing systems. After reading this chapter, readers should be able to quantitatively and qualitatively characterise a queue by a mathematical analysis, and determine the suitable levels of certain queueing system parameters which balance the social waiting cost with the cost associated with the resources consumed.

Decision Theory

With this chapter, readers should be able to understand the nature of risk decision problems and their modelling by means of decision trees, to discover the solution method by calculating the EMV, to apply Bayes Theorem, to calculate revised probabilities according to new information and to interpret solutions by calculating the limit values of the costs of further information.

Games Theory

The purpose of this chapter is to provide mechanisms to understand the cooperative and non-cooperative decision problems; identify the players from a games problem and its feasible strategies; calculate the losses and gain associated with each combination of strategies and solve zero-sum games played by two people by applying Mini-Max for pure strategies, the algebraic method for mixed strategies with 2×2 matrices, the Graph Method for $2 \times M$ and the general linear programming method for any dimension.

Dynamic Programing

After reading this chapter, readers should be able to comprehend the nature of multiphase decision problems that can be modelled by dynamic programming; define the stages of the problem, its input stages and the decisions that can be made; define the transition function between the input state and the output stage according to the decision made in each stage; construct the recursive function of a dynamic programming model; calculate the optimal costs in each stage, as well as the optimal decisions and obtain the optimal solutions that provide dynamic programming for decision making.

Markov Processes

The objective of this book chapter is to help to understand the nature of stochastic systems that can be modelled by a Markov chain; calculate the one-step transition probabilities between the various system states; know the different calculation formulae of the several step transition probabilities, stationary probabilities and the mean first passage times and calculate the mean system operation costs.

Operations Research Problems is related to operations management, production planning and quantitative methods. Readers would be able to identify different operations management problems in order to improve the decision-making process concerning to them. Therefore, it is not only useful for academic purposes but also for industrial practitioners.

We want to acknowledge the valuable help of Lu Qi 吕祺 Da Lian, China 大连, 中国 for the development of this book.

Contents

Synopsis

Operations Research Problems provide readers a valuable compendium of problems as a reference for undergraduate and graduate students, faculty, researchers and practitioners of operations research and management science. These problems can serve as a basis for the development or study of assignments and exams. Also, they can be useful as a guide for the first stage of the model formulation, i.e. the definition of a problem. The chapters address the following topics:

- Linear programming,
- Integer programming,
- Nonlinear programming,
- Network modelling,
- Inventory theory,
- Queueing theory,
- Tree decision,
- Games theory,
- Dynamic programming and
- Markov processes.

The objective of *Operations Research Problems* is to present a considerable number of statements of operations research applications for management decision-making. The solutions of these problems are provided in a concise way although all topics start with a more developed resolution. The proposed problems are based on the research experience of the authors in real-world companies so much as on the teaching experience of the authors in order to develop exam problems for industrial engineering and business administration studies.

Chapter 1
Linear Programming

Abstract This chapter commences with an introduction to linear programming. This is followed by a varied set of linear programming problems with their corresponding solutions. This chapter aims to help learn the formulation, resolution and interpretation of linear programming models and to show some of their applications in the industrial engineering and management area. Hence new variants of basic product mix models and mixtures applied to Industrial Organization Engineering and its management are proposed. Resolution methods are also reviewed: graphic method and Simplex Method. Creating and solving dual problems are also considered. Finally, special attention is paid to the economic interpretation of the results of some of the problems presented through their sensitive analysis.

1.1 Introduction

Operations research employs the scientific method as a basis to deal with decision-making problems by designing and solving mathematical models. One of the most studied and developed is linear programming, which seeks to optimise a linear objective function that is subject to some constraints which are also linear. The name linear programming does not come from producing computer programmes, but originates from planning activities carried out in an organization to optimise available resources.

Linear programming techniques are employed in a large number of problems, such as production planning, financial planning, human resources management, transport problems, distribution, allowances designs, forest planning, scheduling flights, etc. It is also worth stressing the significant progress made by optimization software due to the increased calculation power of computers and to the fall in hardware costs. There is a considerable number of software packages available to solve linear programming problems.

DOI: 10.1007/978-1-4471-5577-5_1,

The basis of linear programming lies in the study of linear inequalities, which appeared for the first time in the work of Fourier (1768–1830). Kantorovich (1939, 1942), of Soviet origin, was among the first to apply linear programming in the field of economy to optimise planning resources. Yet for several years, Kantorovich's work remained unknown in the Western world and was virtually not known in Eastern Europe. Specifically, Stiegler (1945) also contemplated the linear problem to obtain a suitable diet at a minimum cost based on 77 foods by considering nine nutrients. This problem acknowledged the structure of optimising a linear function subject to linear constraints.

Linear programming really began to arouse interest as from 1947 when Dantzig (1953, 1963) developed the Simplex Method to solve the planning problems arising in the U.S. Air Force during World War II. Other parallel concepts to linear programming ones were formulated by von Neumann and Morgensten (1947), who applied the Minimax Theorem to strategy games. As an immediate example of its background, we find the transport planning problem considered by Hitchcock (1941) in the United States.

Koopmans (1951) showed how linear programming provided a suitable model to analyse classic economic theories. During the remainder of the 1950s, linear programming became completely established thanks to the works of Chanrnes (1952) and Chanrnes and Lemke (1952, 1954) on degeneration, to those of Lemke and Chanrnes (1953) and Lemke (1954, 1965, 1968) on duality, and to that of Dantzig(Dantzig 1955) on the compact form and decomposition of large programmes. Ford and Fulkerson (1955, 1957, 1962), who were contracted by the RAND Corporation, established the results on flows in graphs and the primal–dual method for distribution problems. There are also the linear decision rules of Holt et al. (1960), as well as the works of Bowman (1956) to solve production planning problems by employing transport methods.

In 1975, Kantorovich received the Nobel Prize in Economics from the Royal Swedish Academy of Sciences for his contributions to the optimum resources allocation problem. He shared this prize with Tjalling Charles Koopmans. Apparently, the Academy considered that Dantzig's work was too mathematical and there is no Nobel Prize in Mathematics.

Linear programming is a mathematical process to determine the optimum allocation of scarce resources. Allocation problems can come in widely varying forms. There are two types of very common problems: the product mix problem and the mixtures problem. The product mix problem intends to discover which and how many products must be included in the production programme to maximise profits. The mixtures problem attempts to determine the minimum quantity of resources possible to be used to obtain a given level of product or service. Any linear programming problem consists in an objective function and a set of constraints which must satisfy the following conditions: the objective function must be linear; the objective must represent the decision maker's goal and must be the maximisation or the minimization of a linear function; constraints must also be linear.

There are two important object types for most linear programming problems: first, limited resources and second, activities. Each activity consumes or provides additional quantities of resources. The problem consists in determining the best combination of levels of the activities that do not employ more resources than those available. The Simplex Method is a widely used solution algorithm for solving linear programmes. The graphic method is useful when there are only two decision variables.

Linear programming models offer a series of limitations owing to some assumptions created to simplify reality and to represent it by means of a mathematical model. The created assumptions are:

1. Deterministic problems are considered. In other words, all the data are known with certainty.
2. It is assumed that the objective function is linear.
3. Constraints are also considered linear.
4. Decision variables cannot take negative values.
5. The additivity of resources, the total use of each resource, is obtained by summing partial usages of this resource.
6. The divisibility of decision variables; these variables can take fractional values.
7. Independency is assumed among activities and resources for the various decision variables.
8. Both the quantity of the resource employed and the objective function value are proportional to the values of the decision variables. This proportionality requirement is met as the function is objective and constraints are linear.

A linear programming model consists of the following components: decision variables, objective function and constraints. These three model components are linked by mathematical relations.

Decision variables are those factors among which the decision maker must choose and they are controllable variables. The aim of linear programming is to find the best values for these decision variables. In linear programming, variables may take fractional values. In a decision-making process, some factors affect the result variables, but they are not controllable by the decision maker. These input parameters represent the limitations imposed by the environment (interest rates, prices of raw materials, market demand, etc.). The objective function represents the relation between decision variables and uncontrollable variables. The objective could be the maximisation or minimization of any quantity, and is limited by constraints. Constraints express the limitations imposed on management systems owing to the relations with the environment.

Building a linear programming model consists in the following steps: (1) defining decision variables; (2) defining the objective or goal in terms of the decision variables; (3) defining all the system constraints and (4) restricting all the variables so they are not negative. A linear programming model can be expressed canonically as:

$$\begin{array}{ll} \text{Maximise} & c^T x \\ \text{subject to} & Ax \leq b \\ \text{and} & x \geq 0 \end{array}$$

where x represents the vector of decision variables, c and b are vectors of known coefficients and A is a known matrix of coefficients. Objective function $c{\cdot}x$ can be maximised or minimised. Inequalities $A{\cdot}x \leq b$ are the constraints. The non-negativity constraint is represented by $x \geq 0$.

Linear programming is a powerful tool to select alternatives in a decision problem and is, therefore, applied to a wide variety of settings. The number of applications is so high that it would be impossible to list them all. This book chapter provides some of these feasible applications. After reading this chapter, the reader should be capable of: understanding the nature of resources optimization problems and their formulation by linear programming; knowing the resolution graphic model for linear programming problems with two decision variables; comprehending the Simplex Method's logic nature and of applying it to solve linear programming models; formulating the dual problem of a linear programming model; economically interpreting the solutions of a dual problem; understanding the sensitivity analysis concept and of using it for decision making. Selected books for further reading can be found in the Anderson (2009), Bazaraa (1990), Fletcher (2000), Gale (1960), Hillier (2002), Murty (1995), (1983), Rardin (1998), Ravindran (1978), Taha (2010), Von Neumann (1944), Williams (1985) and Winston (2003).

1.2 Bonuses and Merits

The Campus Social Board in New York budgeted \$1,500,000 as the bonuses and merits of 503 university teachers. A Salary Differential Evaluation Board, comprising Campus Social Board members, administrators, supervisors and teachers, determined the three variables that should affect bonuses and merits. If we name these variables as CM_1, CM_2 and CM_3, we obtain:

CM_1: Teachers' performance measured by the university evaluation instrument.
CM_2: General services to the university and its students, including voluntary activities, etc.
CM_3: Professional development, measured by a formula, which includes teaching merits, research, participation in management tasks, etc.

All three variables above are measured on a scale ranging from 1 (low) to 5 (high). The points scored during 1 year determine bonuses and merits for the following year. In 2001, the sum of the points of these variables was:

$CM_1 = 1{,}625$
$CM_2 = 1{,}409$
$CM_3 = 1{,}387$

The Evaluation Board recommended, and the Campus Social Board accepted, the following rules:

- Rule 1: No teacher should receive more than \$5,000 for bonuses and merits in any given year.
- Rule 2: One CM_1 variable point is worth twice a CM_2 variable point; one CM_1 variable point is worth twice a CM_3 variable point.
- Rule 3: The value of one of the CM_1, CM_2, CM_3 variable points should be as high as possible, but subject to the total budget and to these rules.

(a) Formulate a linear programming model that can establish the optimum value of one of the CM_1, CM_2, CM_3 variable points for the year 2002.
(b) Solve the problem.
(c) Explain the meaning of the results obtained: decision variables, the objective function value and surplus/excess variables.

Solution

(a) Formulate a linear programming model that can establish the optimum value of one of the CM_1, CM_2, CM_3 variable points for the year 2002.
Decision variables:

$X_1 =$ value in dollars of one CM_1 variable point
$X_2 =$ value in dollars of one CM_2 variable point
$X_3 =$ value in dollars of one CM_3 variable point.

Objective function:

$$\text{Max}\, z = X_1 + X_2 + X_3$$

Constraints:

$C_1 : 5X_1 + 5X_2 + 5X_3 \leq 5000 (\text{rule 1})$
$C_2 : 1625X_1 + 1409X_2 + 1387X_3 \leq 1500000$ (not to exceed the total budget)
$C_3 : X_1 = 2X_2 (\text{rule 2})$
$C_4 : X_1 = 2X_3 (\text{rule 2})$
$X_1, X_2, X_3 \geq 0$ (non – negativity constraint)

(b) Solving the problem.

$\text{Max}\, z = X_1 + X_2 + X_3 + 0S_1 + 0S_2 - MA_1 - MA_2$
$C_1 : 5X_1 + 5X_2 + 5X_3 + S_1 = 5000$
$C_2 : 1625X_1 + 1409X_2 + 1387X_3 + S_2 = 1500000$
$C_3 : X_1 - 2X_2 + A_1 = 0$
$C_4 : X_1 - 2X_3 + A_2 = 0$

(c) Explain the meaning of the results obtained: decision variables, the objective function value and surplus/excess variables Tables (1.1, 1.2, 1.3, 1.4).

Table 1.1 Simplex method

	C_j	1	1	1	0	0	−M	−M		
C_j	Base	X_1	X_2	X_3	S_1	S_2	A_1	A_2	R	Ratio
0	S_1	5	5	5	1	0	0	0	5,000	1,000
0	S_2	1,625	1,409	1,387	0	1	0	0	1,500,000	923
−M	A_1	1	−2	0	0	0	1	0	0	0
−M	A_2	1	0	−2	0	0	0	1	0	0
	C_j−Z_j	1 + 2M	1 − 2M	1− 2M	0	0	0	0		

Table 1.2 Simplex method

	C_j	1	1	1	0	0	−M	−M		
C_j	Base	X_1	X_2	X_3	S_1	S_2	A_1	A_2	R	Ratio
0	S_1	0	5	15	1	0	0	−5	5,000	333.33
0	S_2	0	1,409	4,637	0	1	0	−1,625	1,500,000	323.49
−M	A_1	0	−2	2	0	0	1	−1	0	0
1	X_1	1	0	−2	0	0	0	1	0	
	C_j−Z_j	0	1−2M	3 + 2M	0	0	0	−1−2M		

Table 1.3 Simplex method

	C_j	1	1	1	0	0	−M	−M		
C_j	Base	X_1	X_2	X_3	S_1	S_2	A_1	A_2	R	Ratio
0	S_1	0	20	0	1	0	−7.5	2.5	5,000	250
0	S_2	0	6,046	0	0	1	−2318.5	693.5	1,500,000	248.1
1	X_3	0	−1	1	0	0	0.5	−0.5	0	
1	X_1	1	−2	0	0	0	1	0	0	
	C_j−Z_j	0	4.0	0	0	0	−1.5−M	0.5−M		

Table 1.4 Simplex method

	C_j	1	1	1	0	0	−M	−M	
C_j	Base	X_1	X_2	X_3	S_1	S_2	A_1	A_2	R
0	S_1	0	0	0	1	−0	0.16953	0.206	38.04
1	X_2	0	1	0	0	0	−0.3835	0.115	248.10
1	X_3	0	0	1	0	0	0.11652	−0.385	248.10
1	X_1	1	0	0	0	0	0.23305	0.229	496.20
	C_j−Z_j	0.0	0.0	0.0	0	0	0.0339−M	0.0412−M	

Each point of variable X_1 is the equivalent to \$496.20, and each point of variables X_2 and X_3 is the equivalent to \$248.10.

Constraint C_1 contains a surplus variable, $S_1 = 38.04$. This implies that the teachers who score (5, 5, 5), will receive \$38.04 less than the \$5,000 fixed as the maximum bonus/merit. In other words, the budgeted amount of \$1,500,000 would be exhausted if all the teachers scored (5, 5, 5) and they all received \$5,000.

The objective function is not of much interest in this case as it simply indicates that the sum of the three compensation variables is $992.39.

1.3 Production Planning in a Textile Firm

A textile firm produces five types of fabric. Each fabric can be woven on one or more of the factory's 38 looms. The Sales Department has foreseen demand for the forthcoming month. The demand details appear in Table 1.5 along with the data on sale price, variable cost and purchase price, which are per metre and for a width of 140 cm. The factory operates 24 h/day and is scheduled for 30 days during the following month.

The factory possesses two loom types: *jacquard* and *ratier*. *Jacquard* looms are more versatile and can be used to produce all five fabrics. *Ratier* looms produce only three of the five fabrics. In all, there are 38 looms: 8 *jacquard* and 30 *ratier*. Table 1.6 indicates the production in fabrics of both loom types. The time required to change fabrics is not significant and is not worth considering.

The firm at least meets all the required demand, be it with its own fabrics or with those acquired from another factory. That is to say, the fabrics which cannot be woven in the factory itself, given the loom capacity limitations, will be be acquired from another factory. The purchase price of each fabric also appears in Table 1.5.

(a) Build a model that can be used to schedule this textile firm's production and which can also determine how many metres of each fabric must be acquired from the other factory.

Table 1.5 Monthly demand, sale price, variable cost and purchase price of fabrics

Fabric	Demand (metres)	Sale price ($/m)	Variable cost ($/m)	Purchase price ($/m)
1	16,500	3.99	2.66	2.86
2	22,000	3.86	2.55	2.70
3	62,000	4.10	2.49	2.60
4	7,500	4.24	2.51	2.70
5	62,000	3.70	2.50	2.70

Table 1.6 Loom speed (m/h)

Fabric	Jacquard	Ratier
1	4.63	–
2	4.63	–
3	5.23	5.23
4	5.23	5.23
5	4.17	4.17

(b) The factory is considering acquiring a ninth *jacquard* loom. In what two ways can profit (if any) be analysed if there is any extra weaving time available? Explain.

(c) How would the model be amended if instead of scheduling production for a single month, the firm had data for the next six months and wished to plan production for these six months?

Solution

(a) Build a model to be used to schedule this textile firm's production and which can also determine how many metres of each fabric must be acquired from the other factory.

Decision variables:

$X1_J =$ metres of fabric 1 woven on a Jacquard loom
$X_{IA} =$ metres of fabric 1 acquired elsewhere
$X_{2J} =$ metres of fabric 2 woven on a Jacquard loom
$X_{2A} =$ metres of fabric 2 acquired elsewhere
$X_{3J} =$ metres of fabric 3 woven on a Jacquard loom
$X_{3R} =$ metres of fabric 3 woven on a Ratier loom
$X_{3A} =$ metres of fabric 3 acquired elsewhere
$X_{4J} =$ metres of fabric 4 woven on a Jacquard loom
$X_{4R} =$ metres of fabric 4 woven on a Ratier loom
$X_{4A} =$ metres of fabric 4 acquired elsewhere
$X_{5J} =$ metres of fabric 5 woven on a Jacquard loom.
$X_{5R} =$ metres of fabric 5 woven on a Ratier loom
$X_{5A} =$ metres of fabric 5 acquired elsewhere.

Objective Function:

$$\begin{aligned}\text{Max}\, z = {} & 1.33X_{1J} + 1.13X_{1A} + 1.31X_{2J} + 1.16X_{2A} + 1.61(X_{3J} + X_{3R}) + 1.5X_{3A} \\ & +1.73(X_{4J} + X_{4R}) + 1.54X_{4A} + 1.2(X_{5J} + X_{5R}) + X_{5A}\end{aligned}$$

Constraints:
(Demand constraints)

$$\begin{aligned}&X_{1J} + X_{1A} \geq 16500 && \text{(fabric 1)}\\ &X_{2J} + X_{2A} \geq 22000 && \text{(fabric 2)}\\ &X_{3J} + X_{3R} + X_{3A} \geq 62000 && \text{(fabric 3)}\\ &X_{4J} + X_{4R} + X_{4A} \geq 7500 && \text{(fabric 4)}\\ &X_{5J} + X_{5R} + X_{5A} \geq 62000 && \text{(fabric 5)}\end{aligned}$$

(Available capacity constraints)

$$X_{1J}/(4.63\cdot 24\cdot 30)\ +X_{2J}/(4.63\cdot 24\cdot 30)\ +X_{3J}/(5.23\cdot 24\cdot 30)$$
$$+X_{4J}/(5.23\cdot 24\cdot 30)\ +X_{5J}/(4.17\cdot 24\cdot 30)\leq 8X_{3R}/(5.23\cdot 24\cdot 30)$$
$$+X_{4R}/(5.23\cdot 24\cdot 30)\ +X_{5R}/(4.17\cdot 24\cdot 30)\leq 30$$

(Non-negativity constraint)

$$X_{1J},X_{1A},X_{2J},X_{2A},X_{3J},X_{3R},X_{3A},X_{4J},X_{4R},X_{4A},X_{5J},X_{5R},X_{5A}\geq 0$$

(b) The factory is considering acquiring a ninth *Jacquard* loom. In what two ways can profit (if any) be analysed if there is any additional weaving time available? Explain.

a. It could substitute the capacity restriction for the *jacquard* loom by the following: $X_{1J}/(4.63\cdot 24\cdot 30)+X_{2J}/(4.63\cdot 24\cdot 30)+X_{3J}/(5.23\cdot 24\cdot 30)+X_{4J}/(5.23\cdot 24\cdot 30)+X_{5J}/(4.17\cdot 24\cdot 30)\leq 9$, and then solve the problem again.
b. By determining the production resource shadow price of the *jacquard* looms (the available capacity constraint of the *jacquard* looms) by either solving the dual problem or starting by analysing the solution obtained by a programme like WinQSB® or Solver® (Excel).

(c) How would the model be amended if instead of scheduling production for a single month, the firm had data for the next 6 months and wished to plan production for these 6 months?

It would be necessary to define as many decision variables as the quantities to be produced of each fabric on each loom or in each subcontract during each period, which would total to 13×6 decision variables. Hence, it would be necessary to work with subindices. Besides, if the demand constraints and the available capacity varied according to the period, it would also be necessary to define each one all for all six periods contemplated.

1.4 Portfolio of Investments

Saúl Cortés, an Industrial Organization Engineer, wishes to form his own portfolio of investments for the purpose of employing the minimum initial investment possible to subsequently generate specific amounts of capital over the next six years (he considers year 1 to year 6). The purpose of Saúl's investments analysis is to plan his daughter's (Susana) expenses when she starts university within 2 years (year 3). Saúl's financial requirements are defined in Table 1.7 below.

The characteristics of the possible investments that Saúl can choose are provided in Table 1.8.

Table 1.7 Saúl Cortés' financial requirements

Year	$
3	20,000
4	22,000
5	24,000
6	26,000

Table 1.8 Characteristics of the investments

Choice	Profitability (%)	Maturity (years)
A	5	1
B	13	2
C	28	3
D	40	4

Products C and D involve risks, so Saúl does not want to invest more than 20 % of the total investment in them.

Build a linear programming model that helps Saúl to optimally solve his investments problem.

Solution

Decision variables:

A_1 = amount in \$ invested in A at the start of year 1
B_1 = amount in \$ invested in B at the start of year 1
C_1 = amount in \$ invested in C at the start of year 1
D_1 = amount in \$ invested in D at the start of year 1
A_2 = amount in \$ invested in A at the start of year 2
B_2 = amount in \$ invested in B at the start of year 2
C_2 = amount in \$ invested in C at the start of year 2
D_2 = amount in \$ invested in D at the start of year 2
A_3 = amount in \$ invested in A at the start of year 3
B_3 = amount in \$ invested in B at the start of year 3
C_3 = amount in \$ invested in C at the start of year 3
A_4 = amount in \$ invested in A at the start of year 4
B_4 = amount in \$ invested in B at the start of year 4
A_5 = amount in \$ invested in A at the start of year 5.

Objective function: to minimise the initial investment

$$\text{Min } z = A_1 + B_1 + C_1 + D_1$$

Constraints:

Amounts which mature at the end of the year (i − 1)—Amount invested at the start of year 1 = amount paid for education in year 1

$1.05A_1 - A_2 - B_2 - C_2 - D_2 = 0$ (investment in A in year 1 + 5 % interest) (*cash flow* of year 2)

$1.13B_1 + 1.05A_2 - A_3 - B_3 - C_3 = 20000$ (*cash flow* of year 3)

$1.28C_1 + 1.13B_2 + 1.05A_3 - A_4 - B_4 = 22{,}000$ (*cash flow* of year 4)
$1.4D_1 + 1.28C_2 + 1.13B_3 + 1.05A_4 - A_5 = 24{,}000$ (*cash flow* of year 5)
$1.4D_2 + 1.28C_3 + 1.13B_4 + 1.05A_5 = 26{,}000$ (*cash flow* of year 6)
Constraints of the 20 % risk
$0.8(C_1 + D_1) - 0.2(A_1 + B_1) \leq 0$ (risk of year 1)
$0.8(C_1 + D_1 + C_2 + D_2) - 0.2(B_1 + A_2 + B_2) \leq 0$ (risk of year 2)
$0.8(C_1 + D_1 + C_2 + D_2 + C_3) - 0.2(B_2 + A_3 + B_3) \leq 0$ (risk of year 3)
$0.8(D_1 + C_2 + D_2 + C_3) - 0.2(B_3 + A_4 + B_4) \leq 0$ (risk of year 4)
$0.8(D_2 + C_3) - 0.2(B_4 + A_5) \leq 0$ (risk of year 5)
Non-negativity restriction

$$A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_4, C_1, C_2, C_3, D_1, D_2 \geq 0$$

1.5 Transferring Currencies

ORGASA, a global firm, operates in the United States, Great Britain, India and Spain. This firm needs to frequently transfer money to cover its cash requirements. These requirements and surpluses in each delegation are provided in Table 1.9.

The currency exchange rates offered by the bank with which ORGASA always works are provided in Table 1.10.

In each column, this table presents the sale prices of the various currencies (which include bank commissions). Hence for instance, when selling a million pounds (sterling), 1.6152 million \$ are received. ORGASA notices that there are several possibilities of transferring currencies which could help cover its requirements, but it is not sure which possibility is the best.

(a) Build a linear programming model which helps find the most convenient method to transfer currencies, and take into account that the aim consists in maximising the value in dollars (USD) at the final position; in other words,

Table 1.9 Currency requirements and surpluses in each delegation

Currency	Symbol	Surplus (in millions)	Requirement (in millions)
Euro	EUR	2.4	
Pound (sterling)	GBP		2.1
US Dollar	USD		5
Indian Rupiah	INR	350	

Table 1.10 Exchange types

	From				
	Currency	USD	GBP	EUR	INR
A	USD	1.00000	1.61520	1.00600	0.02297
	GBP	0.61880	1.00000	0.62230	0.01422
	EUR	0.99350	1.60540	1.00000	0.02282
	INR	43.52200	70.29670	43.78310	1.00000

Table 1.11 The solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_{12}	0	−1.0000	0	−0.0004	At bound	−M	−0.9996
X_{13}	0	−1.0000	0	−1.9995	At bound	−M	0.9995
X_{14}	0	−1.0000	0	−1.9997	At bound	−M	0.9997
X_{21}	0	1.6152	0	−0.0001	At bound	−M	1.6153
X_{23}	0	0	0	−3.2304	At bound	−M	3.2304
X_{24}	0	0	0	−3.2300	At bound	−M	3.2300
X_{31}	2.4000	1.0060	2.4144	0	Basic	1.0057	1.0066
X_{32}	0	0	0	−0.0008	At bound	−M	0.0008
X_{34}	0	0	0	−0.0003	At bound	−M	0.0003
X_{41}	202.3207	0.0230	4.6473	0	Basic	0.0230	0.0230
X_{42}	147.6793	0	0	0	Basic	0.0000	0.0000
X_{43}	0	0	0	0.0000	At bound	−M	0.0000
Objective Function	Max=	7.0617					

Constraint	Left-hand side	Direction	Right-hand side	Slack or Surplus	Shadow price	Allowable Min. RHS.	Allowable Max. RHS.
C1	7.0617	>=	5.0000	2.0617	0	−M	7.0617
C2	2.1000	>=	2.1000	0	−1.6153	0	3.3763
C3	2.4000	<=	2.4000	0	1.0060	0.3506	M
C4	350.0000	<=	350.0000	0	0.0230	260.2435	M

determine the value of the amount of cash available if it has all been changed to dollars and after covering minimum requirements in pounds and dollars. Identify the decision variables, the objective function and the constraints with the data provided by the formulation (note: think of the problem as if it were a variant of a transport model in which money can be transferred from any node to any node, provided an equilibrium remains between inputs and outputs).

Using the model solution obtained with WinQSB® (see Table 1.11), answer the following questions:

(b) Interpret the meaning of the solution (decision variables and objective function).

(c) What would the additional cost come to of requiring one million more monetary units in both the United States and Great Britain?

(d) What surplus is there in each delegation where before there was a surplus? In which of the two delegations with slack would it have been more interesting to have more cash available?

(e) By how much would the final amount available in dollars increase if the requirement of having cash in Great Britain did not exist?

Solution

(a) Build a linear programming model which helps find the most convenient method to transfer currencies, and take into account that the aim consists in maximising the value in dollars (USD) at the final position; in other words, determine the value of the amount of cash available if it has all been changed to dollars and after covering minimum requirements in pounds and dollars. Identify the decision variables, the objective function and the constraints with the data provided by the formulation (note: think of the problem as if it were a variant of a transport model in which money can be transferred from any node to any node, provided there is an equilibrium between inputs and outputs).

Decision variables:

X_{12} = USD units changed to GBP
X_{13} = USD units changed to EUR
X_{14} = USD units changed to INR
X_{21} = GBP units changed to USD
X_{23} = GBP units changed to EUR
X_{24} = GBP units changed to INR
X_{31} = EUR units changed to USD
X_{32} = EUR units changed to GBP
X_{34} = EUR units changed to INR
X_{41} = INR units changed to USD
X_{42} = INR units changed to GBP
X_{43} = INR units changed to EUR.

Objective function:

Max $z = 1.61520X_{21} + 1.00600X_{31} + 0.02297X_{41} - X_{12} - X_{13} - X_{14}$; (the inputs corresponding to the money in USD are maximised, less the outputs of the USD of node 1).

Constraints:

$1.61520X_{21} + 1.00600X_{31} + 0.02297X_{41} - X_{12} - X_{13} - X_{14} \geq 5$; (Everything reaching node 1 (USD), less anything that leaves, must be greater than or equal to 5 million EUR)

$0.61880X_{12} + 0.62230X_{32} + 0.01422X_{42} - X_{21} - X_{23} - X_{24} \geq 2.1$; (Everything reaching node 2 (GBP), less anything that leaves, has to be greater than or equal to 2.1 million GBP)

$X_{31} + X_{32} + X_{34} - 0.99350X_{13} - 1.60540X_{23} - 0.02282X_{43} \leq 2.4$; (Everything leaving node 3 (EUR) has to be lower than or equal to 2.4 million EUR, plus anything that could arrive)

$X_{41} + X_{42} + X_{43} - 43.52200X_{14} - 70.29670X_{24} - 43.78310X_{34} \leq 350$; (Everything leaving node 4 (INR) has to be lower than or equal to 350 million INR, plus anything that could arrive)

$X_{12}, X_{13}, X_{14}, X_{21}, X_{23}, X_{24}X_{31}, X_{32}, X_{34}, X_{41}, X_{42}, X_{43} \geq 0$; (non-negativity constraint)

(b) Interpret the meaning of the solution (decision variables and objective function).
ORGASA should change 2.4 million EUR into dollars to obtain 2.4144 million USD; it should also change 202.3207 million INR into dollars to obtain 4.6473 million \$, and the remaining INR (147.6793) should be changed into 2.1 million pounds (sterling), required by Great Britain. Thus a total of 7.0617 million \$ would be obtained.
(c) What would be the additional cost of requiring another million monetary units in both the United States and Great Britain?
Should the United States require 6 million \$, it would not involve any extra cost, but would imply an extra cost of 1.6153 (in pounds (sterling)) for Great Britain.
(d) What surplus is there in each delegation where before there was a surplus? In which of the two delegations with slack would it have been more interesting to have more cash available?
No surplus and all the available slack is used, so it would have been interesting to have more available in Spain.
(e) By how much would the final amount available in dollars increase if the requirement of having cash in Great Britain did not exist?

This case can be calculated in two ways:
Shadow price (with a positive sign) $\cdot$ 2.1 = 1.6153 $\cdot$ 2.1 = 3.39213;
All the Indian rupiahs destined to be pounds (sterling) are changed into USD: 147.6793 $\cdot$ 0.02297 = 3.39213;

1.6 Production Planning in a Metallurgical Company

A manufacturer in a metal firm of Frankfurt produces four types of products in sequence on two machines. Table 1.12 provides the technical details of this manufacturer's production.

(a) Build a linear programming model that optimises this manufacturer's daily production.
(b) Take into account the Sensitivity Report, as shown in Table 1.13, provided by Solver, Microsoft Excel®, and indicate the optimum solution.

Table 1.12 The production details of the metallurgical manufacturer

Production time per unit (in min)						
Machine	Cost per minute (\$)	Product 1	Product 2	Product 3	Product 4	Daily production capacity (in min)
1	10	2	3	4	2	500
2	5	3	2	1	2	380
Sale price per unit (\$)		65	70	55	45	

Table 1.13 The sensitivity report obtained by solver, Microsoft excel®

Changing cells

Cell	Name	Equal value	Reduced gradient
\$B\$1	X_1	28	0
\$B\$2	X_2	148	0
\$B\$3	X_3	0	−20,00000572
\$B\$4	X_4	0	−9,000011921

Constraints

Cell	Name	Equal value	Lagrange multiplier
\$B\$7	$2X1 + 3X2 + 4X3 + 2X4 \leq 500$	500	6
\$B\$8	$3X1 + 2X2 + 1X3 + 2X4 \leq 380$	380	6

(c) What is the maximum cost per minute which the manufacturer must be willing to incur for any machine?

(d) How much must the machine processing cost be cut per unit of product 3 to become profitable?

Solution

(a) Build a linear programming model that optimises this manufacturer's daily production.

Decision variables:

X_1 = units of product 1 to be produced daily
X_2 = unit of product 2 to be produced daily
X_3 = units of product 3 to be produced daily
X_4 = units of product 4 to be produced daily.

Objective function:

$$\text{To maximize } z = 65X_1 + 70X_2 + 55X_3 + 45X_4 - 10\,(2X_1 + 3X_2 + 4X_3 + 2X_4) - 5(3X_1 + 2X_2 + X_3 + 2X_4)$$

Constraints:

$$2X_1 + 3X_2 + 4X_3 + 2X_4 \leq 500$$

$$3X_1 + 2X_2 + X_3 + 2X_4 \leq 380$$

$$X_1, X_2, X_3, X_4 \geq 0$$

(b) Take into account the Sensitivity Report as shown in Table 1.13, provided by Solver, Microsoft Excel®, and indicate the optimum solution.

Produce 28 units of product 1 and 148 units of product 2.

(c) What is the maximum cost per minute in which the manufacturer must be willing to incur for any machine?
Must not pay more than \$6 per min on each machine.
(d) How much must the machine processing cost be cut per unit of product 3 to become profitable?
\$20 per unit of product 3.

1.7 Production Planning in a Cosmetics Firm

A firm from Milan sells chemical products for professional cosmetics. It is planning the production of three products, GCA, GCB and GCC, for a given period of time by mixing two different components: C1 and C2. All the end products must contain at least one of the two components, and not necessarily both.

For the next planning period, 10,000 l of C1 and 15,000 l of C2 are available. The production of GCA, GCB and GCC must be scheduled to at least cover the minimum demand level of 6,000, 7,000 and 9,000 l, respectively. It is assumed that when chemical components are mixed, there is no loss or gain in volume.

Each chemical component, C1 and C2, has a proportional critical element, 0.4 and 0.2, respectively. That is to say, each litre of C1 contains 0.4 l of the critical element. To obtain GCA, the mixture must proportionally contain at least a 0.3 fraction of the critical element. Another requirement is that the quantity of the critical element is seen in GCB, an 0.3 fraction at the most.

Furthermore, the minimum ratio of C1 with C2 in product GCC must be 0.3. The profit expected for the sale of each litre of GCA, GCB and GCC is \$125, \$135 and \$155, respectively.

(a) Build a linear programming model which optimises the production planning of this firm in Milan.
Based on the model solution obtained with WinQSB® (see Table 1.14), answer the following questions:
(b) Interpret the meaning of the solution (decision variables and objective function).
(c) For its image, this firm believes that all end products should contain at least 1,000 l of each component for both C1 and C2. If this plan is to go ahead, what impact would this have on the optimum profit?
(d) If the firm could stock up on more litres of component C1 or C2, which do you think it should buy? Why?
(e) It is believed that the firm could increase the minimum demand of product GCA to 6,200 l thanks to a good promotion. Could this improve profits?

Table 1.14 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_{11}	3000.0000	125.0000	375000.0000	0	Basic	−M	151.6667
X_{21}	3000.0000	125.0000	375000.0000	0	Basic	98.3333	171.6667
X_{12}	0	135.0000	0	−26.6667	At bound	−M	161.6667
X_{22}	9666.6670	135.0000	1305000.0000	0	Basic	65.0000	155.0000
X_{13}	7000.0000	155.0000	1085000.0000	0	Basic	128.3333	M
X_{23}	2333.3330	155.0000	361666.7000	0	Basic	135.0000	M
Objective function	Max=	3501667.0000					

Constraint	Left-hand side	Direction	Right-hand side	Slack or Surplus	Shadow price	Allowable Min. RHS.	Allowable Max. RHS.
C1	10000.0000	<=	10000.0000	0	161.6667	9750.0000	18000.0000
C2	15000.0000	<=	15000.0000	0	135.0000	12333.3300	M
C3	6000.0000	>=	6000.0000	0	−23.3333	0	6500.0000
C4	9666.6670	>=	7000.0000	2666.6670	0	−M	9666.6670
C5	9333.3330	>=	9000.0000	333.3333	0	−M	9333.3330
C6	0	>=	0	0	−13.3333	−4000.000	500.0000
C7	−9666.6670	<=	0	9666.6670	0	−9666.6670	M
C8	0.0002	>=	0	0	−6.6667	−8000.0000	1000.0000

(f) With the intention of selling more litres of products GCA and GCB, the idea is to raise the price of product GCC to achieve a profit of \$200/l. How would this idea affect component types C1 and C2 which would form part of the mixture of each end product?

Solution

(a) Build a linear programming model that optimises this Milan-based firm's production planning.

Decision variables:

X_{11} = Litres of C1 to be included in product GCA
X_{21} = Litres of C2 to be included in product GCA
X_{12} = Litres of C1 to be included in product GCB
X_{22} = Litres of C2 to be included in product GCB
X_{13} = Litres of C1 to be included in product GCC
X_{23} = Litres of C2 to be included in product GCC

Objective function:

$$\text{Max}\, z = 125(X_{11} + X_{21}) + 135(X_{12} + X_{22}) + 155(X_{13} + X_{23})$$

Constraints:

$X_{11} + X_{12} + X_{13} \leq 10000$	(availability of component C1)
$X_{21} + X_{22} + X_{23} \leq 15000$	(availability of component C2)
$X_{11} + X_{21} \geq 6000$	(demand of product GCA)
$X_{12} + X_{22} \geq 7000$	(demand of product GCB)
$X_{13} + X_{23} \geq 9000$	(demand of product GCC)
$0.4X_{11} + 0.2X_{21} \geq 0.3(X_{11} + X_{21})$	(Product GCA must contain at least one 0.3 fraction of the critical element)
$0.4X_{12} + 0.2X_{22} \leq 0.3(X_{12} + X_{22})$	(Product GCB must contain one 0.3 fraction of the critical element at the most)
$X_{13} \geq 0.3X_{23}$	(the minimum ratio between C1 and C2 must be 0.3)
$X_{11}, X_{21}, X_{12}, X_{22}, X_{13}, X_{23} \geq 0$	(non-negativity of the decision variables)

(b) Interpret the meaning of the solution (decision variables and objective function).

The firm should generate product GCA with 3,000 L of each component C1 and C2. Product GCB should be created with 9,666.667 L of C2. Product GCC should be made with 7,000 L of C1 and 2,333.333 L of C2. All this would give a profit of \$3,501,667.

(c) For its image, this firm believes that all the end products should contain at least 1,000 L of each component for both C1 and C2. If this plan is to go ahead, what impact would this cause on the optimum profit?

From the current profit of 3,501,667, −26.6667 would have to be deducted from the reduced cost of variable X_{12} for the 1,000 L proposed. This would entail a profit of $3,475,000.3.

(d) If the firm could stock up on more litres of component C1 or C2, which do you think it should buy? Why?

Component C1 because it presents the higher shadow price.

(e) It is believed that the firm could increase the minimum demand of product GCA to 6,200 l thanks to a good promotion. Could this improve profits?

No. In fact the opposite applies as it would entail a drop in its shadow price for R3, −23.3333, multiplied by the 200 L of increased demand. This would involve a profit of 3,497,000$.

(f) With the intention of selling more litres of products GCA and GCB, the idea is to raise the price of product GCC to achieve a profit of $200/L. How would this idea affect component types C1 and C2 which would form part of the mixture of each end product?

There would be no effect because the profit coefficient for variables X_{13} and X_{23} is of an unlimited range, which means that although this profit coefficient increases, the optimum solution structure would not change. That is to say, products GCA and GCC would be formed by C1 and C2, while product GCB would be formed only by C2.

1.8 Product Mix of Aluminium Windows

The REGIORG firm in Athens manufactures aluminium windows. This firm supplies three quality types called: normal, economic and luxury. The relevant information about its production process is provided in Table 1.15.

The unit profit of each quality type is $30, $20 and $40, respectively. This firm works with an optimum product mix determined by means of linear programming solved with WinQSB® (see Table 1.16).

Despite this, the firm's Organization Engineer is concerned about it. Answer the following questions:

(a) The Engineer believes that, for the firm's image, it must product at least 40 luxury quality units per month. If this plan goes ahead, how would it affect the current profit?

Table 1.15 Production data

Operation	Hours/month required			Hours/month available
	Normal	Economic	Luxury	
1	1	3	2	400
2	2	0	3	600
3	1	4	0	600

Table 1.16 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or Profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_1	300.0000	30.0000	9000.0000	0	Basic	24.4444	M
X_2	33.3333	20.0000	666.6666	0	Basic	0	90.0000
X_3	0	40.0000	0	−8.3333	At bound	−M	48.3333
Objective function	Max=	9666.6670					

Constraint	Left-hand side	Direction	Right-hand side	Slack or Surplus	Shadow price	Allowable Min. RHS.	Allowable Max. RHS.
C1	400.0000	<=	400.0000	0	6.6667	300.0000	525.0000
C2	600.0000	<=	600.0000	0	11.6667	0	800.0000
C3	433.3333	<=	600.0000	166.6667	0	433.3333	M

(b) The Engineer thinks that the operation time for the economic quality type can be cut to only 3 h. Could this improve the current profit?
(c) With the intention of producing more luxury windows, the current prices policy has been reviewed and it has been concluded that they can change to 30, 10 and \$50 for the normal, economic and luxury quality types, respectively. Do you think that the firm's current profit would increase if this plan went ahead?
(d) The Engineer has informed the management that the capacities available for each operation will be 360, 600 and 500 per month for operations 1, 2 and 3, respectively, owing to the wear of the machines. How would this affect the current product mix?
(e) The Engineer is studying the possibility of producing a new type of standard window which would entail 2, 0 and 2 h for operations 1, 2 and 3, respectively. It is estimated that this window quality could provide a profit of \$20 per unit. What can be recommended in relation to this?

Solution

(a) The Engineer believes that, for the firm's image, it must produce at least 40 luxury quality units per month. If this plan goes ahead, how would it affect on the current profit?
Profit would lower by \$333.33.
(b) The Engineer thinks that the operation time 3 for the economic quality type can be cut to only 3 h. Could this improve the current profit?
No.
(c) With the intention of producing more luxury windows, the current prices policy has been reviewed and it has been concluded that they can change to 30, 10 and \$50 for the normal, economic and luxury quality types, respectively. Do you think that the firm's current profit would increase if this plan went ahead?
It would be necessary to once again solve the model to answer this question.

(d) The Engineer has informed the management that the capacities available for each operation will be 360, 600 and 500 per month for operations 1, 2 and 3, respectively, owing to the wear of the machines. How would this affect the current product mix?
Normal and economic windows would continue to be produced.
(e) The Engineer is studying the possibility of producing a new type of standard window which would entail 2, 0 and 2 h for operations 1, 2 and 3, respectively. It is estimated that this window quality could provide a profit of \$20 per unit. What can be recommended in relation to this?
It would be necessary to once again solve the model to answer this question.

1.9 Production Planning in the Automobile Industry

An automobile assembly plant assembles two types of vehicles: a four-door saloon and a people carrier. Both vehicle types must pass through a painting plant and an assembly plant. If the painting plant only paints four-door saloons, it can paint some 2,000 vehicles each day, whereas if it paints only people carriers, it can paint some 1,500 vehicles each day. Moreover, if the assembly plant only assembles either four-door saloons or people carriers, it can assemble some 2,200 vehicles every day. Each people carrier implies an average profit of \$3,000, whereas a four-door saloon implies an average profit of \$2,100.

(a) Use linear programming and indicate the daily production plan that would maximise the vehicle assembly plant's daily profit.
(b) What surpluses would be produced in the painting plant? And in the assembly plant?

Solution

(a) Use linear programming and indicate the daily production plan that would maximise the vehicle assembly plant's daily profit.

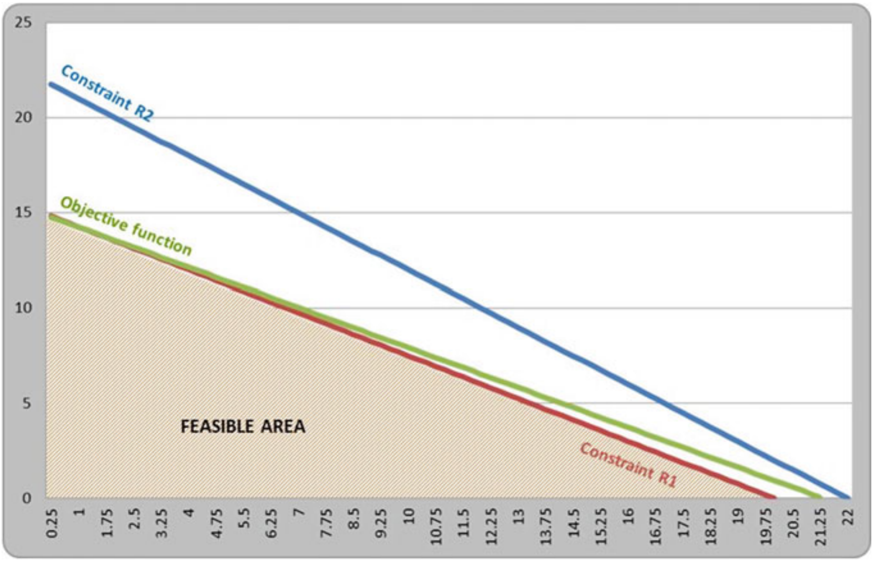
Decision variables:

X_1 = Hundreds of four-door saloons produced daily
X_2 = Hundreds of people carriers produced daily
Objective function:
To maximise $z = 21\ X_1 + 30\ X_2$

Constraints

R1: The fraction of the day during which the painting plant is occupied is equal to or less than 1:
R1: The fraction of the day during which the painting plant works on four-door saloons: 1/2.000
R1: The fraction of the day during which the painting plant works on people carriers: 1/1.500

R1: $\frac{1}{20}X_1 + \frac{1}{15}X_2 \leq 1$
R2: The fraction of the day during which the assembly plant is occupied is equal to or less than 1: The fraction of the day during which the assembly plant works on four-door saloons or people carriers: 1/2.200.
R2: $\frac{1}{22}X_1 + \frac{1}{22}X_2 \leq 1$
R3: The non-negativity constraint
R3: $X_1, X_2 \geq\ = 0.$
As this model has two decision variables, the following solution can be solved graphically:



Thus, the assembly plant should not assemble four-door saloons, but 1,500 people carriers each day, which entails a daily profit of $4,500,000.

(b) What surpluses would be produced in the painting plant? And in the assembly plant?
The painting plant is saturated, that is, it has no surplus, yet the assembly plant has a surplus of 0.3.

1.10 Portfolio of Investments

ORGASA has a portfolio of investments in shares, bonds and other investment alternatives. Now it has available funds amounting to $200,000 which must be considered for new investments. The four investment alternatives that ORGASA is contemplating are offered in Table 1.17.

Table 1.17 Investments data

Financial details	Telefónita	Sankander	Ferrofial	Gamefa
Price per share ($)	100	50	80	40
Annual rate of return	0.12	0.08	0.06	0.10
Measure of risk per $ invested	0.10	0.07	0.05	0.08

The measure of risk indicates uncertainty associated with the share in terms of its capacity to reach the annual return foreseen; the higher the value, the greater the risk.

ORGASA has stipulated the following conditions for its investment:

- Rule 1: The annual rate of return from this portfolio must be at least 9 %.
- Rule 2: No value can represent more than 50 % of the total investment in dollars.

(a) Use the linear programming model to develop a portfolio of investments which minimises the risk.
(b) If the firm ignores the risk involved and uses a maximum return strategy on its investment, how will the former model be amended?

Solution

(a) Use the linear programming model to develop a portfolio of investments which minimises the risk.
Decision variables:

$X_1 =$ Number of Telefónita shares
$X_2 =$ Number of Sankander shares
$X_3 =$ Number of Ferrofial shares
$X_4 =$ Number of Gamefa shares.

Objective function:

$$\text{To minimize } z = 0.1 \cdot 100X_1 + 0.07 \cdot 50X_2 + 0.05 \cdot 80X_3 + 0.08 \cdot 40X_4$$

Constraints:

$$0.12 \cdot 100X_1 + 0.08 \cdot 50X_2 + 0.06 \cdot 80X_3 + 0.1 \cdot 40X_4 \geq 18000; \text{ (rate of return} \geq 9\,\%)$$

$100X_1 \leq 100000$; (no value can represent more than 50 % of the total investment)

$$50X_2 \leq 100000;$$

$$80X_3 \leq 100000;$$

$$40X_4 \leq 100000;$$

$100X_1 + 50X_2 + 80X_3 + 40X_4 = 200.000$; (the investment must be equal to the available $200,000)
$X_1, X_2, X_3, X_4 \geq 0$ non-negativity.

(b) If the firm ignores the risk involved and uses a maximum return strategy on its investment, how will be the former model be amended?

Decision variables:

X_1 = Number of Telefónita shares
X_2 = Number of Sankander shares
X_3 = Number of Ferrofial shares
X_4 = Number of Gamefa shares.

Objective function:

$$\text{To maximize } z = 0.12 \cdot 100X_1 + 0.08 \cdot 50X_2 + 0.06 \cdot 80X_3 + 0.1 \cdot 40X_4$$

Constraints:
$0.12 \cdot 100X_1 + 0.08 \cdot 50X_2 + 0.06 \cdot 80X_3 + 0.1 \cdot 40X_4 \geq 18.000$; (rate of return $\geq$9 %)
$100X_1 \leq 100.000$; (no value can represent more than 50 % of the total investment)

$$50X_2 \leq 100.000;$$

$$80X_3 \leq 100.000;$$

$$40X_4 \leq 100.000;$$

$100X_1 + 50X_2 + 80X_3 + 40X_4 = 200000$; (the investment must be the same as the $200,000 available)
$X_1, X_2, X_3, X_4 \geq 0$(non-negativity)

1.11 Investment Funds

A small investor has $12,000 to invest and three different funds to choose from. Guaranteed investment funds offer an expected rate of return of 7 %, mixed funds (part is guaranteed capital) have an expected rate of return of 8 %, while an investment on the Stock Exchange involves an expected rate of return of 12 %, but without guaranteed investment capital. In order to minimise the risk, the investor has decided to not invest more than $2,000 on the Stock Exchange. Moreover for tax reasons, the investor needs to invest at least three times more in guaranteed investment funds than in mixed funds. Let us assume that at the end of the year the returns are those expected; what are the optimum investment amounts?

(a) Consider this problem as if it were a linear programming model with two decision variables
(b) Solve the problem with the graphic method and indicate the optimum solution.

Solution

(a) Consider this problem as if it were a linear programming model with two decision variables

Decision variables:

X: amount (in thousands of \$) invested in guaranteed funds;
Y: amount (in thousands of \$) invested in mixed funds;

Objective function:

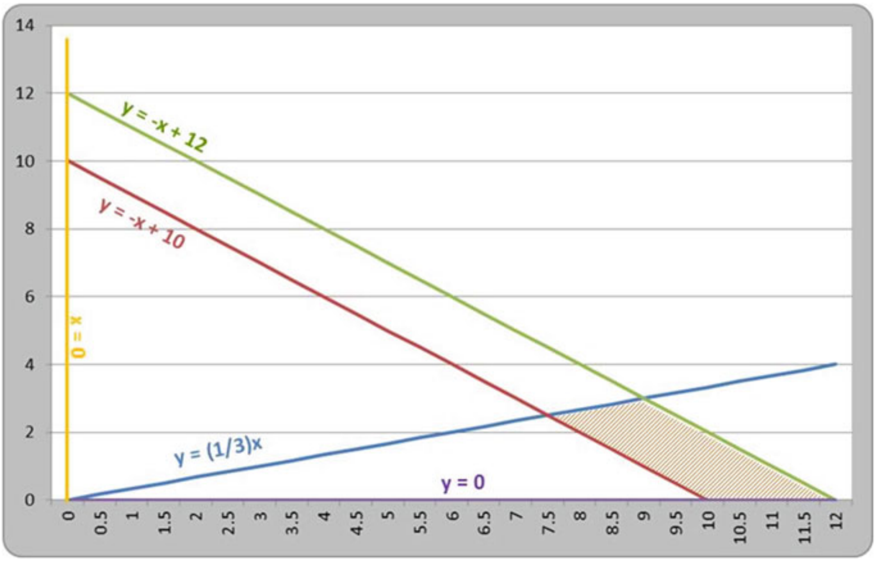
$$\text{Max } z = 0.07X + 0.08Y + 0.12(12 - X - Y) = 1.44 - 0.05X - 0.04Y$$

Constraints:

$12 - X - Y \geq 0$ The non-negativity constraint of the amount invested
$(12 - X - Y) \leq 2$ Upper limit of the amount invested
$Y \leq (1/3)X$ Constraint for tax reasons
$X, Y \geq 0$

(b) Solve the problem with the graphic method and indicate the optimum solution.

The feasible region would be as follows:



When testing the feasible region vertices in (9, 3), (12, 0), (10, 0), and (7.5, 2.5), an optimum investment return of \$965 is obtained when investing \$7,500 in mixed funds, \$2,500 in guaranteed funds and \$2,000 on the Stock Exchange.

1.12 Renting Warehouses

A firm has realised that it will not have sufficient storage space for the next three months. Details of the additional storage requirements for the next 3 months are provided in Table 1.18.

To cover its requirements, the firm plans to rent additional space on a short-term basis. At the beginning of each month, the firm can rent any amount of space for any number of months. It can pay separate rents of different quantities of space and/or over different periods of time. For instance, during the first month it could rent 20,000 m^2 for 2 months, and rent 5,000 m^2 separately for 1 month. It could also acquire new rented spaces before the previous ones expire. The costs per 1,000 m^2 of rented space in accordance with duration are provided in Table 1.19.

Build a linear programming model whose solution provides a rent policy that covers space requirements at a minimum cost.

Solution

Decision variables:

X_{11} = Quantity of space (m^2) the firm rents in month 1 for month 1.
X_{12} = Quantity of space (m^2) the firm rents in month 1 for month 2.
X_{13} = Quantity of space (m^2) the firm rents in month 1 for month 3.
X_{21} = Quantity of space (m^2) the firm rents in month 2 for month 1.
X_{22} = Quantity of space (m^2) the firm rents in month 2 for month 2.
X_{31} = Quantity of space (m^2) the firm rents in month 3 for month 1.

Table 1.18 Additional storage space requirements

Month	January	February	March
Space required (1,000 m^2)	25	10	20

Table 1.19 Rent costs

Rent duration	1 month	2 months	3 months
Cost (\$ per 1,000 m^2)	280	450	600

Table 1.20 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_1	0	30.0000	0	−12.5000	At bound	−M	42.5000
X_2	12.0000	40.0000	480.0000	0	Basic	17.5000	70.0000
X_3	21.0000	35.0000	735.0000	0	Basic	20.0000	80.0000
Objective function	Max=	1215.0000					
Constraint	**Left-hand side**	**Direction**	**Right-hand side**	**Slack or Surplus**	**Shadow price**	**Allowable Min. RHS.**	**Allowable Max. RHS.**
C1	90.0000	<=	90.0000	0	7.5000	54.0000	112.5000
C2	54.0000	<=	54.0000	0	10.0000	22.5000	90.0000
C3	78.0000	<=	93.0000	15.000	0	78.0000	M

Objective function:

To minimise $z = 280\ (X_{11} + X_{21} + X_{31}) + 450(X_{12} + X_{22}) + 600X_{13}$

Constraints:

$$
\begin{aligned}
&X_{11} + X_{12} + X_{13} \geq 25000\\
&X_{12} + X_{13} + X_{21} + X_{22} \geq 10000\\
&X_{13} + X_{22} + X_{31} \geq 20000\\
&X_{11}, X_{12}, X_{13}, X_{22}, X_{22}, X_{31} \geq 0
\end{aligned}
$$

The objective function minimises the cost of the space to be rented. The first three constraints ensure that the quantity of space required is rented for months 1, 2 and 3, respectively. The last constraint imposes non-negativity to all the variables.

1.13 Production Planning in a Tubes Factory

A steel tubes factory produces three sizes of tubes: small (1), medium (2) and large (3). These tubes can be produced in any of three machines: A, B or C. Each unit of product 1 requires 3 h in machine A, 2 h in machine B and 1 h in machine C. Each unit of product 2 requires 4 h in machine A, 1 h in machine B and 3 h in machine C. Each unit of product 3 requires 2 h in machine A, 2 h in machine B and 2 h in machine C. Profit is \$30 for product 1, \$40 for product 2 and \$35 for product 3. Machine A can be used for a maximum of 90 h, machine B for a maximum of 54 h and machine C for a maximum of 93 h.

The solution of the proposed problem was obtained by WinQSB® (see Table 1.20). Answer the questions below:

(a) What would the minimum profit be for product 1 for its production to be profitable?
(b) Assume that the profit of product 2 is increased to \$50/unit; would the objective function value change? If so, by how much?
(c) Assume that the number of hours available in machine 1 is increased to 100 h. Would the optimum solution change? What would the objective function value be?
(d) Assume that the number of hours available in machine 3 is reduced to 80 h. Would the optimum solution change? What would the objective function value be?
(e) Assume that it is possible to rent an identical machine to 2 for \$12/h. Should the manufacturer take this option? Why?
(f) Assume that the objective function changes to max $60X_1 + 80X_2 + 70X_3$. Would the optimum solution change? What about the objective function value?

Solution

(a) What should the minimum profit of product 1 be for its production to be profitable?
Its production is not profitable if its profit is under \$42.5. If it was over \$42.5, the optimum solution structure would change, although it would be necessary to resolve the problem again to check this.
(b) Assume that the profit of product 2 is increased to \$50/unit; would the objective function value change? If so, by how much?
\$120
(c) Assume that the number of hours available in machine 1 is increased to 100 h. Would the optimum solution change? What would the objective function value be?
The optimum solution structure would not change, but the objective function value would become \$75.
(d) Assume that the number of hours available in machine 3 is reduced to 80 h. Would the optimum solution change? What would the objective function value be?
No, it would remain the same.
(e) Assume that it was possible to rent an identical machine to 2 for \$12/h. Should the manufacturer take this option? Why?
No because, on the one hand, the generated profit would be \$10/h as opposed to \$12/h. On the other hand, it would be necessary to solve the problem again for rents over 90 available hours because the optimum solution structure would change.
(f) Assume that the objective function changes to max $60X_1 + 80X_2 + 70X_3$. Would the optimum solution change? What about the objective function value?
No. It would increase twofold.

1.14 Production Planning of a Wires Manufacturer

A firm in Valencia manufactures aluminium and copper wires. Each kg of aluminium wire requires 5 kWh of electricity and 0.25 h of labour. Each kg of copper wire requires 2 kWh of electricity and 0.5 h of labour. Copper wire production would be restricted by the fact that the quantity of raw materials available permits the maximum production of 60 kg/day. Electricity is limited to 500 kWh/day and labour to 40 h/day. The aluminium wire profit is \$0.25/kg, while the copper wire profit is \$0.4/kg. What quantity of each wire should be produced to maximise the profit and what would the profit be?

Solution

Decision variables:

$X =$ kg manufactured of aluminium wire
$Y =$ kg manufactured of copper wire

Objective function:

To maximise $z = 0.25X + 0.40Y$

Constraints:

$Y \leq 60$
$5X + 2Y \leq 500$
$0.25X + 0.5Y \leq 40$
$X \geq 0$
$Y \geq 0$

As it is a problem with two decision variables, it can be solved graphically to obtain the solution below:

$X = 85$
$Y = 37.5$
$z = 36.25.$

1.15 Mixed Investments

The firm Inversiones Internacionales, S.A.U. has up to 5 million \$ available to invest in six possible investments. Table 1.21 shows the characteristics of each investment.

From experience in such investments, the firm knows it is not recommended to invest more than 25 % of the total investment in any of these investment options. Besides, it is necessary to invest at least 30 % in precious metals, and at least 45 % between trade credits and corporate bonds. Finally, an overall risk limit of no more than 2.0 is required.

Table 1.21 Profitability and risk of each investment option

Investments	Profitability (%)	Risk
Trade credits	7	1.7
Corporate bonds	10	1.2
Stocks of gold	19	3.7
Stocks of platinum	12	2.4
Mortgage bonds	8	2.0
Building loans	14	2.9

Build a linear programming model whose solution indicates how much the firm should invest in all six possible investment options to maximise the profitability of its investments.

Solution

Decision variables:

X_1 = Dollars invested in trade credits
X_2 = Dollars invested in corporate bonds
X_3 = Dollars invested in stocks of gold
X_4 = Dollars invested in stocks of platinum
X_5 = Dollars invested in mortgage bonds
X_6 = Dollars invested in building loans.

Objective function (profitability gained in dollars):

$$\text{Max } z = 0.07X_1 + 0.10X_2 + 0.19X_3 + 0.12X_4 + 0.08X_5 + 0.14X_6$$

Constraints:

$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \leq 5000000$ (Invest up to 5 million \$)
$X_1 \leq 0.25(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$ (No more than 25 % in any investment)

$$X_2 \leq 0.25\ (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$$

$$X_3 \leq 0.25\ (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$$

$$X_4 \leq 0.25\ (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$$

$$X_5 \leq 0.25\ (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$$

$$X_6 \leq 0.25\ (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$$

$X_3 + X_4 \geq 0.30\,(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$ (At least 30 % in precious metals).
$X_1 + X_2 \geq 0.45\ (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$ (At least 45 % in trade credits and corporate bonds).

$1.7X_1 + 1.2X_2 + 3.7X_3 + 2.4X_4 + 2.0X_5 + 2.9X_6 \leq 2.0\,(X_1 + X_2 + X_3 + X_4 + X5 + X_6)$ (The overall risk limit must not exceed 2.0. Use the weighted mean to calculate the portfolio of investments risk).

$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \geq 0$ (Non-negativity constraint).

1.16 Production Planning in a Carpentry Firm

A carpentry firm produces tables and chairs. A table requires 4 m^2 of wood panels, while a chair requires 3 m^2. Wood costs 1$ per m^2 and 4,000 m^2 of wood panel are available (Constraint 1 or C1). Moreover, 2 h of skilled labour is required to produce one unfinished table or chair. In order to transform an unfinished table into a finished one, 3 more hours are needed, and it takes 2 additional hours to change an unfinished chair into a finished one. There are 6,000 h of skilled labour available (Constraint 2 or C2). The sale prices are provided in Table 1.22.

The solution of the proposed problem was obtained by WinQSB® (see Table 1.23). Answer the questions that follow:

(a) What would happen if the sale price of unfinished chairs (X_3) increased?
(b) What would happen if the sale price of unfinished tables (X_1) increased?
(c) What would happen if the sale price of finished chairs lowered to 100 $?
(d) How would profit change if the availability of wood panel varied?
(e) How much would it be willing to pay for another carpenter?
(f) Assume that industrial regulations complicate the finishing process and it would take one extra hour to finish a table or a chair. How would this affect the production plans?

Solution

(a) What would happen if the sale price of unfinished chairs (X_3) increased?
Presently, unfinished chairs are sold for 60 $. As the increase permitted on the profit coefficient is 50 $, it would not be profitable to produce them, even if they were sold for the same amount as the finished chairs. If the price lowered, the solution would not change.

Table 1.22 Sale prices

Product	Sale price ($)
Unfinished table (X_1)	70
Finished table (X_2)	140
Unfinished chair (X_3)	60
Finished chair (X_4)	110

(b) What would happen if the sale price of unfinished tables (X_1) increased?
The acceptable increase is above 70. That is, even though unfinished tables could be sold for more than finished tables, there would be no wish to sell them. It does not suffice to make unfinished tables more profitable than finished tables; they should be made more profitable than finished chairs.
(c) What would happen if the sale price of finished chairs lowered to 100 $?
This change would alter the production plan because this would involve lowering the price of finished chairs by 10 $ and the acceptable reduction is only 5 $. To know what could happen, it would be necessary to solve the problem again.
(d) How would profit change if the availability of wood panel varied?
The shadow price of wood restriction is 26.7 $. The range of values for the optimum solution basis to remain unaltered is 0–4,500. This means that it would still be interesting to specialise in finished chairs for an availability of up to 4,500. If the existing availability lowered, manufacturing finished chairs would still be an interesting option.
(e) How much would it be willing to pay for another carpenter?
Not all the labour available is being used, so nothing would be paid for further labour.
(f) Assume that industrial regulations make complicate finishing process and it would take one extra hour to finish a table or a chair. How would this affect the production plans.

It would be necessary to solve the problem again to obtain the solution. Nonetheless, common sense tells us that it would not work because, to produce 1,333.3 finished chairs, these additional hours of labour would be required, and they are not available.

Table 1.23 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_1	0	30.0000	0	−76.6667	At bound	−M	106.6667
X_2	0	100.0000	0	−6.6667	At bound	−M	106.6667
X_3	0	30.0000	0	−50.0000	At bound	−M	80.0000
X_4	1333.3330	80.0000	106666.7000	0	Basic	75.0000	M
Objective function	Max=	106666.7000					
Constraint	**Left-hand side**	**Direction**	**Right-hand side**	**Slack or Surplus**	**Shadow price**	**Allowable Min. RHS.**	**Allowable Max. RHS.**
C1	4000.0000	<=	4000.0000	0	26.6667	0	4500.0000
C2	53333.3340	<=	6000.0000	666.6667	0	5333.3340	M

Table 1.24 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_1	0	6.0000	0	−2.0000	At bound	−M	8.0000
X_2	4.0000	5.0000	20.0000	0	Basic	4.0000	8.0000
X_3	8.0000	8.0000	64.0000	0	Basic	6.0000	10.0000
Objective function	Max=	84.0000					

Constraint	Left-hand side	Direction	Right-hand side	Slack or Surplus	Shadow price	Allowable Min. RHS	Allowable Max. RHS
C1	12.0000	<=	12.0000	0	2.0000	10.0000	20.0000
C2	20.0000	<=	20.0000	0	3.0000	12.0000	24.0000

1.17 Product Mix

Consider a product mix model where X_1, X_2 y X_3 represent the number of units of each product to be manufactured and that the objective function coefficients are the unit profit of the products. The production of all three products requires a certain number of units of the two limited resources, as shown below:

$$\text{To maximize } z = 6X_1 + 5X_2 + 8X_3$$

Subject to
$X_1 + X_2 + X_3 \leq 12$ (mechanised section)
$2X_1 + X_2 + 2X_3 \leq 20$ (finishing section)

$$X_1, X_2, X_3 \geq 0$$

The solution of the proposed problem has been obtained by WinQSB® (see Table 1.24). Answer the following questions:

(a) What is the range of variation for the profit coefficient (c_1) of product X_1, provided the optimum solution does not vary?
(b) What happens if $c_1 = 6$ is increased by 25 %?
(c) Assess the effect of increasing the current profit of the first product by 3 \$/unit.
(d) What is the impact of an error of −37.5 % on the estimated unit profit of article 3?
(e) Which of the resources is worth increasing?

Solution

(a) What is the range of variation for the profit coefficient (c_1) of product X_1, provided the optimum solution does not vary?
[−M, 8]

(b) What happens if $c_1 = 6$ is increased by 25 %?
The optimum solution structure remains the same.

(c) Assess the effect of increasing the current profit of the first product by 3$/unit
It would be necessary to solve the problem again because the optimum solution structure would change.

(d) What is the impact of an error of −37.5 % on the estimated unit profit of article 3?
It would be necessary to solve the problem again because the optimum solution structure would change.

(e) Which of the resources is worth increasing?
The finishing section, which has a higher shadow price.

1.18 Transport Planning in a Olive Production Firm

A firm from Jaén has three production plants located in Jaén, Seville and Almería. The production capacities in en kg estimated for the next three months are provided in Table 1.25.

The firm distributes olives by means of four regional distribution centres located in Valencia, Madrid, Barcelona and La Coruña. The forecasted demand for these distribution centres for the next three months is provided in Table 1.26.

The firm's management wishes to determine how much of its production must be sent from each plant to each distribution centre. The unit cost in dollars per kg sent by means of each route is shown in Table 1.27.

- Build a linear programming model to help the firm in its decision making
- For strategical reasons, the firm has adopted the following policies for its transport planning:

Table 1.25 The forecasted production capacities in each plant

Plant	Production capacity (kg)
Jaén	5,000
Seville	6,000
Almería	2,500

Table 1.26 Forecasted demand in each distribution centre

Distribution centre	Demand (kg)
Valencia	6,000
Madrid	4,000
Barcelona	2,000
La Coruña	1,500

Table 1.27 Transport costs between plants and distribution centres ($)

Origin/Destination	Valencia	Madrid	Barcelona	La Coruña
Jaén	30	20	70	60
Seville	70	50	20	30
Almería	20	50	40	50

– At least 60 % of the total production from Jaén must be sent to Valencia.
– Deliveries from Seville to Valencia will have a fixed cost of $200.
– Only Seville or Almería can deliver to La Coruña, but never both.

Include these constraints in the model of section (a).
Solution

- Build a linear programming model to help the firm in its decision making
  Decision variables:

X_{11} = kg to be sent from Jaén to Valencia in the next 3 months
X_{12} = kg to be sent from Jaén to Madrid in the next 3 months
X_{13} = kg to be sent from Jaén to Barcelona in the next 3 months
X_{14} = kg to be sent from Jaén to La Coruña in the next 3 months
X_{21} = kg to be sent from Seville to Valencia in the next 3 months
X_{22} = kg to be sent from Seville to Madrid in the next 3 months
X_{23} = kg to be sent from Seville to Barcelona in the next 3 months
X_{24} = kg to be sent from Seville to La Coruña in the next 3 months
X_{31} = kg to be sent from Almería to Valencia in the next 3 months
X_{32} = kg to be sent from Almería to Madrid in the next 3 months
X_{33} = kg to be sent from Almería to Barcelona in the next 3 months
X_{34} = kg to be sent from Almería to La Coruña in the next 3 months.

Objective function:

$$\text{To minimize } z = 30X_{11} + 20X_{12} + 70X_{13} + 60X_{14} + 70X_{21} + 50X_{22} + 20X_{23} + 30X_{24} + 20X_{31} + 50X_{32} + 40X_{33} + 50X_{34}$$

Capacity constraints:

$$X_{11} + X_{12} + X_{13} + X_{14} \leq 5000$$

$$X_{21} + X_{22} + X_{23} + X_{24} \leq 6000$$

$$X_{31} + X_{32} + X_{33} + X_{34} \leq 2500$$

Demand constraints:

$$X_{11} + X_{21} + X_{31} + X_{41} \geq 6000$$

$$X_{12} + X_{22} + X_{32} + X_{42} \geq 4000$$

$$X_{13} + X_{23} + X_{33} + X_{43} \geq 2000$$

$$X_{14} + X_{24} + X_{34} + X_{44} \geq 1500$$

Non-negativity constraints:

$$X_{11}, X_{12}, X_{13}, X_{14}, X_{21}, X_{22}, X_{23}, X_{24}, X_{31}, X_{32}, X_{33}, X_{34} \geq 0$$

- For strategical reasons, the firm has adopted the following policies for its transport planning:
  - At least 60 % of the total production from Jaén must be sent to Valencia.
  - Deliveries from Seville to Valencia will have a fixed cost of \$200.
  - Only Seville or Almería can deliver to La Coruña, but never both.

  Include these constraints in the model of section (a).

At least 60 % of the total production from Jaén must be sent to Valencia:

$$X_{11} \geq 0.6(X_{11} + X_{12} + X_{13} + X_{14})$$

Deliveries from Seville to Valencia will have a fixed cost of \$200:

A new binary decision variable, Y_{21}, is defined, which takes a value of 0 if there is no delivery from Seville to Valencia, and a value of 1 otherwise. The objective function is amended as follows:

$$\begin{aligned}\text{Minimize } z = {} & 30X_{11} + 20X_{12} + 70X_{13} + 60X_{14} + 70X_{21} + 50X_{22} + 20X_{23} \\ & + 30X_{24} + 20X_{31} + 50X_{32} + 40X_{33} + 50X_{34} + 200Y_{21}\end{aligned}$$

Moreover, the following constraint is added:

$$X_{21} \leq 6000Y_{21}$$

Two new binary decision variables, Y_{24} and Y_{34}, are defined which take a value of 0 if a delivery is made to La Coruña from the corresponding origin, and 1 otherwise. The following constraints are added:

$$X_{24} \leq 6000Y_{24}$$

$$X_{34} \leq 2500Y_{34}$$

$$Y_{24} + Y_{34} \leq 1$$

1.19 Programming Weekly Production in a Metallurigcal Company

The PRODA, S.A. industrial products firm has to face the problem of scheduling the weekly production of its three products (P1, P2 and P3). These products are sold to large industrial firms and PRODA, S.A. wishes to supply its products in quantities that are more profitable for it.

Each product entails three operations: smelting; mechanisation; assembly and packaging. The smelting operations for products P1 and P2 could be subcontracted, but the smelting operation for product P3 requires special equipment, thus preventing the use of subcontracts. The direct costs of all three operations and the sale prices of the respective products are provided in Table 1.28.

Each unit of product P1 requires 6 min of smelting time (if performed at PRODA, S.A.), 6 min of mechanisation time and 3 min of assembly and packaging time, respectively. For product P2, the times are 10, 3 and 2 min, respectively. One unit of product P3 needs 8 min of smelting time, 8 min of mechanisation and 2 min for assembly and packaging.

PRODA, S.A. has weekly capacities of 8,000 min of smelting time, 12,000 min of mechanisation time and 10,000 min of assembly and packaging time.

(a) Build a linear programming model which maximises PRODA, S.A.'s weekly profits.
(b) Pose the dual problem of (a) by indicating the meaning of the variables, the objective function and the dual constraints.

The solution of the proposed problem has been obtained by WinQSB® (see the sample in Table 1.29). Answer the questions below:

(c) Given the optimum product mix, should the firm follow what the solution provided by linear programming states? If your answer is no, how can this be introduced into the linear programming model?
(d) If product 3 generates more unit profits, why is it not included in the optimum product mix?
(e) PRODA, S.A. is thinking about updating its assembly and packaging line to achieve better times. Is this a suitable strategy?
(f) There are two processes available to smelt product 2 and both are employed in the optimum problem solution. How do you assess the impact of preventing subcontracting the smelting process on product 2?

Table 1.28 Cost of operations and sale prices in dollars

Direct costs and sale prices ($)	P1	P2	P3
Cost of smelting at PRODA, S.A.	0.30	0.50	0.40
Cost of subcontracted smelting	0.50	0.60	–
Cost of mechanisation	0.20	0.10	0.27
Cost of assembly and packaging	0.30	0.20	0.20
Sale price	1.50	1.80	1.97

Table 1.29 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_1	0	0.7000	0	−1.1600	At bound	−M	1.8600
X_2	0	0.5000	0	−1.3000	At bound	−M	1.8000
X_3	800.0000	1.0000	800.0000	0	Basic	0.9000	M
X_4	3200.0000	0.9000	2880.0000	0	Basic	0.2500	1.0000
X_5	0	1.1000	0	−1.3800	At bound	−M	2.4800
Objective function	Max=	3680.0000					

Constraint	Left-hand side	Direction	Right-hand side	Slack or Surplus	Shadow price	Allowable Min. RHS.	Allowable Max. RHS.
C1	8000.0000	<=	8000.0000	0	0.0100	0	40000.0000
C2	12000.0000	<=	12000.0000	0	0.3000	2400.0000	15000.0000
C3	8000.0000	<=	10000.0000	2000.0000	0	8000.0000	M

Solution

(a) Design a linear programming model that maximises PRODA, S.A.'s weekly profits

Decision variables:

X_1 = units of product P1 with smelting performed at PRODA, S.A.
X_2 = units of product P1 with a subcontracted smelting process.
X_3 = units of product P2 with smelting performed at PRODA, S.A.
X_4 = units of product P2 with a subcontracted smelting process.
X_5 = units of product P3.

Objective function:

$$\text{Max } z = 0.7X_1 + 0.5X_2 + X_3 + 0.9X_4 + 1.1X_5$$

Constraints:

$6X_1 + 10X_3 + 8X_5 \leq 8000$ (Capacity available for smelting)
$6X_1 + 6X_2 + 3X_3 + 3X_4 + 8X_5 \leq 12000$ (Capacity available for mechanisation)
$3X_1 + 3X_2 + 2X_3 + 2X_4 + 2X_5 \leq 10000$ (Capacity available for assembly and packaging)
$X_1, X_2, X_3, X_4, X_5 \geq 0$ (Non-negativity constraint)

(b) Pose the dual problem of (a) by indicating the meaning of the variables, the objective function and the dual constraints.

Decision variables:

U_1 = Shadow price, marginal value or opportunity cost of the minutes of smelting time at PRODA, S.A.

U_2 = Shadow price, marginal value or opportunity cost of the minutes of mechanisation time at PRODA, S.A.
U_3 = Shadow price, marginal value or opportunity cost of the minutes of assembly and packaging time at PRODA, S.A.

Objective function:

$$\text{Min } z = 8000U_1 + 12000U_2 + 10000U_3$$

(To minimise what is no longer earned due to not having more resources or the total value of the resources employed)
Constraints:
$6U_1 + 6U_2 + 3U_3 \geq 0.7$ (The value of the resources invested in product 1 must be at least equal to the profit generated)

$$6U_2 + 3U_3 \geq 0.5$$

$$10U_1 + 3U_2 + 2U_3 \geq 1$$

$$3U_2 + 2U_3 \geq 0.9$$

$$8U_1 + 8U_2 + 2U_3 \geq 1.1$$

(c) Given the optimum product mix, should the firm follow what the solution provided by linear programming states? If your answer is no, how can this be introduced into the linear programming model?
The optimum product mix is: 800 units of product 2 (smelted at PRODA, S.A.) and 3,200 units of product 2 (by a subcontracted smelting process). This implies that the firm becomes a producer of a single product; in other words, it should change strategy. Should it wish to offer a wider variety of products (2 or 3), this should be reflected with additional minimum productions constraints.
(d) If product 3 generates more unit profits, why is it not included in the optimum product mix?
This is because product P3 employs scarcer resources more intensively. For example, P3 takes 8 min mechanisation time, while P2 takes only 3.
(e) PRODA, S.A. is thinking about updating its assembly and packaging line to achieve better times. Is this a suitable strategy?
No because there is currently a slack variable of 2,000 min, so increasing the time available would not improve the current situation.
(f) There are two processes available to smelt product 2 and both are employed in the optimum problem solution. How do you assess the impact of preventing subcontracting the smelting process on product 2?
The corresponding variable (X_4) would be removed from the problem structure and the problem would need to be solved again.

1.20 Production Planning in a Bricks Factory

A bricks factory produces four types of breeze blocks. The manufacturing process comprises three stages: mixing, vibration and inspection. During the following month, there are 800 machining hours available for mixing, 1,000 machining hours available for vibration and 340 man/hours for inspection purposes. The factory wishes to maximise its profits within this period. To go about this, it has built the following linear programming model:

$$\text{Max } z = 8X_1 + 14X_2 + 30X_3 + 50X_4$$

Subject to

$$X_1 + 2X_2 + 10X_3 + 16X_4 \le 800$$

$$1.5X_1 + 2X_2 + 4X_3 + 5X_4 \le 1000$$

$$0.5X_1 + 0.6X_2 + X_3 + 2X_4 \le 340$$

$$X_1,\ X_2, X_3, X_4 \ge 0$$

The solution of the proposed problem has been obtained by WinQSB® (see the sample in Table 1.30). Answer the following questions:

(a) What is the optimum solution?
(b) Is this optimum solution the only one?
(c) How much should the profit coefficient of product 3 be increased by at least to make its production worthwhile?
(d) How much could the profit coefficient of product 2 be lowered without changing the optimum basis?

Table 1.30 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_1	400.0000	8.0000	3200.0000	0	Basic	7.0000	9.8182
X_2	200.0000	14.0000	2800.0000	0	Basic	11.8947	16.0000
X_3	0	30.0000	0	−28.0000	At bound	−M	58.0000
X_4	0	50.0000	0	−40.0000	At bound	−M	90.0000
Objective function	Max=	6000.0000					

Constraint	Left-hand side	Direction	Right-hand side	Slack or Surplus	Shadow price	Allowable Min. RHS.	Allowable Max. RHS.
C1	800.0000	<=	800.0000	0	5.0000	666.6667	1000.0000
C2	1000.0000	<=	1000.0000	0	2.0000	800.0000	1050.0000
C3	320.0000	<=	340.0000	20.0000	0	320.0000	M

(e) Within what range could the number of machining hours for mixing vary without changing the optimum basis?
(f) How much would the factory be willing to pay for man/hour of additional inspection?
(g) A competitor offers the factory the chance to rent the additional capacity for mixing at 4 monetary units per hour. Should it accept this offer?
(h) At what price would it be willing to rent 1 additional hour of vibration to its competitor? Up to how many hours (without changing the optimum solution)?
(i) How much can the inspection time be reduced without changing the optimum solution?
(j) What is the new objective function value if the vibration hours are increased to 1,020?
(k) Would it accept the production of a type 5 brick if it requires 2 h of each activity and its profit is 30?

Solution

(a) What is the optimum solution?
Producing 200 type 2 bricks and 400 type 3 ones to obtain a total profit of 6,000 monetary units. Besides, 20 h in the inspection section would be left over.
(b) Is this the only optimum solution?
Yes because all the non-basic variables take a negative *Cj–Zj* value. Should any of the non-basic variables take a null *Cj-Zj* value, this would mean that the problem has alternative solutions.
(c) How much should the profit coefficient of product 3 be increased by at least to make its production worthwhile?
$C3\ (\text{new}) \geq Z3 = 28$
The contribution of the profit of X_3 should be increased from a minimum of 30 up to above 58. Then the basic solution structure would change and X_3 could be inputted to form part of it.
(d) How much could the profit coefficient of product 2 be lowered without changing the optimum basis?
It would be necessary to calculate the range for basic variable X_2: [11.8947 – 16].
(e) Within what range could the number of machining hours for mixing vary without changing the optimum basis?
It would be necessary to calculate the range for Constraint 1: [666.6667 – 1000.0000].
(f) How much would the factory be willing to pay for a man/hour of additional inspection?
Nothing because there is a 20-hour slack, so the shadow price of Constraint 3 is zero.

(g) A competitor offers the factory the chance to rent the additional capacity for mixing at 4 monetary units per hour. Should it accept this offer?
Yes because for each additional hour dedicated to mixing, the objective function value would increase by 5 monetary units.

(h) At what price would it be willing to rent 1 additional hour of vibration to its competitor? Up to how many hours (without changing the optimum solution)?
Up to 2 monetary units, this being the shadow price value of Constraint 2. It could rent up to a maximum of 50 h, according to the range calculated for Constraint 2 [800 – 1,050].

(i) How much can the inspection time be reduced without changing the optimum solution?
According to the range calculated for Constraint 3: [320 – M], 20 h could be obtained.

(j) What is the new objective function value if the vibration hours are increased to 1,020?
6,040

(k) Would it accept the production of a type 5 brick if it requires 2 h of each activity and its profit is 30?
Yes because, on the one hand, type 2 brick has a profit of 14 monetary units and consumes 2 h of the mixing and vibrating resource. On the other hand, although the inspection resource consumes more hours than type 1 and 2 bricks, there are hours in this resource not being used with the current optimum solution.

1.21 Sensitivity Analysis

Assume that you are the person in charge of the production planning in a firm that produces four product types. There are three limited production resources, and all four product types require a certain quantity of these three resources. The intention is to optimise production by applying a linear programming model so that:

To maximise $z = cx$

Subject to

$$Ax \leq b$$

$$x \geq 0$$

After solving the problem with WinQSB®, the solution and sensitivity information provided in Table 1.31 were obtained. Some data have been hidden with a letter *X*. Answer the following questions and provide reasons for your answers:

Table 1.31 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
Product 1	0	12	−9	At bound	*X1*	*X2*
Product 2	27	6	*X3*	Basic	1	Infinite
Product 3	0	12	0	At bound	8	20
Product 4	0	6	−7	At bound	-Infinite	13
Constraint	**Left-hand side**	**Direction**	**Right-hand side**	**Shadow price**	**Allowable Min. RHS.**	**Allowable Max. RHS.**
Resource 1	100	$\leq$	100	14	50	150
Resource 2	200	$\leq$	300	*X4*	*X5*	*X6*
Resource 3	100	$\leq$	100	75	20	190

(a) If it were possible to purchase 10 other units of resource 3 for a total of $700, would it be worthwhile?
(b) If it were possible to purchase 100 other units of resource 3 for a total of $8,000, would it be worthwhile?
(c) Consider that the firm's profit coefficient of product 3 is within the minimum and maximum range permitted. What is the minimum and maximum that would be produced of product 3?
(d) What is the minimum profit coefficient in the objective function for which product 4 could be produced?
(e) If the supply of any of the production resources could be increased by 10 units without incurring any cost, which resource would you choose?
(f) Complete the six boxes labelled with *X* with the correct values.
(g) Is this a degenerated solution?

Solution

(a) If it were possible to purchase 10 other units of resource 3 for a total of $700, would it be worthwhile?
Yes because it would increase the objective function by $750, so an additional profit of $50 would be obtained.
(b) If it were possible to purchase 100 other units of resource 3 for a total of $8,000, would it be worthwhile?
This we do not know because if we purchased 100 further units, we would exceed the maximum amount permitted for the independent constraint term of 190 units, and it would be necessary to solve the problem again.

(c) Consider that the firm's profit coefficient of product 3 is within the minimum and maximum range permitted. What is the minimum and maximum that would be produced of product 3?
Nothing because within this range of profit, the optimum solution structure obtained does not vary. Therefore, nothing of product 3 is produced.

(d) What is the minimum profit coefficient in the objective function for which product 4 could be produced?
Within the given range of profit, the optimum solution structure obtained does not vary. Therefore between [-infinite 13], product 4 is not produced. With a coefficient over 13, it would be necessary to solve the problem again with a view to checking if product 4 is produced.

(e) If the supply of any of the production resources could be increased by 10 units without incurring any cost, which resource would you choose?
Resource 3 as it has a higher shadow price.

(f) Complete the six boxes labelled with X with the correct values.
$X_1 =$ - *Infinite*
$X_2 = 21$
$X_3 = 0$
$X_4 = 0$
$X_5 = 200$
$X_6 =$ *Infinite*

(g) Is this a degenerated solution?
A solution is degenerated when there are many solutions. In this case it is because a variable that is not basic, product 3, has a reduced cost of 0. This means that should this occur, the objective function value would not come down.

1.22 Crop Planning

A citrus firm in Valencia possesses 120 acres (1 acre = 4,074 m^2) and it is planning to sow at least three crops. The seeds for crops A, B and C cost \$40, \$20 and \$30 per acre, respectively. It intends to invest a maximum of \$3,400 in seeds. Seeds A, B and C require 1, 2 and 1 working day(s) per acre, respectively, and 170 working days available are considered. If the owner can obtain a profit of \$100 per acre with seed A, \$300 per acre with seed B and \$200 per acre with seed C, how many acres should the owner sow of each seed to maximise the profit?

(a) Model the programme like a linear programming model.
The solution to the proposed problem has been obtained with WinQSB® (see the sample in Table 1.32). Answer the questions set out below and provide reasons for your answers:

Table 1.32 Solution obtained by WinQSB®

Decision variable	Solution value	Unit cost or profit c(j)	Total contribution	Reduced cost	Basic status	Allowable Min. c(j)	Allowable Max. c(j)
X_1	0	100.0000	0	−100.0000	At bound	−M	200.0000
X_2	50.0000	300.0000	15000.0000	0	Basic	200.0000	400.0000
X_3	70.0000	200.0000	14000.0000	0	Basic	150.0000	300.0000
Objective Function	Max=	29000.0000					

Constraint	Left-hand Side	Direction	Right-hand side	Slack or Surplus	Shadow price	Allowable Min. RHS.	Allowable Max. RHS.
C1	120.0000	<=	120.0000	0	100.0000	85.0000	127.5000
C2	3100.0000	<=	3400.0000	300.0000	0	3100.0000	M
C3	170.0000	<=	170.0000	0	100.0000	140.0000	240.0000

(b) Is it possible to sow 40 acres with seed A, 50 acres with seed B and 30 acres with seed C?
(c) Is it possible to not sow seed A, and to sow 40 acres with seed B and 70 acres with seed C?
(d) What is the optimum solution? What is the optimum objective function value?
(e) What capacity is completely used with the optimum combination?
(f) Where are the idle capacities?
(g) If the budget was increased by \$1, how much would the total profit increase by? Why?
(h) If the working days available increased by one extra day, how much would the total profit increase by? Why?
(i) If the initial budget was changed to \$3,401, what would happen to the optimum objective function value?
(j) If the budget was limited to \$800 for seed B, what would happen to the optimum objective function value?
(k) If the profit per acre with seed B was \$400, would the optimum amounts in acres be the same? Why?
(l) If the profit per acre with seed C were \$310, would the optimum amounts in acres be the same? Why?

Solution

(a) Model the programme like a linear programming model.

Decision variables:

X_1 = number of acres to sow with seed A.
X_2 = number of acres to sow with seed B.
X_3 = number of acres to sow with seed C.

Objective function:

Max $z = 100X_1 + 300X_2 + 200X_3$ (maximise the profit)

Constraints:

$X_1 + X_2 + X_3 \leq 120$ (total number of acres available)
$40X_1 + 20X_2 + 30X_3 \leq 3400$ (budget available for seeds)
$X_1 + 2X_2 + X_3 \leq 170$ (number of working days available)

$$X_1 + X_2 + X_3 \geq 0$$

(b) Is it possible to sow 40 acres with seed A, 50 acres with seed B and 30 acres with seed C?
It would not be the optimum solution.
(c) Is it possible to not sow seed A, and to 40 acres with seed B and 70 acres with seed C?
It would be the optimum solution.
(d) What is the optimum solution? What is the optimum objective function value?
Not sowing seed A, sowing 40 acres with seed B and 70 acres with seed C to obtain a profit of $29,000.
(e) What capacity is completely used with the optimum combination?
The total number of acres available and the total number of working days.
(f) Where are the idle capacities?
In the available budget for seeds.
(g) If the budget was increased by $1, how much would the total profit increase by? Why?
It would increase by nothing because it is an idle capacity.
(h) If the working days available increased by one extra day, how much would the total profit increase by? Why?
By $100, which corresponds to the shadow price of the corresponding constraint.
(i) If the initial budget was changed to $3,401, what would happen to the optimum objective function value?
Nothing because there is already a surplus of $300.
(j) If the budget was limited to $800 for seed B, what would happen to the optimum objective function value?
It would be necessary to add a new constraint to the initial model, $20X_2 \leq 800$, and the problem would have to be solved again in order to answer this question.
(k) If the profit per acre with seed B was $400, would the optimum amounts in acres be the same? Why?
All we can state with any certainty is that the optimum solution structure would be the same. This is because 400 is within the limit of the ranges for the profit coefficient of X_2.

(l) If the profit per acre with seed C were \$310, would the optimum amounts in acres be the same? Why?

All we can state with any certainty is that the optimum solution structure would be the same. This is because 310 is within the limit of the ranges for the profit coefficient of X_3.

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Chapter 2
Integer Programming

Abstract This chapter begins by introducing integer linear programming. Then, it proposes a mixed set of integer linear programming problems and provides their solutions. This chapter aims to provide a better understanding of the formulation of integer linear programming models. It pays special attention to the use of binary decision variables to express the conditions or dichotomies in the constraints of the problems. Thus, it sets out different problem formulations with their solutions in relation to Industrial Organisation Engineering and the management setting.

2.1 Introduction

Integer linear programming models are an extension of continuous linear programming models but, unlike these, they consider that several of or all the defined decision variables cannot take fractional values. It is worth stressing the consideration of binary or zero–one variables in the integer linear programming models proposed by Balas (1965). Binary variables are appropriate whenever decision variables can obtain one or two values, i.e. disjunctive programming.

While a function in linear programming is maximised or minimised over a region of convex feasibility, in integer linear programming a function is maximised over a region of feasibility, which is not generally convex. Thus, a solution for integer linear programming models is more complex than it is for linear programming models. Based on the Simplex Method, Gomory (1958, 1960a, b, c, 1963) obtained the general expression to approach the convex covering of a set of feasible solutions employing secant plans. On the other hand, branch and bound algorithms for integer/real variables are published by Land and Doig (1960), Driebeck (1966) and Little et al. (1963).

There are lots of variants of algorithms that take simplex and branch and bound as a basis (branch and cut, branch and price, benders decomposition, etc.) Many of

R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5_2, © Springer-Verlag London 2014

these methods have theoretically demonstrated their convergence to an optimal integer solution. However, for real world problems, this convergence can prove so slow that, for practical purposes, the method might prove useless. In this context, a plethora of heuristic and meta-heuristic techniques has also emerged that pursues the search for solutions in an acceptable computing time. Thus, a non-exhaustive search for algorithms does not explore the whole space of solutions. These algorithms start with an initial solution and iteratively move on to another better solution by usually finding reasonably good solutions quickly, but not guaranteeing that all the solutions are explored. Genetic algorithms, simulated annealing, ant colony algorithms, among others, belong to this category. In this kind of techniques (e.g. genetic algorithms), a fitness function (objective function) must be able to guide the search towards feasible solutions and optimal solutions. However, highly constrained problems imply very complex fitness functions that not always guide the search efficiently towards feasibility and optimality. As these techniques are beyond the scope of this book, an interesting reference to check the state of the art in optimisation algorithms is the COIN-OR initiative (COmputational Infrastructure for Operations Research project, see COIN-OR 2011), an initiative to spur the development of open-source software for the operations research community.

Integer linear programming models are employed in a large number of problems with intrinsically integer variables: problems with manufacturing product units; transport problems; allocation and optimisation networks; sequencing problems; travelling salesman problems; knapsack problems; investment problems; fixed costs problems; set covering and partition problems; dichotomies and approach problems; the production lines balance; problems involving location of plants; among others.

After reading this chapter, the reader should be capable of: knowing and modelling different integer linear programming prototype problems and formulating these models with binary variables.

Selected books for fur„ther read„ing can be found in Andeson et al. (2009), Fletcher (2000), Hillier and Lieberman (2002), Murty (1995), Rardin (1998), Taha (2010) and Winston (2003).

2.2 Postmen/Women's Shifts

A Post Office needs a different number of fulltime postmen and postwomen for different days of the week. The number of fulltime postmen/women required per day, which is calculated according to the number of items to be delivered, is provided in Table 2.1.

Table 2.1 Number of postmen/women required

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Number of fulltime postmen/women required	17	13	15	19	14	16	11

According to Trade Union rules, each fulltime employee has to work 5 days running, and must then rest for 2 days. For instance, an employee who works Monday–Friday must rest on Saturday and Sunday. The Post Office wishes to meet its daily requirements and to use only fulltime employees.

(a) Formulate a mathematical model that the Post Office can use to obtain the minimum number of fulltime postmen/women that it must work with.
(b) Calculate the two first tables of the Simplex Method that are required to obtain the solution to this problem.

Solution

(a) Formulate a mathematical model that the Post Office can use to obtain the minimum number of fulltime postmen/women that it must work with.

Decision variables:

X_1 = No. of fulltime postmen/women who must start their 5-day shift on Mondays
X_2 = No. of fulltime postmen/women who must start their 5-day shift on Tuesdays
X_3 = No. of fulltime postmen/women who must start their 5-day shift on Wednesdays
X_4 = No. of fulltime postmen/women who must start their 5-day shift on Thursdays
X_5 = No. of fulltime postmen/women who must start their 5-day shift on Fridays
X_6 = No. of fulltime postmen/women who must start their 5-day shift on Saturdays
X_7 = No. of fulltime postmen/women who must start their 5-day shift on Sundays.

Objective Function:

$$\text{Min } z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7.$$

Constraints:

The number of postmen/women who work on the same day must at least equal the number of postmen/women required for this day. For example, the number of postmen/women who work on Fridays will be those who start their shifts on Monday, Tuesday, Wednesday, Thursday and Friday, and must be a minimum of 14, and so forth with all the days of the week:

$$\begin{aligned}
X_1 + X_2 + X_3 + X_4 + X_5 &\geq 14 \\
X_2 + X_3 + X_4 + X_5 + X_6 &\geq 16 \\
X_3 + X_4 + X_5 + X_6 + X_7 &\geq 11 \\
X_4 + X_5 + X_6 + X_7 + X_1 &\geq 17 \\
X_5 + X_6 + X_7 + X_1 + X_2 &\geq 13 \\
X_6 + X_7 + X_1 + X_2 + X_3 &\geq 15 \\
X_7 + X_1 + X_2 + X_3 + X_4 &\geq 19
\end{aligned}$$

where $X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0$ and integer values.

Table 2.2 Minimization problem

	C_j	1	1	1	1	1	1	1	0	0	0	0	0	0	0	M	M	M	M	M	M	M		
C_j	Base	X_1	X_2	X_3	X_4	X_5	X_6	X_7	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	A_4	A_5	A_6	A_7	R	Ratio
M	A_1	1	1	1	1	1	0	0	−1	0	0	0	0	0	0	1	0	0	0	0	0	0	14	14
M	A_2	0	1	1	1	1	1	0	0	−1	0	0	0	0	0	0	1	0	0	0	0	0	16	M
M	A_3	0	0	1	1	1	1	1	0	0	−1	0	0	0	0	0	0	1	0	0	0	0	11	M
M	A_4	1	0	0	1	1	1	1	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	17	17
1	A_5	1	1	0	0	1	1	1	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	13	13
M	A_6	1	1	1	0	0	1	1	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	15	15
M	A_7	1	1	1	1	0	0	1	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	19	19
	$C_j - Z_j$	$1 - 5M$	$1 - 5M$	$1 - 5M$	$1 - 5M$	$1 - 5M$	$1 - 5M$	$1 - 5M$	M	M	M	M	M	M	M	0	0	0	0	0	0	0		

Table 2.3 Minimization problem

	C_j	1	1	1	1	1	1	1	0	0	0	0	0	0	0	M	M	M	M	M	M	M		
C_j	Base	X_1	X_2	X_3	X_4	X_5	X_6	X_7	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	A_4	A_5	A_6	A_7	R	Ratio
M	A_1	0	0	1	1	0	−1	−1	−1	0	0	0	1	0	0	1	0	0	0	−1	0	0	1	1
M	A_2	0	1	1	1	1	1	0	0	−1	0	0	0	0	0	0	1	0	0	0	0	0	16	16
M	A_3	0	0	1	1	1	1	1	0	0	−1	0	0	0	0	0	0	1	0	0	0	0	11	11
M	A_4	0	−1	0	1	0	0	0	0	0	0	−1	1	0	0	0	0	0	1	−1	0	0	4	M
1	X_1	1	1	0	0	1	1	1	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	13	M
M	A_6	0	0	1	0	−1	0	0	0	0	0	0	1	−1	0	0	0	0	0	−1	1	0	2	2
M	A_7	0	0	1	1	−1	−1	0	0	0	0	0	1	0	−1	0	0	0	0	−1	0	1	6	6
	$C_j - Z_j$	0	0	1 − 5 M	1 − 5 M	0	0	0	M	M	M	M	1 − 4 M	M	M	0	0	0	0	5 M − 1	0	0		

(a) Calculate the first two Simplex tables that are required to obtain the solution to the problem (Tables 2.2 and 2.3).

The solution to the problem will be 1 postman/woman who begins his or her shift on Mondays; 3 start on Tuesdays, Wednesdays and Saturdays; 7 start on Thursdays and 6 on Sundays. In all, a minimum of 23 postmen/women are required.

2.3 Distribution of Air-Conditioning Units

A North American air-conditioning firm has production plants in Portland and Flint. It has to supply a certain number of units in its distribution centres located in Los Angeles and Atlanta. The delivery costs are summarised in Table 2.4.

The supply and demand data are provided in Table 2.5 as number of units:

(a) Consider an integer linear programming model that determines how the deliveries from Portland and Flint should be carried out in order to minimise delivery costs.
(b) Solve the problem and interpret the results.

Solution

(a) Consider an integer linear programming model that determines how the deliveries from Portland and Flint should be carried out in order to minimise delivery costs.

Decision variables:

Table 2.4 Delivery costs between distribution centres

Production plants	Distribution centres	Delivery costs
Portland	Los Angeles	\$30
	Atlanta	\$40
Flint	Los Angeles	\$60
	Atlanta	\$50

Table 2.5 Supply and demand data

Supply	Available units	Demand	Required units
Portland	200	Los Angeles	300
Flint	600	Atlanta	400

X_1 = Units delivered from Portland to Los Angeles.
X_2 = Units delivered from Portland to Atlanta.
X_3 = Units delivered from Flint to Los Angeles.
X_4 = Units delivered from Flint to Atlanta.

Objective function:

$$\text{Minimize } z = 30X_1 + 40X_2 + 60X_3 + 50X_4$$

Constraints:

$$\begin{aligned} X_1 + X_2 &\leq 200 \\ X_3 + X_4 &\leq 600 \\ X_1 + X_3 &\geq 300 \\ X_2 + X_4 &\geq 400 \end{aligned}$$

$X_1, X_2, X_3, X_4 \geq 0$ and integer values

(b) Solve the problem and interpret the results.

Step 1: Standardising the problem.
A minimisation problem is dealt with by multiplying the objective function by –1 and we transform it into a maximisation problem.

$$\begin{aligned} \text{Min } z = -\ \text{Max } w &= -30X_1 - 40X_2 - 60X_3 - 50X_4 \\ &+ 0S_1 + 0S_2 - 0S_3 - 0S_4 - MA_3 - MA_4 \\ X_1 + X_2 + S_1 &= 200 \\ X_3 + X_4 + S_2 &= 600 \\ X_1 + X_3 - S_3 + A_3 &= 300 \\ X_2 + X_4 - S_4 + A_4 &= 400 \end{aligned}$$

Step 2: Generating an initial solution (Tables 2.6, 2.7, 2.8 and 2.9).

$$X_1 = X_2 = X_3 = X_4 = S_3 = S_4 = 0$$

The solution to the problem is:

$$\begin{aligned} X_1 &= 200 \\ X_2 &= 0 \\ X_3 &= 100 \\ X_4 &= 400 \\ z &= 32{,}000 \end{aligned}$$

In other words, send 200 units from Portland to Los Angeles, 100 go from Flint to Los Angeles, and 400 leave Flint to Atlanta, with a total cost of \$32,000.

Table 2.6 Simplex Method

	C_j	−30	−40	−60	−50	0	0	0	0	−M	−M		
C_j	Base	X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	A_3	A_4	R	Ratio
0	S_1	1	1	0	0	1	0	0	0	0	0	200	200
0	S_2	0	0	1	1	0	1	0	0	0	0	600	600/0
−M	A_3	1	0	1	0	0	0	−1	0	1	0	300	300
−M	A_4	0	1	0	1	0	0	0	−1	0	1	400	400/0
	$C_j - Z_j$	−30+M	−40 + M	−60 + M	−50 + M	0	0	−M	−M	0	0		

Table 2.7 Simplex method

	C_j	−30	−40	−60	−50	0	0	0	0	−M	−M		
C_j	Base	X_1	X_2	X_3	X_4	S_1	S_2	S_3	S_4	A_3	A_4	R	Ratio
−30	X_1	1	1	0	0	1	0	0	0	0	0	200	200/0
0	S_2	0	0	1	1	0	1	0	0	0	0	600	600
−M	A_3	0	−1	1	0	−1	0	−1	0	1	0	100	100/0
−M	A_4	0	1	0	1	0	0	0	−1	0	1	400	400
	$C_j - Z_j$	0	−10	−60 + M	−50 + M	30 − M	0	−M	−M	0	0		

Table 2.8 Simplex method

C_j	C_j Base	-30 X_1	-40 X_2	-60 X_3	-50 X_4	0 S_1	0 S_2	0 S_3	0 S_4	$-M$ A_3	$-M$ A_4	R
−30	X_1	1	1	0	0	1	0	0	0	0	0	200
0	S_2	0	−1	1	0	0	1	0	1	0	−1	200
−M	A_3	0	−1	1	0	−1	0	−1	0	1	0	100
−50	X_4	0	1	0	1	0	0	0	−1	0	1	400
	$C_j - Z_j$	0	40 − M	−60 + M	0	30 − M	0	−M	−50	0	−M + 50	

Table 2.9 Simplex method

C_j	C_j Base	-30 X_1	-40 X_2	-60 X_3	-50 X_4	0 S_1	0 S_2	0 S_3	0 S_4	$-M$ A_3	$-M$ A_4	R
−30	X_1	1	1	0	0	1	0	0	0	0	0	200
0	S_2	0	0	0	0	1	1	1	1	−1	−1	200
−60	X_3	0	−1	1	0	−1	0	−1	0	1	0	100
−50	X_4	0	1	0	1	0	0	0	−1	0	1	400
	$C_j - Z_j$	0	−20	0	0	−30	0	−60	−50	60 − M	50 − M	

2.4 Contracting Carpenters

Thanks to a suitable marketing strategy and the quality of its star product, the Pombal bookcase, the carpentry business has received more orders than it can actually produce. Over the next 4 weeks, 52, 65, 70 and 85 bookcases, respectively, must be produced. Currently there are six artisan carpenters.

The general carpentry management decided to contract new staff to meet their commercial commitments. As artisans are short, inexperienced staff must be contracted. A novice can be trained on an apprenticeship for a week. The novice works a second week as an apprentice to gain experience. At the beginning of the third week (after 2 weeks of work), he/she becomes an artisan.

Estimated production and employees' salaries are provided in Table 2.10.

Each artisan can train up to two novices per week (training a novice takes only 1 week). Any surplus weekly production can be kept to meet the following commercial commitments.

Table 2.10 Estimated production and employees' weekly salaries

	Production Bookcases/week	Salaries $/week
Artisan working on production only	10	300
Artisan working on production and training	5	400
Apprentice	5	150
Novice	1	50

The firm's analysts estimate that it proves quite difficult to surpass the weekly demand for 90 bookcases. Thus, a decision was made to finish the period with no novices and apprentices, but with at least nine artisans. The firm's Trade Union rules forbid dismissals due to layoffs.

Formulate a linear programming model that defines the contracts to be issued for the purpose of meeting the commercial commitments at a minimum cost.

Solution

Decision variables:

X_{ij} : type "*i*" staff working in week "*j*"
Z_j : overproduction of week "*j*"

where

$i = 1$: producing artisan
$i = 2$: training artisan
$i = 3$: apprentice
$i = 4$: novice
$j = 1.4$: weeks.

Objective function: to meet commitments at a minimum cost.

$$\text{Min}\, z = \sum_{j=1}^{4}\sum_{i=1}^{4}\alpha_i \cdot x_{ij} \text{ where } \alpha_i : \text{is the salary of a type } i \text{ employee}$$

Constraints:

$$\begin{aligned}
&(\text{Week } 1)\\
&X_{11} + X_{21} = 6\\
&10X_{11} + 5X_{21} + X_{41} \geq 52\\
&X_{41} \leq 2X_{21}\\
&Z_1 = 10X_{11} + 5X_{21} + X_{41} - 52
\end{aligned}$$

$$\begin{aligned}
&(\text{Week } 2)\\
&X_{32} = X_{41}\\
&X_{12} + X_{22} = X_{11} + X_{21}\\
&10X_{12} + 5X_{22} + X_{42} + 5X_{32} + Z_1 \geq 65\\
&X_{42} \leq 2X_{22}\\
&Z_2 = 10X_{12} + 5X_{22} + X_{42} + 5X_{32} + Z_1 - 65
\end{aligned}$$

(Week 3)
$X_{33} = X_{42}$
$X_{13} + X_{23} = X_{12} + X_{22} + X_{32}$
$10X_{13} + 5X_{23} + X_{43} + 5X_{33} + Z_2 \geq 70$
$X_{43} \leq 2X_{23}$
$Z_3 = 10X_{13} + 5X_{23} + X_{43} + 5X_{33} + Z_2 - 70$

(Week 4)
$X_{34} = X_{43}$
$X_{14} = X_{13} + X_{23} + X_{33}$
$10X_{14} + 5X_{34} + Z_3 \geq 85$
$X_{14} \geq 9$

$X_{ij}, Z_j \geq 0$ and integer values.

2.5 Planning Production and Inventories

Consider the production of a single product in a planning horizon in T periods. If production during a given period t $(t = 1, \ldots, T)$ is decided, a fixed cost cf_t is incurred. Any excess products manufactured during early periods can be stored to meet the demand for later periods. Besides, all the demand must be met during each period. Production capacity constraints are not considered.

As $t = 1,.., T$:

- d_t is the demand for this product during each period,
- cp_t are the cost profits of production during each period, and
- ca_t are the cost profits of storage during each period.

(a) Formulate an integer linear programming model which minimizes the total costs of production, storage and fixed costs
(b) Assume that it permits a delay in delivering demand at cost crd_t per demand unit not delivered on time during each period. However, all the demand must be met during the last period T, or in other words, a delay in the demand during period T must be null. Amend the model in the former section to contemplate this option.
(c) Assume that production can take place in a maximum of five periods, although these periods cannot occur consecutively. Amend the model in the former section to contemplate this option.

Solution

(a) Formulate an integer linear programming model which minimizes the total costs of production, storage and fixed costs

Decision variables:
$X_t =$ Units of the product during period t to be produced.
$I_t =$ Units of the product during period t to be stored.
$Y_t =$ Binary variable that is 1 if the product is manufactured during period t, and 0 otherwise. This variable is employed to apply fixed costs.
Objective function:

$$\text{Minimize}\, z = \sum_{t=1}^{T} \text{cp}_t X_t + \text{ca}_t I_t + \text{cf}_t Y_t$$

Constraints:
$I_{t-1} + X_t - I_t = d_t$ (Demand must be met during each period)
$X_t \geq d_t Y_t$ (If there is any production, the corresponding binary variable is 1)
$X_t, I_t \geq 0$ (The non-negativity constraint)

(b) Assume that it permits a delay in delivering demand at cost crd_t per demand unit not delivered on time during each period. However, all the demand must met during the last period T, or in other words, a delay in the demand during period T must be null. Amend the model in the former section to contemplate this option.

The decision variables to be added:
$\text{Rd}_t =$ Delayed product units during period t
Objective function:

$$\text{Minimize}\, z = \sum_{t=1}^{T} \text{cp}_t X_t + \text{ca}_t I_t + \text{cf}_t Y_t + \text{crd}_t \text{Rd}_t$$

Constraints:
The first constraint is amended
$I_{t-1} + X_t - I_t - \text{Rd}_{t-1} + \text{Rd}_t = d_t$ (Demand can be delayed during each period)
The following constraint is added
$\text{Rd}_T = 0$ (The delay during the last period must be null)

(c) Assume that production can take place in a maximum of five periods, although these periods cannot occur consecutively. Amend the model in the former section to contemplate this option.

Constraints:
The following constraints are added:
$\sum_t Y_t = 5$ (Production can take place during a maximum of five periods)
$Y_t - Y_{t-1} \leq 1$ (Production cannot occur over consecutive periods)

2.6 Tarmacking Shifts

The firm MMM has obtained a contract to tarmac the streets in the centre of Exeter. The Department of Traffic Engineering and Planning estimated that at least the number of employees indicated in Table 2.11 are required for each 4-h interval over a standard 24-h period.

All of MMM's members of staff work 8-h ongoing shifts. There are six feasible shifts which began on the hour that each 4-h period shown in the table begins. All the members of staff are paid the same salary per hour, except those working the 20:00 h and 00:00 h shifts, who receive a 50 % increase. Moreover, the salary per hour is 100 % higher between 00:00 h and 06:00 h.

(a) Consider a linear programming model that can determine how many employees are required in all six shifts in order to minimise the cost of MMM's salaries to meet the personnel requirements.
(b) The temporary work ETS firm offers the members of staff more flexible working hours, with 4-h shifts beginning on the hour at the start of all the 4-h periods in the table. The cost of using ETS' staff is the same at all the times considered, but is 80 % higher than the daily cost of MMM's members of staff for each shift. MMM wishes to know if there would be any profit made if ETS was used. Amend the former model to determine the optimum mix of using MMM's and ETS' staff in order to minimise the costs incurred to MMM through salaries and to meet the personnel requirements.

Solution

(a) Consider a linear programming model that can determine how many employees are required in all six shifts in order to minimise the cost of MMM's salaries to meet the personnel requirements.

Decision variables:

X_1 = no. of MMM workers who begin at 06:00 h.
X_2 = no. of MMM workers who begin at 10:00 h.
X_3 = no. of MMM workers who begin at 14:00 h.
X_4 = no. of MMM workers who begin at 18:00 h.
X_5 = no. of MMM workers who begin at 22:00 h.

Table 2.11 Personnel requirements

Time period	Personnel
06–10	70
10–14	20
14–18	80
18–22	30
22–02	10
02–06	10

X_6 = no. of MMM workers who begin at 02:00 h.

Objective function:

As the salary paid per hour is the same, except for some shifts, the objective function can disregard the constant value of 1 h's work, as follows:

$$\text{Min}\, z = 8X_1 + 8X_2 + 6X_3 + 2\cdot 1.5X_3 + 2X_4 + 4\cdot 1.5X_4 + 2\cdot 2X_4 + 2\cdot 1.5X_5 \\ + 6\cdot 2X_5 + 4\cdot 2X_6 + 4X_6$$

Constraints:

The number of workers during each period must exceed that required:

$$X_6 + X_1 \geq 70$$
$$X_1 + X_2 \geq 20$$
$$X_2 + X_3 \geq 80$$
$$X_3 + X_4 \geq 30$$
$$X_4 + X_5 \geq 10$$
$$X_5 + X_6 \geq 10$$

$X_1, X_2, X_3, X_4, X_5, X_6 \geq 0$ (non-negativity) (Integers)

(b) The temporary work ETS firm offers the members of staff more flexible working hours, with 4-h shifts beginning on the hour at the start of all the 4-h periods in the table. The cost of using ETS' staff is the same at all the times considered, but is 80 % higher than the daily cost of MMM's members of staff for each shift. MMM wishes to know if there would be any profit made if ETS was used. Amend the former model to determine the optimum mix of using MMM's and ETS' staff in order to minimise the costs incurred to MMM through salaries and to meet the personnel requirements.

Decision variables:

Six new decision variables corresponding to ETS' workers are added:

X_7 = no. of ETS workers who begin at 06:00 h.
X_8 = no. of ETS workers who begin at 10:00 h.
X_9 = no. of ETS workers who begin at 14:00 h.
X_{10} = no. of ETS workers who begin at 18:00 h.
X_{11} = no. of ETS workers who begin at 22:00 h.
X_{12} = no. of ETS workers who begin at 02:00 h.

Objective function:

To the former objective function, it is necessary to add the cost of the ETS workers. As previously seen, the constant value of 1 h's work is not added, but the corresponding increase in salary is:

$$\begin{aligned}\text{Min}\, z = {} & 8X_1 + 8X_2 + 6X_3 + 2 \cdot 1.5X_3 + 2X_4 + 4 \cdot 1.5X_4 + 2 \cdot 2X_4 + 2 \cdot 1.5X_5 \\ & + 6 \cdot 2X_5 + 4 \cdot 2X_6 + 4X_6 + 4 \cdot 1.8X_7 + 4 \cdot 1.8X_8 + 4 \cdot 1.8X_9 + 2 \cdot 1.8X_{10} \\ & + 2 \cdot 1.5X_{10} + 2 \cdot 1.5 \cdot 1.8X_{11} + 2 \cdot 2 \cdot 1.8X_{11} + 4 \cdot 2 \cdot 1.8X_{12}\end{aligned}$$

Constraints:
The constraints are also amended to include the ETS workers:

$$\begin{aligned}X_6 + X_1 + X_7 &\geq 70 \\ X_1 + X_2 + X_8 &\geq 20 \\ X_2 + X_3 + X_9 &\geq 80 \\ X_3 + X_4 + X_{10} &\geq 30 \\ X_4 + X_5 + X_{11} &\geq 10 \\ X_5 + X_6 + X_{12} &\geq 10\end{aligned}$$

$X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}, X_{12} \geq 0$ (non-negativity) (Integers)

2.7 Transport Planning

A firm must transport machines from production plants *A*, *B* and *C* to warehouses *X*, *Y* and *Z*. Five machines are required in *X*, 4 in *Y* and 3 in *Z*, whereas 8 machines are available in *A*, 5 in *B* and 3 in *C*. The transport costs (in dollars) between sites are provided in Table 2.12.

(a) Formulate an integer linear programming model that minimizes transport costs.
(b) Assume that the cost of transporting a machine from plant B increases by $10 for all the machines as of the third one; that is, the 4th, the 5th, etc. Reformulate the model in Section (a) by considering this assumption.

Solution

(a) Formulate an integer linear programming model that minimizes transport costs.

Decision variables:

Table 2.12 Transport costs

Plant/Warehouse	X	Y	Z
A	50	60	30
B	60	40	20
C	40	70	30

X_{AX} = units to transport from plant A to warehouse X
X_{AY} = units to transport from plant A to warehouse Y
X_{AZ} = units to transport from plant A to warehouse Z
X_{BX} = units to transport from plant B to warehouse X
X_{BY} = units to transport from plant B to warehouse Y
X_{BZ} = units to transport from plant B to warehouse Z
X_{CX} = units to transport from plant C to warehouse X
X_{CY} = units to transport from plant C to warehouse Y
X_{CZ} = units to transport from plant C to warehouse Z

Objective function:

$$\text{Minimise } z = 50X_{AX} + 60X_{AY} + 30X_{AZ} + 60X_{BX} + 40X_{BY} + 20X_{BZ} + 40X_{CX} + 70X_{CY} + 30X_{CZ}$$

Constraints:

Availability constraints

$$X_{AX} + X_{AY} + X_{AZ} \leq 8$$
$$X_{BX} + X_{BY} + X_{BZ} \leq 5$$
$$X_{CX} + X_{CY} + X_{CZ} \leq 3$$

Requirements constraints

$$X_{AX} + X_{BX} + X_{CX} \geq 5$$
$$X_{AY} + X_{BY} + X_{CY} \geq 4$$
$$X_{AZ} + X_{BZ} + X_{CZ} \geq 3$$

The non-negativity constraint
$X_{AX}, X_{AY}, X_{AZ}, X_{BX}, X_{BY}, X_{BZ}, X_{CX}, X_{CY}, X_{CZ} \geq 0$ and integer

(a) Assume that the cost of transporting a machine from plant B increases by \$10 for all the machines as of the third one; that is, the fourth, the fifth, etc. Reformulate the model in Section (a) by considering this assumption.

Decision variables:

X_{BX2} = units to transport from plant B to warehouse X if there are more than three
X_{BY2} = units to transport from plant B to warehouse Y if there are more than three
X_{BZ2} = units to transport from plant B to warehouse Z if there are more than three

Objective function:

$$\text{Minimise}\, z = 50X_{AX} + 60X_{AY} + 30X_{AZ} + 60X_{BX} + 40X_{BY} + 20X_{BZ} + 40X_{CX} + 70X_{CY} + 30X_{CZ} + 70X_{BX2} + 50X_{BY2} + 30X_{BZ2}$$

Add the following constraints:

$$X_{BX} + X_{BY} + X_{BZ} \leq 3$$

This constraint limits to three the number of machines to be considered for transport by variables X_{BX}, X_{BY} and X_{BZ}.

$X_{BX2}, X_{BY2}, X_{BZ2} \geq 0$ and integers

Amend the following availability and requirements constraints:

$$X_{BX} + X_{BY} + X_{BZ} + X_{BX2} + X_{BY2} + X_{BZ2} \leq 5$$
$$X_{AX} + X_{BX} + X_{BX2} + X_{CX} \geq 5$$
$$X_{AY} + X_{BY} + X_{BY2} + X_{CY} \geq 4$$
$$X_{AZ} + X_{BZ} + X_{BZ2} + X_{CZ} \geq 3$$

2.8 Production Planning of Automobile Seats

A firm that produces automobile seats manufactures three seat types on two different production lines. Up to 30 workers can be used at the same time on each line to produce any seat type. Each worker is paid \$400 per week on production line 1, and \$600 per week on production line 2. One week of production on production line 1 costs \$1,000 to organise it and costs \$2,000 on production line 2. Table 2.13 provides the seat units that each worker produces of each seat type in one week on each production line.

The weekly demand of seats is at least 120,000 units of seat 1, 150,000 units of seat 2 and 200,000 units of seat 3.

Consider an integer linear programming model to minimise the total cost of the production plan to meet weekly production demands.

Solution

Decision variables:

X_1 = no. of workers employed on production line 1

Table 2.13 Weekly production of seats per worker

	Seats (thousands of units)		
Production line	1	2	3
1	20	30	40
2	50	35	45

X_2 = no. of workers employed on production line 2
Y_1 = 1 if line 1 is used, 0 otherwise
Y_2 = 1 if line 2 is used, 0 otherwise

Objective function:

$$\text{Minimise}\, z = 1000Y_1 + 2000Y_2 + 400X_1 + 600X_2$$

Constraints:

$$20X_1 + 50X_2 = 120$$
$$30X_1 + 35X_2 = 150$$
$$40X_1 + 45X_2 = 200$$
$$X_1 = 30Y_1$$
$$X_2 = 30Y_2$$
$$X_1, X_2 = 0$$
$$Y_1, Y_2 = 0\,1$$

2.9 Production Planning in the Shoe Industry

An auxiliary company from the footwear sector produces different types of materials to make shoes: leather, canvas and rubber. Each of these materials requires the following production times to produce 100 square feet in three different sections of the factory (A, B and C), as set out in Table 2.14.

Production times of 4,000, 5,000 and 6,000 h are available in each factory section (A, B and C), respectively. The profit of selling one square foot of leather, one square foot of canvas and one square foot of rubber is \$15, 7 and 3, respectively.

(a) Formulate an integer linear programming model which maximises the gross profit margin.
(b) Assume that a second working shift can be used with an additional cost of 500 dollars/shift in each factory section (A, B and C) that would add 3,500, 2,500 and 3,000 h in production time, respectively. Amend the former model to contemplate whether a second shift in each factory section is an interesting option or not.

Table 2.14 Production times in each section

Material	A	B	C
Leather	7 h	8 h	6 h
Canvas	3 h	2 h	5 h
Rubber	5 h	5 h	3 h

(c) In addition, apart from a second shift, assume that an additional third shift can be used which costs 700 dollars/shift in each factory section (A, B and C) that would add 1,500, 2,000 and 1,000 h in production time, respectively. Amend the model in Section (b) to consider whether a third shift in each factory section is an interesting option or not.
(d) Based on the model in Section (c), how would it be formulated if a third shift could not be used, unless the second shift was used, in each factory section?
(e) Based on the model in Section (c), how would it be formulated if only one additional shift could be used, either a second or third one, in each factory section?

Solution

(a) Formulate an integer linear programming which maximises the gross profit margin.

Decision variables:

X_1 = No. of batches of 100 sq ft. of leather;
X_2 = No. of batches of 100 sq ft. of canvas;
X_3 = No. of batches of 100 sq ft. of rubber;

Objective function:

$$\text{Maximise}\, z = 1500X_1 + 700X_2 + 300X_3$$

Constraints:

$$\begin{aligned}
&7X_1 + 3X_2 + 5X_3 \leq 4000\\
&8X_1 + 2X_2 + 5X_3 \leq 5000\\
&6X_1 + 5X_2 + 3X_3 \leq 6000\\
&X_1, X_2, X_3 \geq 0,\ \text{integers}
\end{aligned}$$

(b) Assume that a second working shift can be used with an additional cost of 500 dollars/shift in each factory section (A, B and C) that would add 3,500, 2,500 and 3,000 h in production time, respectively. Amend the former model to contemplate whether a second shift in each factory section is an interesting option or not.

Three new binary variables will be added:

Y_1 = 1 if a second shift is used in A, and 0 otherwise;
Y_2 = 1 if a second shift is used in B, and 0 otherwise;
Y_3 = 1 if a second shift is used in C, and 0 otherwise;

The constraints are amended as follows:

$$7X_1 + 3X_2 + 5X_3 \le 4000 + 3500Y_1$$
$$8X_1 + 2X_2 + 5X_3 \le 5000 + 2500Y_2$$
$$6X_1 + 5X_2 + 3X_3 \le 6000 + 3000Y_3$$

The objective function is amended as so:

$$\text{Maximise } z = 1500X_1 + 700X_2 + 300X_3 - 500(Y_1 + Y_2 + Y_3)$$

(c) In addition, apart from a second shift, assume that an additional third shift can be used which costs 700 dollars/shift in each factory section (A, B and C) that would add 1,500, 2,000 and 1,000 h in production time, respectively. Amend the model in Section (b) to consider whether a third shift in each factory section is an interesting option or not.

Three new binary variables will be added:

$Z_1 = 1$ if a third shift is used in A, and 0 otherwise;
$Z_2 = 1$ if a third shift is used in B, and 0 otherwise;
$Z_3 = 1$ if a third shift is used in C, and 0 otherwise;

The constraints are amended as follows:

$$7X_1 + 3X_2 + 5X_3 \le 4000 + 3500Y_1 + 1500Z_1$$
$$8X_1 + 2X_2 + 5X_3 \le 5000 + 2500Y_2 + 2000Z_2$$
$$6X_1 + 5X_2 + 3X_3 \le 6000 + 3000Y_3 + 1000Z_3$$

The objective function is amended as so:

$$\begin{aligned}\text{Maximise } z = {} & 1500X_1 + 700X_2 + 300X_3 - 500(Y_1 + Y_2 + Y_3) \\ & - 700(Z_1 + Z_2 + Z_3)\end{aligned}$$

(d) Based on the model in Section (c), how would it be formulated if a third shift could not be used, unless the second shift was used, in each factory section?

$$Z_1 \le Y_1$$
$$Z_2 \le Y_2$$
$$Z_3 \le Y_3$$

(e) Based on the model in Section c), how would it be formulated if only one additional shift could be used, either a second or third one, in each factory section?

$$Y_1 + Z_1 \leq 1$$
$$Y_2 + Z_2 \leq 1$$
$$Y_3 + Z_3 \leq 1$$

2.10 Allocating Orders to Machines

The firm Orgasa has received five orders (P1, P2, P3, P4, P5), which have to be carried out. To meet them, there are five machines available (M1, M2, M3, M4, M5). Each machine can carry out every task at the cost shown in Table 2.15. The problem consists in determining optimum allocation which minimises the total cost of carrying out orders by assuming that each machine can do only one order and that all the orders must be carried out.

(a) Formulate an integer linear programming model to determine the optimum allocation plan.
(b) Assume that processing each order on each machine requires an average preparation cost of 10 units. How is the model in the former sector amended?

Solution

(a) Formulate an integer linear programming model to determine the optimum allocation plan.

Decision variables:

$Y_{11} = 1$ if M1 carries out P1, and 0 otherwise;
$Y_{12} = 1$ if M1 carries out P2, and 0 otherwise;
$Y_{13} = 1$ if M1 carries out P3, and 0 otherwise;
$Y_{14} = 1$ if M1 carries out P4, and 0 otherwise;
$Y_{15} = 1$ if M1 carries out P5, and 0 otherwise;
$Y_{21} = 1$ if M2 carries out P1, and 0 otherwise;
$Y_{22} = 1$ if M2 carries out P2, and 0 otherwise;
$Y_{23} = 1$ if M2 carries out P3, and 0 otherwise;
$Y_{24} = 1$ if M2 carries out P4, and 0 otherwise;

Table 2.15 Cost of carrying out tasks on each machine

Order/Machine	M1	M2	M3	M4	M5
P1	16	4	9	5	6
P2	2	14	7	5	13
P3	8	10	3	12	11
P4	3	7	6	10	5
P5	3	6	8	11	7

$Y_{25} = 1$ if M2 carries out P5, and 0 otherwise;
$Y_{31} = 1$ if M3 carries out P1, and 0 otherwise;
$Y_{32} = 1$ if M3 carries out P2, and 0 otherwise;
$Y_{33} = 1$ if M3 carries out P3, and 0 otherwise;
$Y_{34} = 1$ if M3 carries out P4, and 0 otherwise;
$Y_{35} = 1$ if M3 carries out P5, and 0 otherwise;
$Y_{41} = 1$ if M4 carries out P1, and 0 otherwise;
$Y_{42} = 1$ if M4 carries out P2, and 0 otherwise;
$Y_{43} = 1$ if M4 carries out P3, and 0 otherwise;
$Y_{44} = 1$ if M4 carries out P4, and 0 otherwise;
$Y_{45} = 1$ if M4 carries out P5, and 0 otherwise;
$Y_{51} = 1$ if M5 carries out P1, and 0 otherwise;
$Y_{52} = 1$ if M5 carries out P2, and 0 otherwise;
$Y_{53} = 1$ if M5 carries out P3, and 0 otherwise;
$Y_{54} = 1$ if M5 carries out P4, and 0 otherwise;
$Y_{55} = 1$ if M5 carries out P5, and 0 otherwise;

Objective function:
It represents the total cost of carrying out tasks:

$$\begin{aligned}\text{Minimize } z = {} & 16Y_{11} + 4Y_{12} + 9Y_{13} + 5Y_{14} + 6Y_{15} + 2Y_{21} + 14Y_{22} + 7Y_{23} \\ & + 5Y_{24} + 13Y_{25} + 8Y_{31} + 10Y_{32} + 3Y_{33} + \\ & 12Y_{34} + 11Y_{35} + 3Y_{41} + 7Y_{42} + 6Y_{43} + 10Y_{44} + 5Y_{45} + 3Y_{51} \\ & + 6Y_{52} + 8Y_{53} + 11Y_{54} + 7Y_{55}\end{aligned}$$

Constraints:
It must be ensured that each machine does only one of the tasks:

$$Y_{11} + Y_{12} + Y_{13} + Y_{14} + Y_{15} \leq 1$$
$$Y_{21} + Y_{22} + Y_{23} + Y_{24} + Y_{25} \leq 1$$
$$Y_{31} + Y_{32} + Y_{33} + Y_{34} + Y_{35} \leq 1$$
$$Y_{41} + Y_{42} + Y_{43} + Y_{44} + Y_{45} \leq 1$$
$$Y_{51} + Y_{52} + Y_{53} + Y_{54} + Y_{55} \leq 1$$

And that all the orders are carried out:

$$Y_{11} + Y_{21} + Y_{31} + Y_{41} + Y_{51} \geq 1$$
$$Y_{12} + Y_{22} + Y_{32} + Y_{42} + Y_{52} \geq 1$$
$$Y_{13} + Y_{23} + Y_{33} + Y_{43} + Y_{53} \geq 1$$
$$Y_{14} + Y_{24} + Y_{34} + Y_{44} + Y_{54} \geq 1$$
$$Y_{15} + Y_{25} + Y_{35} + Y_{45} + Y_{55} \geq 1$$

The non-negativity constraint:

$$Y_{11}, Y_{12}, Y_{13}, Y_{14}, Y_{15}, Y_{21}, Y_{22}, Y_{23}, Y_{24}, Y_{25}, Y_{31}, Y_{32}, Y_{33}, Y_{34}, Y_{35}, Y_{41}, Y_{42}, Y_{43}, Y_{44}, Y_{45}, Y_{51}, Y_{52}, Y_{53}, Y_{54}, Y_{55} \geq 0$$

(b) Assume that processing each order on each machine requires an average preparation cost of 10 units. How is the model in the former sector amended?

The objective function is amended as follows:

$$\begin{aligned}\text{Minimize}\, z = {} & (16+10)Y_{11} + (4+10)Y_{12} + (9+10)Y_{13} + (5+10)Y_{14} \\ & + (6+10)Y_{15} + (2+10)Y_{21} + (14+10)Y_{22} + (7+10)Y_{23} + (5+10)Y_{24} \\ & + (13+10)Y_{25} + (8+10)Y_{31} + (10+10)Y_{32} + (3+10)Y_{33} + (12+10)Y_{34} \\ & + (11+10)Y_{35} + (3+10)Y_{41} + (7+10)Y_{42} + (6+10)Y_{43} + (10+10)Y_{44} \\ & + (5+10)Y_{45} + (3+10)Y_{51} + (6+10)Y_{52} + (8+10)Y_{53} + \\ & (11+10)Y_{54} + (7+10)Y_{55}\end{aligned}$$

2.11 Opening LED TV Production Plants

SAMSUNG, a manufacturer of LED televisions, is considering opening a new assembly plant to produce three TV models: high-, medium- and low-range models. There are two possible locations: 1 and 2. The investment required to construct the factory at location 1 is \$2,000,000 and \$1,750,000 at location 2. The unit profits of production are \$15, 13 and 10 , respectively, for the high- (a), medium- (m) and low- (b) range models at location 1 and \$16, 12 and 9, respectively, at location 2.

At least 75,000 units of the high-, 100,000 units of the medium- and 200,000 units of the low-range models are to be produced annually.

(a) If only one assembly plant is to be constructed, model the problem with a view to minimising costs.
(b) If there is a possibility of constructing two plants (locations 1 and 2), model the problem with a view to minimising costs by also considering the following constraints:

– Should low-range television be produced at location 1, a subsidy of \$1,000,000 will be granted.
– The high-range model will be produced only at one of the two locations.

Solution

(a) If only one assembly plant is to be constructed, model the problem with a view to minimising costs.

Decision variables:

X_{1a} = no. of televisions of range a produced at location 1 annually;
X_{1m} = no. of televisions of range m produced at location 1 annually;
X_{1b} = no. of televisions of range b produced at location 1 annually;
X_{2a} = no. of televisions of range a produced at location 2 annually;
X_{2m} = no. of televisions of range m produced at location 2 annually;
X_{2b} = no. of televisions of range b produced at location 2 annually;
$Y_1 = 1$ if produced at location 1, and 0 otherwise;
$Y_2 = 1$ if produced at location 2, and 0 otherwise;

Objective function:

$$\begin{aligned}\text{Minimize}\, z = {} & 15X_{1a} + 13X_{1m} + 10X_{1b} + 16X_{2a} + 12X_{2m} + 9X_{2b} + 2000000Y_1 \\ & + 1750000Y_2\end{aligned}$$

Constraints:

$$\begin{aligned}
& X_{1a} + X_{2a} \geq 75000; \\
& X_{1m} + X_{2m} \geq 100000; \\
& X_{1b} + X_{2b} \geq 200000; \\
& Y_1 + Y_2 = 1; \\
& X_{1a} \leq MY_1 \\
& X_{2a} \leq MY_2 \\
& X_{1m} \leq MY_1 \\
& X_{2m} \leq MY_2 \\
& X_{1b} \leq MY_1 \\
& X_{2b} \leq MY_2 \\
& X_{1a}, X_{1m}, X_{1b}, X_{2a}, X_{2m}, X_{2b} \geq 0 \\
& \text{and integers (the non - negativity constraint)} \\
& Y_1, Y_2 = \ 0\,\text{o}\,1
\end{aligned}$$

where M is a sufficiently positive number.

(b) If there is a possibility of constructing two plants (locations 1 and 2), model the problem with a view to minimising costs by also considering the following constraints:

- Should low-range television be produced at location 1, a subsidy of \$1,000,000 will be granted.
- The high-range model will be produced only at one of the two locations.

Decision variables:

X_{1a} = no. of televisions of range a produced at location 1 annually;
X_{1m} = no. of televisions of range m produced at location 1 annually;

X_{1b} = no. of televisions of range b produced at location 1 annually;
X_{2a} = no. of televisions of range a produced at location 2 annually;
X_{2m} = no. of televisions of range m produced at location 2 annually;
X_{2b} = no. of televisions of range b produced at location 2 annually;
Y_{1a} = 1 if the televisions of range a are produced at location 1;
Y_{1m} = 1 if the televisions of range m are produced at location 1;
Y_{1b} = 1 if the televisions of range b are produced at location 1;
Y_{2a} = 1 if the televisions of range a are produced at location 2;
Y_{2m} = 1 if the televisions of range m are produced at location 2;
Y_{2b} = 1 if the televisions of range b are produced at location 2;
Z_1 = 1 if produced at location 1, and 0 otherwise;
Z_2 = 1 if produced at location 2, and 0 otherwise;

Objective function:

$$\text{Minimize } z = 15X_{1a} + 13X_{1m} + 10X_{1b} + 16X_{2a} + 12X_{2m} + 9X_{2b} + 2000000Z_1 + 1750000Z_2 - 1000000Y_{1b}$$

Constraints:

$$\begin{aligned}
&X_{1a} + X_{2a} \geq 75000;\\
&X_{1m} + X_{2m} \geq 100000;\\
&X_{1b} + X_{2b} \geq 200000;\\
&Y_{1a} + Y_{2a} = 1;\\
&Z_1 + Z_2 \geq 1;\\
&Y_{1a} \leq Z_1;\\
&Y_{2a} \leq Z_2;\\
&Y_{1m} \leq Z_1;\\
&Y_{2m} \leq Z_2;\\
&Y_{1b} \leq Z_1;\\
&Y_{2b} \leq Z_2;\\
&X_{1a} \leq MY_{1a};\\
&X_{2a} \leq MY_{2a};\\
&X_{1m} \leq MY_{1m};\\
&X_{2m} \leq MY_{2m};\\
&X_{1b} \leq MY_{1b};\\
&X_{2b} \leq MY_{2b};
\end{aligned}$$

$X_{1a}, X_{1m}, X_{1b}, X_{2a}, X_{2m}, X_{2b} \geq 0\, Y$, integers (The non-negativity constraint)

$$Y_{1a}, Y_{2a}, Y_{1m}, Y_{2m}, Y_{1b}, Y_{2b}, Z_1, Z_2 = 0, 1$$

where M is a sufficiently positive number.

2.12 Planning Surveys

The firm Orgasa Estudios de Mercado, S.A.(OEM) specialises in evaluating customers' reactions to new products, services and advertising campaigns. A client company has required OEM's services to determine customers' reactions to a recently advertised product of domestic use. It has been agreed that during door-to-door surveys will be used to obtain information from homes with and without children, and that surveys will be conducted in the morning and afternoon. Specifically, the client contracted OEM to conduct 1,000 surveys according to the following guidelines:

- At least 400 homes with children will be surveyed.
- At least 400 homes without children will be surveyed.
- The total number of homes surveyed in the afternoon will be as large as the number of homes surveyed in the morning.
- At least 40 % of the surveys conducted at homes with children will be held in the morning.
- At least 60 % of the surveys conducted at homes with children will be held in the afternoon.

Since the surveys conducted in homes with children require extra interviewer time, and as the interviewers working afternoons are paid more than those working mornings, the cost of one survey varies according to the home type, as Table 2.16 shows.

Formulate a model to determine the optimum surveys plan.

Solution

Decision variables:

X_1 = No. of surveys done at homes with children in the morning;
X_2 = No. of surveys done at homes with children in the afternoon;
X_3 = No. of surveys done at homes without children in the morning;
X_4 = No. of surveys done at homes without children in the afternoon;

Objective function:

$$\text{Minimize } z = 20X_1 + 25X_2 + 18X_3 + 20X_4$$

Constraints:

Table 2.16 Cost of each survey type

Type of home	Cost of a survey in the morning ($)	Cost of a survey in the afternoon ($)
With children	20	25
Without children	18	20

$$X_1 + X_2 + X_3 + X_4 = 1000$$
$$X_1 + X_2 \geq 400$$
$$X_3 + X_4 \geq 400$$
$$X_2 + X_4 \geq X_1 + X_3$$
$$X_1 \geq 0.4(X_1 + X_2)$$
$$X_4 \geq 0.6(X_3 + X_4)$$
$$X_1, X_2, X_3, X_4 \geq 0 \text{ (Integer)}$$

2.13 Production Planning in a Toys Firms

A toys firm is planning the production of two new toys. The fixed cost of configuring the production plant and the unit profit per toy type are provided in Table 2.17 below.

The firm has two factories that are capable of producing these toys. The production rates per toy type are provided in Table 2.18.

Factories 1 and 2 can provide 480 and 720 h, respectively, to produce these toys. The firm wishes to know what, where and how many toys should be produced to maximise its profits.

Introduce decision variables 0–1 and formulate the above problem as an integer linear programming problem. To avoid duplicating costs, the model must decide which factories must be configured for each toy type and the corresponding fixed configuration costs must apply.

Solution

Decision variables:

We must decide whether to configure a factory to produce one toy or not; thus:

$F_{11} = 1$ if factory 1 is configured to produce toy 1, and 0 otherwise.
$F_{12} = 1$ if factory 1 is configured to produce toy 2 and 0 otherwise.
$F_{21} = 1$ if factory 2 is configured to produce toy 1 and 0 otherwise.
$F_{22} = 1$ if factory 2 is configured to produce toy 2 and 0 otherwise.
We must decide how many of each type must be produced in each factory; thus:

Table 2.17 Cost of the configuration and unit profit per toy type

Toy	Cost of configuration ($)	Unit profit ($/toy)
1	45,000	12
2	76,000	16

Table 2.18 Production rates per toy type and factory

	Toy 1	Toy 2
Factory 1	52 units/hour	38 units/hour
Factory 2	42 units/hour	23 units/hour

X_{11} = No. of toys produced in factory 1 of type 1.
X_{12} = No. of toys produced in factory 1 of type 2.
X_{21} = No. of toys produced in factory 2 of type 1.
X_{22} = No. of toys produced in factory 2 of type 2.

Objective function:
The objective is to maximise total profit:

$$\text{Maximise}\, z = 12(X_{11} + X_{21}) + 16(X_{12} + X_{22}) - 45000(F_{11} + F_{21}) \\ - 76000(F_{21} + F_{22})$$

Constraints:
The production time available in each factory cannot be exceeded:

$$\mathrm{X}_{11}/52 + \mathrm{X}_{12}/38 \leq 480$$
$$\mathrm{X}_{21}/42 + \mathrm{X}_{22}/23 \leq 720$$

No toy can be produced unless the factory has been configured for this purpose:

$$\mathrm{X}_{11} \leq 52(480)\ \mathrm{F}_{11}$$
$$\mathrm{X}_{12} \leq 38(480)\ \mathrm{F}_{12}$$
$$\mathrm{X}_{21} \leq 42(720)\ \mathrm{F}_{12}$$
$$\mathrm{X}_{22} \leq 23(720)\ \mathrm{F}_{22}$$

The non-negative integer values constraint:
$\mathrm{X}_{11},\ \mathrm{X}_{12},\ \mathrm{X}_{21},\ \mathrm{X}_{22} \geq 0$ and integers

2.14 Allocation in a Lawyers' Office

A lawyers' office has taken on five new cases, and each one can be adequately handled by any of the five more recent partners. Given the difference in experience and practice, lawyers will spend different times on their cases. One of the most experienced partners has estimated requirements in terms of time (in hours; see Table 2.19).

Table 2.19 Estimation of the time required per case and lawyer

	Case 1	Case 2	Case 3	Case 4	Case 5
Lawyer 1	45	22	30	65	15
Lawyer 2	180	163	85	148	68
Lawyer 3	21	17	193	59	65
Lawyer 4	18	83	16	70	115
Lawyer 5	97	75	20	70	71

In order to determine the optimum way of allocating cases to lawyers so that each one takes on a different case and that the total time of hours spent is the minimum formulate this problem as an integer linear programming model

Solution

Decision variables:

This problem can be modelled as an allocation problem. As it is a matter of deciding which lawyer takes on each case, we define X_{ij} which takes a value of 1 if lawyer i is allocated case j, and 0 otherwise, where $i, j = 1,\ldots,5$. These 25 variables are the decision variables of the problem.

Objective function:

Our objective is to optimally allocate lawyers to cases (or vice versa). Evidently, one allocation is better than another if the total number of hours spent on preparing cases is lower. Hence, the objective function must minimise the number of hours that the five lawyers spend on preparing the five cases.

$$\begin{aligned}
\text{Minimize } z = {} & 45X_{11} + 22X_{12} + 30X_{13} + 65X_{14} + 15X_{15} + 180X_{21} + 163X_{22} \\
& + 85X_{23} + 148X_{24} + 68X_{25} + 21X_{31} + 17X_{32} + 193X_{33} + 59X_{34} \\
& + 65X_{35} + 18X_{41} + 83X_{42} + 16X_{43} + 70X_{44} + 115X_{45} + 97X_{51} \\
& + 75X_{52} + 20X_{53} + 70X_{54} + 71X_{55}
\end{aligned}$$

Constraints:

The constraints of this problem emerge because each lawyer is expected to take on a different case. If lawyer 1 can take on only one case, then one, and only one, of variables $X_{1j}, j = 1,\ldots,5$ takes a value of 1, while the other four should take a value of zero:

$$\begin{aligned}
X_{11} + X_{12} + X_{13} + X_{14} + X_{15} &= 1\,(1 \text{ case for lawyer } 1) \\
X_{21} + X_{22} + X_{23} + X_{24} + X_{25} &= 1\,(1 \text{ case for lawyer } 2) \\
X_{31} + X_{32} + X_{33} + X_{34} + X_{35} &= 1\,(1 \text{ case for lawyer } 3) \\
X_{41} + X_{42} + X_{43} + X_{34} + X_{45} &= 1\,(1 \text{ case for lawyer } 4) \\
X_{51} + X_{52} + X_{53} + X_{54} + X_{55} &= 1\,(1 \text{ case for lawyer } 5)
\end{aligned}$$

Besides, together with the fact that there are as many lawyers as cases, each case can be prepared by only one lawyer:

$$\begin{aligned}
X_{11} + X_{21} + X_{31} + X_{41} + X_{51} &= 1(1 \text{ lawyer for case } 1) \\
X_{12} + X_{22} + X_{32} + X_{42} + X_{52} &= 1(1 \text{ lawyer for case } 2) \\
X_{13} + X_{23} + X_{33} + X_{43} + X_{53} &= 1(1 \text{ lawyer for case } 3) \\
X_{14} + X_{24} + X_{34} + X_{44} + X_{54} &= 1(1 \text{ lawyer for case } 4) \\
X_{15} + X_{25} + X_{35} + X_{45} + X_{55} &= 1(1 \text{ lawyer for case } 5) \\
X_{ij} \quad \{0, 1\}, i, j &= 1, \ldots, 5
\end{aligned}$$

2.15 Production Planning

An Organization Engineer is working on a firm's monthly production for the next 6 months. The firm can work each month using a normal shift or an extended shift. A normal shift costs \$100,000 a month and can produce up to 5,000 units per month. An extended shift costs up to \$180,000 a month and can produce up to 7,500 units per month. It is necessary to remember that the cost incurred for each shift type is fixed and is, therefore, independent of the quantity produced. If the firm decides to not produce in a given month, the incurred costs are zero.

It is estimated that changing from a normal shift in 1 month to an extended shift in the next month incurs an additional cost of \$15,000. Additional costs are not incurred when changing from an extended shift in 1 month to a normal shift in the next month.

The cost of storing stock is estimated at \$2 per unit and month (based on existing stock at the end of each month) and the initial stock is 3,000 units (produced from a normal shift). The quantity of stock at the end of month 6 should be at least 2,000 units.

The demand of the firm's product in all of the next 6 months is indicated in Table 2.20.

The production constraints are such that if the firm produces something in a particular month, it must produce at least 2,000 units.

The firm needs a production plan for the next 6 months to avoid stockouts.

Formulate a mathematical model that helps the Organisation Engineer to devise a production plan for the next 6 months that avoids stockouts.

Solution

Decision variables:

$X_t = 1$ if a normal shift is used in month t ($t = 1, 2,.., 6$), or 0 otherwise
$Y_t = 1$ if an extended shift is used in month t ($t = 1, 2,.., 6$), or 0 otherwise
$Z_t = 1$ if a normal shift is used in month $t - 1$, but changes to an extended shift in month t ($t = 1, 2,.., 6$), or 0 otherwise.
$W_t = 1$ if there is production in month t ($t = 1, 2,.., 6$), or 0 otherwise.
I_t = inventory at the end of period t ($t = 1, 2,.., 6$).
P_t = quantity produced in month t ($t = 1, 2,\ldots, 6$).

Objective function:

$$\text{Minimise}\, z = \sum_{t=1}^{6} (100000X_t + 180000Y_t + 15000Z_t + 2I_t)$$

Table 2.20 The firm's demand in the next 6 months

Month	1	2	3	4	5	6
Demand (in units)	6,000	6,500	7,500	7,000	6,000	6,000

Constraints:
Only one shift type can be used each month:

$$X_t + Y_t = 1\ (t = 1, .., 6)$$

It is not possible to go beyond the production limits:

$$P_t \leq 5000X_t + 7500Y_t$$

Stockouts are not permitted:

$$I_t \geq 0$$

The continuity equation for inventory is:

$$I_t = I_{t-1} + P_t - D_t$$

The quantity of stock at the end of month 6 should be at least 2,000 units:

$$I_6 \geq 2000$$

The constraints that relate P_t with binary variable W_t:
$P_t \leq 7500W_t$ (7500 represents the larger quantity that can be produced irrespectively of the shift employed)

$$P_t \geq 2000W_t$$

The constraint that relates the variable of changing shift Z_t with the planned changes:

$$Z_t = X_{t-1}Y_t$$

This constraint is not linear, but can be transformed into the two following linear equations:

$$Z_t > \ = X_{t-1} + Y_t - -1$$
$$Z_t < \ = \ (X_{t-1} + Y_t)/2$$

P_t, $I_t \geq 0$ and integers

$$X_{t-1}, Y_t, Z_{t,} W_t \in \{0, 1\}$$

2.16 Planning University Schedules

The Studies Coordinator for the Industrial Organisation Engineering degree at EPSA is attempting to solve the "academic schedule problem". The objective is to associate the classrooms and times of subject matters with an academic programme divided into two courses.

It is considered that three classrooms and 5 h, respectively, are available to teach eight subject matters. These subject matters are grouped into two teachers and two academic courses.

The set of all the subject matters is: A = {a1, a2, a3, a4, a5, a6, a7, a8}

Teacher 1 teaches three subject matters. The set of teacher 1's subject matters is: AP1 = {a1, a2, a8}

Teacher 2 teaches five subject matters. The set of teacher 2's subject matters is: AP2 = {a3, a4, a5, a6, a7}

The set of the subject matters in academic course 1 is: AC1 = {a1, a2, a3, a4}

The set of the subject matters in academic course 2 is: AC2 = {a5, a6, a7, a8}

We must bear in mind that:

- Each teacher teaches all his/her subject matters.
- Each teacher teaches one subject matter each hour at the most.
- Each subject matter is taught only once.
- Only one subject matter is taught in each classroom and each hour at the most.
- One subject matter of each academic course is taught each hour at the most.

Create an integer linear programming model that associates three classrooms and 5 h to the eight subject matters of the academic programme divided into two courses by accomplishing a compact schedule.

Solution

Decision variables:

v(a, c, h): a binary variable that takes a value of 1 if subject matter *a* is given in classroom *c* at time *h,* and 0 otherwise.

Objective function:

$$\sum_{a\in A}\sum_{c=1}^{3}\sum_{h=1}^{5}(c+h)v(a,c,h)$$

Constraints:

Each teacher teaches all his/her subject matters.

$$\sum_{a\in AP1}\sum_{c=1}^{3}\sum_{h=1}^{5}v(a,c,h)=3$$

$$\sum_{a\in AP2}\sum_{c=1}^{3}\sum_{h=1}^{5}v(a,c,h)=5$$

Each teacher teaches one subject matter each hour at the most.

$$\sum_{a\in AP1}\sum_{c=1}^{3}v(a,c,h)\leq 1,\forall h$$

$$\sum_{a \in AP2} \sum_{c=1}^{3} v(a,c,h) \leq 1, \forall h$$

Each subject matter is taught only once.

$$\sum_{c=1}^{3} \sum_{h=1}^{5} v(a,c,h) = 1, \forall a$$

Only one subject matter is taught in each classroom and each hour at the most.

$$\sum_{a \in A} v(a,c,h) \leq 1, \forall c, \forall h$$

One subject matter of each academic course is taught each hour at the most.

$$\sum_{a \in AC1} \sum_{c=1}^{3} v(a,c,h) \leq 1, \forall h$$

$$\sum_{a \in AC2} \sum_{c=1}^{3} v(a,c,h) \leq 1, \forall h$$

2.17 The Shortest Path

Define the decision variables, the objective function and the constraints to be considered to find a graph with N nodes (A, B, …, N) and the shortest path from A to N by integer linear programming.

Solution

Decision variables:

X_{ij} = 1 if the arc corresponds to the shortest path, 0 if the arc does not belong to the shortest path

The cost coefficients of the objective function are:

C_{ij} = distance between i and j (infinite if there is no direct connection)

Objective function:

$$\text{Minimize}\, z = C_{ij} \cdot X_{ij}$$

With these data, the problem involves the minimum cost of flow, with an inward and an outward flux of 1 per node. Hence, for instance, the constraints of the outward flow for node A will be:

$$\sum_{j=1}^{n} X_{Aj} = 1 \quad \text{(outward flux of node A is 1)}$$

where n represents the total number of arcs from node i to node n

For node B:

$$\sum_{j=1}^{n} X_{Bj} = 1 \quad \text{(the outward flux of node B is 1)}$$

$$\sum_{i=1}^{n} X_{iB} = 1 \qquad \text{(the inward flux of node B is 1)}$$

And so forth for the remaining nodes. Only the constraints corresponding to the inward flux are considered for the last node N.

$$\sum_{i=1}^{n} X_{iN} = 1 \qquad \text{(the inward flux of node N is 1)}$$

2.18 Planning Sales

A firm produces two products, A and B, using two imported components, C and D, which it also sells to its customers. The sales prices of these four products are $69, $57, $4.5 and $3.2, respectively. The assembling of product A requires 4 units of product C and 3 units of product D, while product B requires 6 units of product C and 9 units of product D. The purchase price of 1 unit of component C is $4, whereas component D costs $3. Product A and product B require 3 and 2 h of assembling time, respectively. The cost of the assembling hours allocated to each product unit is $1/hour.

It is estimated that 10,000 and 14,000 units of products C and D are available, with a total of 7,000 assembling hours for the next 6 months.

(a) Formulate an integer linear programming model that maximises this firm's gross profit margin for the next 6 months.
(b) Assume that a commercial constraint exists whereby one, and only one, of products A and B and either component C or D can be sold. Amend the original model in Section (a) to bear in mind this consideration.
(c) Assume that only 3 of the 4 products available are expected to be sold. Add the appropriate constraints to the original model to contemplate this new commercial constraint.
(d) Amend the original problem to consider a fixed cost of $6,000 for product A and of $4,000 for product B which is incurred when selling one product unit. These costs are not incurred if a decision is made to not sell these products.

Solution

(a) Formulate an integer linear programming which maximises this firm's gross profit margin for the next 6 months.

Decision variables:

X_1 = Units of product A to sell
X_2 = Units of product B to sell
X_3 = Units of product C to sell
X_4 = Units of product D to sell

Objective function:

$$\begin{aligned}\text{Maximise } z = & (69 - 4 \cdot 4 - 3 \cdot 3 - 1 \cdot 3)X_1 + (57 - 4 \cdot 6 - 3 \cdot 9 - 1 \cdot 2)X_2 \\ & + (4.5 - 4)X_3 + (3.2 - 3)X_4\end{aligned}$$

Constraints:

$$\begin{aligned}&X_3 + 4X_1 + 6X_2 \le 10000\\&X_4 + 3X_1 + 9X_2 \le 14000\\&3X_1 + 2X_2 \le 7000\\&X_1, X_2, X_3, X_4 \ge 0 \text{ and integers}\end{aligned}$$

(b) Assume that a commercial constraint exists whereby one, and only one, of products A and B and either component C or D can be sold. Amend the original model in Section (a) to bear in mind this consideration.

A new binary variable is added:
$Y_1 = 1$ if product A is sold and 0 if product B is sold;
The following constraints are added:

$$\begin{aligned}&X_1 \le (7000/3)Y_1\\&X_2 \le (7000/2) \cdot (1 - Y_1)\end{aligned}$$

(c) Assume that only 3 of the 4 products available are expected to be sold. Add the appropriate constraints to the original model to contemplate this new commercial constraint.

Four new binary variables are added:

$Z_1 = 1$ if product A is sold, 0 otherwise;
$Z_2 = 1$ if product B is sold, 0 otherwise;
$Z_3 = 1$ if product C is sold, 0 otherwise;
$Z_4 = 1$ if product D is sold, 0 otherwise;

The following constraints are added:

$$\begin{aligned}
X_1 &\leq (7000/3)Z_1 \\
X_2 &\leq (7000/2)Z_2 \\
X_3 &\leq 10000Z_3 \\
X_4 &\leq 14000Z_4 \\
Z_1 + Z_2 + Z_3 + Z_4 &= 3
\end{aligned}$$

(d) Amend the original problem to consider a fixed cost of \$6,000 for product A and of \$4,000 for product B which is incurred when selling one product unit. These costs are not incurred if a decision is made to not sell these products.

Four new binary variables are added:
Binary variables Z_1 and Z_2, from the previous section, are employed.
The objective function is amended as so:

$$\text{Maximise}\, z = 41X_1 + 4X_2 + 0.5X_3 + 2X_4 - 6000Z_1 - 4000Z_2$$

The following restrictions, as indicated in the former section, are added:

$$\begin{aligned}
X_1 &\leq (7000/3)Z_1 \\
X_2 &\leq (7000/2)Z_2
\end{aligned} .]] >$$

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Chapter 3
Non-Linear Programming

Abstract This chapter begins by introducing non-linear programming. Next, it proposes the formulation of a series of non-linear programming problems with their corresponding solutions. Specifically, multi-modal and multi-variable problems with inequality constraints are modelled. The solution is done by applying the Kuhn-Tucker conditions. It sets out different non-linear programming problems with their solutions in relation to Industrial Organisation Engineering and the management setting.

3.1 Introduction

A non-linear programming problem occurs when decision variables are continuous and objective function, or any of the constraints, is not linear. In such problems, the feasible solution region is not convex and the solution may be found from within. Besides, the direction that the objective function shifts in cannot always take increasing or decreasing values.

In the solution for non-linear programming models, the fundamental results originate from the development of mathematical calculation dating back to the eighteenth century, and the basic concept is Lagrangian. The characterisation of the optimality conditions required in restricted problems is generalised from the Lagrange results in the well-known Kuhn and Tucker Theorem (1951), which compiles and structures a set of investigations carried out by many authors in the 1940s, among which John (1948) is cited. Non-linear programming progressed in the 1960s and 1970s, and it was possible to tackle medium-sized problems with several tens of constraints and a few hundred variables. Yet research into the search for efficient algorithms actively continued because those that existed were not altogether satisfactory. Regarding interior algorithms, Karmarkar (1984) stands out. These algorithms are an alternative to the traditional, well-known Simplex; notwithstanding, it is quite proper to frame it in the non-linear optimisation line where it may be considered a particular case of penalty algorithms.

R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5_3,

If any bibliographic references on such algorithms were cited, the works of Nemhauser and Wolsey (1988) and Parker and Radin (1988), among others, would stand out.

In this chapter, the Kuhn-Tucker conditions (1951) are employed as a resolution algorithm for non-linear programming problems. The minimisation problem is provided canonically:

$$\begin{aligned} &\min f(x) \\ &g_i(x) \le 0;\ i = 1, \ldots, m \end{aligned}$$

and the Lagrangian function as follows:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

The Kuhn-Tucker conditions are: gradient condition, feasibility condition, orthogonality condition and non-negativity condition.

$$\begin{aligned} &\frac{\partial L}{\partial x_j} = 0;\ j = 1, \ldots, n \\ &\frac{\partial L}{\partial \lambda_i} = g_i(x) \le 0;\ i = 1, \ldots, m \\ &\lambda_i g_i(x) = 0;\ i = 1, \ldots, m \\ &\lambda_i \ge 0;\ i = 1, \ldots, m \end{aligned}$$

The point (x^*, λ^*) that verifies them is known as the Kuhn-Tucker point.

If in the previous canonical minimisation problem f and g_i are functions with partial continuous first-order derivatives and x^* is a regular point, which is the local minimum for the given problem, then vector $\lambda \in \Re v$ exists, which (x^*, λ^*) fulfils the Kuhn-Tucker conditions.

In the previous canonical minimisation problem, (x^*, λ^*) is a Kuhn-Tucker point and f and g_i are convex functions, then x^* is the global minimum. In a maximisation case, if x^* is a local maximum, f is concave and g_i is convex; then x^* is an overall maximum.

The applied solution method is as follows:

Step 0. Verify that f and g_i have partial continuous first-order derivatives.
Step 1. Construct the Lagrangian function.
Step 2. Form the system of algebraic equations and in equations (Kuhn-Tucker conditions). Establish the solutions (x^*, λ^*) of the system (Kuhn-Tucker points).
Step 3. If f is convex (concave) and g_i are convex, then points x^* are global minimums (maximums). In another case, each solution must be examined individually.

With Lagrange multipliers, the value in the Lagrange multiplier optimum associated with the k-th constraint is equal to the value, in the optimum, of the partial derivative of the objective function in relation to the second member of this constraint; that is to say, it is equal to the modification that the objective function undergoes in the optimum when the second member of the corresponding constraint is slightly amended. It is for this very reason that Lagrange multipliers receive the name of calculation prices, or shadow prices, as they determine the consequences of marginally modifying a constraint and, accordingly, of evaluating the interest of an operation consisting in allocating resources to efficiently shift constraints.

The objective of this book chapter is to help learn the formulation of non-linear programming models and of presenting some of their applications in the industrial engineering and management domain.

After reading this chapter, the reader should be capable of formulating different prototype non-linear programming problems, and of modelling multi-variant and multi-model functions with inequality constraints by the Kuhn-Tucker conditions.

Selected books for further reading can be found in the References section.

3.2 Selecting Investments

The manager of the firm TRONIC must decide how to distribute an investment of 100 million dollars for the following year among four concepts: R&D, Advertising, Technological Resources and Human Resources.

For all four concepts, his consultant has established that the profit after 1 year of investing X million dollars is obtained according to the formulation below (see Table 3.1).

In accordance with the collective bargaining agreement signed 1 month ago, the minimum amount to be invested in Human Resources (promotions, bonuses, etc.) over the next year is 20 million dollars.

When analysing the formulae, it may be deduced that, although important investments in Technological Resources offer good investment revenue, when investments are small or moderate, the investment revenue offered by R&D or advertising is significantly higher. However, due to pressure from competitors, the investment made in Technological Resources should be such that the contribution of this concept to the investment revenue must not be negative.

Table 3.1 Profits to be obtained according to the amounts invested

R&D	$10 \cdot \left(\sqrt[3]{2 \cdot X} - 2\right)$
Advertising	$10 \cdot (Ln(X) - 2)$
Technological resources	$10 \cdot \left(e^{X/50} - 2\right)$
Human resources	$10 \cdot \frac{X}{50}$

(a) Model the problem
(b) Consider the Kuhn-Tucker conditions for this problem
(c) Solve the problem
(d) Interpret the result.

Solution

(a) Model the problem

Decision variables:

X_i: million monetary units invested in R&D, Advertising, Technological Resources and Human Resources, respectively, where $i = 1, 2, 3, 4$.

Objective function:

$$\text{Max}\, f = 10\left[\sqrt[3]{2 \cdot X_1} - 2 + \text{Ln}(X_2) - 2 + e^{X_3/50} - 2 + \frac{X_4}{50}\right]$$

Constraints:

$$\begin{aligned} & X_1 + X_2 + X_3 + X_4 - 100 \leq 0 \\ & 20 - X_4 \leq 0 \\ & -10 \cdot \left(e^{X_3/50} - 2\right) \leq 0 \\ & X_i \geq 0 \end{aligned}$$

(b) Consider the Kuhn-Tucker conditions for this problem

Lagrangian function:

$$L = f + \lambda_1 \cdot g_1 + \lambda_2 \cdot g_2 + \lambda_3 \cdot g_3$$

The Kuhn-Tucker conditions:

$$\begin{aligned} & \frac{\partial L}{\partial X_1} = 0 \Rightarrow 10 \cdot \frac{\sqrt[3]{2}}{3} \cdot X_1^{-2/3} + \lambda_1 = 0 \\ & \frac{\partial L}{\partial X_2} = 0 \Rightarrow \frac{10}{X_2} + \lambda_1 = 0 = 0 \\ & \frac{\partial L}{\partial X_3} = 0 \Rightarrow 10 \cdot e^{X_3/50} \cdot \frac{1}{50} + \lambda_1 - \lambda_3 \cdot e^{X_3/50} \cdot \frac{10}{50} = 0 \\ & \frac{\partial L}{\partial X_4} = 0 \Rightarrow \frac{10}{50} + \lambda_1 - \lambda_2 = 0 \\ & \lambda_1 \cdot (X_1 + X_2 + X_3 + X_4 - 100) = 0 \\ & \lambda_2 \cdot (20 - X_4) = 0 \\ & \lambda_3 \cdot \left(-10 \cdot \left(e^{X_3/50} - 2\right)\right) = 0 \\ & \lambda_i \leq 0 \end{aligned}$$

(c) Solve the problem

By assuming that the Human Resources condition is saturated, we obtain $X_4 = 20$.

By assuming that the Technological Resources condition is saturated, we obtain $X_3 = 34.65$.

With these values and based on the described equations, we obtain:

$$\begin{aligned} X_1 &= 25 \\ X_2 &= 20.35 \\ \lambda_1 &= -0.49 \\ \lambda_2 &= -0.29 \\ \lambda_3 &= -0.228 \\ F &= 30.968 \end{aligned}$$

To check whether the solution found is an optimum solution, we obtain the Hessian Matrix of f in the solution:

Second derivative:

$$\begin{aligned} \frac{\partial f}{\partial X_1^2} &= 10 \cdot \frac{\sqrt[3]{2}}{3} \cdot \frac{-2}{3} X_1^{-5/3} \\ \frac{\partial f}{\partial X_2^2} &= -\frac{10}{X_2^2} \\ \frac{\partial f}{\partial X_3^2} &= 10 \cdot e^{X_3/50} \cdot \frac{1}{50} \cdot \frac{1}{50} - \lambda_3 \cdot e^{X_3/50} \cdot \frac{10}{50} \cdot \frac{1}{50} \\ \frac{\partial f}{\partial X_4^2} &= 0 \end{aligned}$$

All the other second derivatives are null.

The Hessian Matrix is:

$$\begin{bmatrix} - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is neither positive semi-defined nor negative semi-defined; in other words, the objective function is neither concave nor convex; thus it is not a global maximum.

(d) Interpret the result

- Having calculated the investment mix, a profit of 31 million monetary units is obtained.
- λ_1: The marginal benefit of each monetary unit invested over the 100 million is 49 % (if an additional 1 million were invested, an additional profit of 0.49 million would be obtained).

- λ_2: If the Hum. Res. constraint was lowered to 19 million, the profit would increase by 0.29 million.
- λ_3: If a loss in the profit produced by Tech. Res. of 1 dollar was permitted, the profit would increase by 0.228 dollars (until the limit of 1 million is reached).

3.3 Location of a Stationery Store

A future businessperson wishes to locate the spot to place a stationery store in the vicinity of a primary school and a secondary school. On a map of coordinates, both schools are situated at points A (20, 0) and B (0, 20). At point (0, 0), there is a store with similar characteristics and, according to municipal laws, the positioning of the new stationery store would have to be at a distance of at least 400 m from the already existing one.

Determine the point at which the new stationery store must be located to minimise the sum of the distances to the square of the two schools.

Solution

Decision variables:

X coordinate on the x-axis of the stationary store location
Y coordinate on the y-axis of the stationary store location

Objective function:

$$\begin{aligned} \text{Min f} &= (X-0)^2 + (Y-20)^2 + (X-20)^2 + (Y-0)^2 \\ &= 2X^2 + 2Y^2 - 40X - 40Y + 800 \end{aligned}$$

Constraints:

$$X^2 + Y^2 \geq 400^2 \rightarrow g: -X^2 - Y^2 + 400^2 \leq 0$$

Solve this by applying the Kuhn-Tucker conditions:

Step 0:

Verify that f and g have partial continuous first-order derivatives. Functions f and g are second-degree polynomials with which the conditions are fulfilled.

Step 1:

Construct the Lagrangian function:

$$\text{L}\ (x, \lambda) = 2X^2 + 2Y^2 - 40X - 40Y + 800 + \lambda\left(400^2 - X^2 - Y^2\right)$$

Step 2:

Form the system of algebraic equations and inequations (Kuhn-Tucker conditions)

$$\partial L/\partial x = 4X - 40 - 2\,\lambda\,X = 0$$
$$\partial L/\partial y = 4Y - 40 - 2\lambda\;Y = 0$$
$$\partial L/\partial \lambda\; = 400^2 - X^2 - Y^2 \leq 0$$
$$\lambda\left(400^2 - X^2 - Y^2\right) = 0$$
$$\lambda \geq 0$$

If $\lambda = 0$, then X = Y = 10, and constraint $400^2 - X^2 - Y^2 \leq 0$ is not fulfilled. If $\lambda \neq 0$, then the sought point is (282.84, 282.84) and $\lambda = 1.93$.

3.4 Placing a Handrail

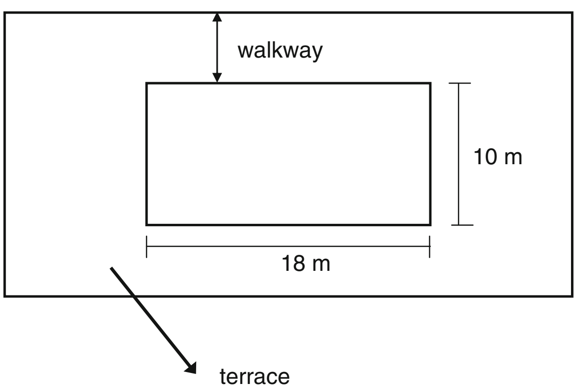
The intention is to place a handrail on a terrace with an empty space in its centre in such a way that the handrail goes around the interior and exterior terrace perimeters. This empty space, which is rectangular in shape, measures 10 m wide by 18 m long (see Fig. 3.1). Between the interior and the exterior terrace perimeters, a walkway must be left, which is the same width on all sides. If the handrail measures 250 m:

(a) Formulate a non-linear programming model that maximises the area occupied by the walkway to improve circulation in it
(b) Apply the Kuhn-Tucker conditions to the model in the previous section and indicate the metres that the outer terrace sides and the walkway width measure
(c) This problem could have been formulated by linear programming and by maximising the walkway width in this case, but how?

Solution

(a) Formulate a non-linear programming model that maximises the area occupied by the walkway to improve circulation in it

Fig. 3.1 Sketch of the terrace



Decision variables:

X_1 metres that the outer terrace side measures in parallel with the 10-m side of the empty space.

X_2 metres that the outer terrace side measures in parallel with the 18-m side of the empty space.

Objective function: (maximise the walkway area)

Max $f = X_1 X_2 - 180$ (rectangular outer area−empty space)

Constraints:
$(X_1 - 10)/2 = (X_2 - 18)/2$ (The walkway width is equal on all sides)
$2 (X_1 + X_2) + 56 \leq 250$ (Only a 250-m handrail is available)
$X_1, X_2 \geq 0$

(b) Apply the Kuhn-Tucker conditions to the model in the previous section and indicate the metres that the outer terrace sides and the walkway width measure

The canonical form of the problem:

$$\begin{aligned}
&\text{Max} f = X_1 X_2 - 180 \\
&0.5X_1 - 0.5X_2 + 4 \leq 0 \\
&-0.5X_1 + 0.5X_2 - 4 \leq 0 \\
&2X_1 + 2X_2 - 194 \leq 0
\end{aligned}$$

The Lagrangian function:

$$\begin{aligned}
L = X_1 \cdot X_2 - 180 &+ \lambda_1(0.5X_1 - 0.5X_2 + 4) + \lambda_2(-0.5X_1 + 0.5X_2 - 4) \\
&+ \lambda_3(2X_1 + 2X_2 - 194)
\end{aligned}$$

Kuhn-Tucker conditions:

Gradient condition:

$$\begin{aligned}
\partial L/\partial X_1 &= X_2 + 0.5\lambda_1 - 0.5\lambda_2 + 2\lambda_3 = 0 \\
\partial L/\partial X_2 &= X_1 - 0.5\lambda_1 + 0.5\lambda_2 + 2\lambda_3 = 0
\end{aligned}$$

Feasibility condition:

$$\begin{gathered}
5x_1 - 0.5x_2 + 4 \leq 0 \\
-0.5X_1 + 0.5X_2 - 4 \leq 0 \\
2x_1 + 2x_2 - 194 \leq 0
\end{gathered}$$

Orthogonality condition:

$$\lambda_1(0.5X_1 - 0.5X_2 + 4) = 0$$
$$\lambda_2(-0.5X_1 + 0.5X_2 - 4) = 0$$
$$\lambda_3(2X_1 + 2X_2 - 194) = 0$$

Non-positivity condition

$$\lambda_1,\ \lambda_2,\ \lambda_3 \leq 0$$

Solution:

If $\lambda_1 = \lambda_2 = \lambda_3 = 0 \rightarrow X_1 = X_2 = 0$ (not possible)
If $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0 \rightarrow X_1 = X_2$ (not possible)
If $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 \neq 0 \rightarrow X_1 = -X_2$ (not possible)
If $\lambda_1 = \lambda_2 \neq 0$ and $\lambda_3 = 0 \rightarrow X_1 = -X_2$ (not possible)
If $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$:
$(0.5X_1 - 0.5X_2 + 4) = 0$
$(-0.5X_1 + 0.5X_2 - 4) = 0$
$(2X_1 + 2X_2 - 194) = 0.$

There are three equations with two unknown quantities, where $X_1 = 44.5$, $X_2 = 52.5$ and the walkway width is 17.25 m.
The Hessian Matrix of $f(x1, x2) = (0, 1, 1, 0)$. Thus f is semi-defined negative, the function is concave and, therefore, the values obtained are a global maximum of f.

(c) This problem could have been formulated by linear programming and by maximising the walkway width in this case, but how?

Decision variables:

X = metres of walkway width.
X_1 metres that the outer terrace side measures in parallel with the 10-m side of the empty space.
X_2 metres that the outer terrace side measures in parallel with the 18-m side of the empty space.

Objective function:

Max $f = X$

Constraints:

$X = (X_1 - 10)/2$ (walkway width equal on all sides)
$X = (X_2 - 18)/2$ (walkway width equal on all sides)
$2\ [(2X + 18) + (2X + 10)] + 56 \leq 250$ (outer perimeter + inner perimeter smaller than or equal to 250 m)
$X, X_1, X_2 \geq 0.$

3.5 Planning the Construction of Homes

On a housing estate on the Alicante coastline, two types of homes are being built: apartments and penthouse, whose prices are p_1 and p_2, respectively. The curve of demand for apartments is $d_1 = 100 - 2p_1$ and is $d_2 = 150 - 3p_2$ for penthouses. The builder in charge, who has already sold 60 apartments, which lowers market demand, d_1, wishes to completely adjust demand so that no homes are left unsold. He has also calculated that, owing to the orders that he has already sent to his raw materials suppliers, it would be worth him building 15 times more apartments than penthouses. He has also calculated that building an apartment costs him 5 million dollars, while a penthouse costs him 3 million dollars. Knowing that the builder has a budget of 350 million dollars for homes still to be sold, work out the following:

(a) Calculate how many homes of each type must be built, and what prices must be set for the builder to maximise his profit
(b) Would it be more convenient for him to increase the available budget?

Solution

(a) Calculate how many homes of each type must be built, and what prices must be set for the builder to maximise his profit

Objective function:

$$\begin{aligned} \text{Max } z &= (p_1 - 5)(100 - 60 - 2p_1) + (p_2 - 3)(150 - 3p_2) \\ &= -2p_1^2 - 3p_2^2 + 50p_1 + 159p_2 - 650 \end{aligned}$$

Constraints:

$$5(40 - 2p_1) + 3(150 - 3p_2) \le 350 \to -10p_1 - 9p_2 + 300 \le 0 \quad \text{(budget constraint)}$$
$$15(150 - 3p_2) \le (40 - 2p_1) \to -2p_1 - 45p_2 + 2210 \le 0 \quad \text{(market constraint)}$$
$$p_1, p_2 \ge 0$$

The Kuhn-Tucker conditions
Lagrangian function

$$\begin{aligned} L(\lambda) = {} & -2p_1^2 - 3p_2^2 + 50p_1 + 159p_2 - 650 + \lambda_1(-10p_1 - 9p_2 + 300) \\ & + \lambda_2(-2p_1 - 45p_2 + 2210) \end{aligned}$$

Gradient condition:

$$\frac{\partial L}{\partial p_1} = -4p_1 + 50 - 10\lambda_1 - 2\lambda_2 = 0$$
$$\frac{\partial L}{\partial p_2} = -6p_2 + 159 - 9\lambda_1 - 45\lambda_2 = 0$$

Orthogonality condition:

$$\lambda_1(-10p_1 - 9p_2 + 300) = 0$$
$$\lambda_2(-2p_1 - 45p_2 + 2210) = 0$$
$$\lambda_1, \lambda_2 \leq 0 \text{ (for max.non-positivity)}$$

By taking $\lambda_1 = 0$ and $\lambda_2 \neq 0$, the following results are obtained:

$$p1 = 10.96$$
$$p2 = 49.60$$
$$\lambda_1 = 0$$
$$\lambda_2 = 3.08$$

From the demand functions, it is discovered that 78 apartments and 1.2 penthouses should be built.

(b) Would it be more convenient for him to increase the available budget?

No because, as the Lagrange multiplier, $\lambda_1 = 0$, indicates for the budget constraint, the object function would not increase at all for every one million dollars that the budget is increased by.

3.6 Production Planning in a Drinks Firm

A firm that packs refreshments and beers, situated in the province of Valencia (E Spain) employs the same syrup to produce its 1.5 l COLI and PEPSA products on its S1 production line. Once processed, each hectolitre of syrup produces 40 units of the 1.5 l COLI product and 20 units of the 1.5 l PEPSA product. If X_1 units of the 1.5 l COLI product and X_2 units of the 1.5 l PEPSA product are produced, the firm estimates that the daily income obtained in dollars would be given by the following function:

$$f(X_1, X_2) = 49000X_1 - X_1^2 + 30X_2 - 2X_2^2$$

It costs 150 dollars to buy and process each hectolitre of syrup. The S1 packaging line has a net capacity of producing 7,100 1.5 l product units every hour. The firm works 5 days a week in 8-h shifts.

Given its weekly target coverage, the firm is committed to produce at least half the amount of PEPSA than COLI.

Although priority orders tend to amend its production planning, the firm wishes to have a basic product planning that optimises its daily profits.

(a) Formulate a non-linear programming model that helps the firm create its basic daily production plan for its S1 packaging line.

(b) Use the Kuhn-Tucker conditions to determine how the firm can maximise its profits.
(c) What is the maximum quantity that the firm would be willing to pay for 1 h of overtime production? How much would the daily profit increase per additional COLI unit produced?

Solution

(a) Formulate a non-linear programming model that helps the firm create its basic daily production plan for its S1 packaging line.

Decision variables:

X_1 = Units of the 1.5 l COLI product to be produced
X_2 = Units of the 1.5 l PEPSA product to be produced

Objective Function:

$$\text{Max } z = 49000X_1 - X_1^2 + 30X_2 - 2X_2^2 - 150(40X_1 + 20X_2)$$

Constraints:

$$\begin{aligned} &X_1 + X_2 \leq 56800 \\ &X_1 \leq 2X_2 \\ &X_1, X_2 \geq 0 \end{aligned}$$

(b) Use the Kuhn-Tucker conditions to determine how the firm can maximise its profits.

Lagrangian Function:

$$\begin{aligned} L = {} & 49000X_1 - X_1^2 + 30X_2 - 2X_2^2 - 6000X_1 - 3000X_2 + \lambda_1(X_1 + X_2 - 56800) \\ & + \lambda_2(X_1 - 2X_2) \end{aligned}$$

Gradient Condition:

$$\begin{aligned} \frac{\partial L}{\partial X_1} &= 43000 - 2X_1 + \lambda_1 + \lambda_2 = 0; \\ \frac{\partial L}{\partial X_2} &= -2970 - 4X_2 + \lambda_1 - 2\lambda_2 = 0; \end{aligned}$$

Feasibility Condition:

$$\begin{aligned} &X_1 + X_2 - 56800 \leq 0; \\ &X_1 - 2X_2 \leq 0; \end{aligned}$$

Orthogonality Condition:

$$\lambda_1(X_1 + X_2 - 56800) = 0;$$
$$\lambda_2(X_1 - 2X_2) = 0;$$

Non-positivity Condition:

$$\lambda_1, \lambda_2 \leq 0;$$

To solve this, we do $\lambda_1 = \lambda_2 = 0$; this is not possible because X_1 and X_2 would take negative values.

For $\lambda_1 = 0, \lambda_2 \neq 0$;

$$X_1 = 13838.33$$
$$X_2 = 6919.16$$
$$Z = 287249204.2$$
$$\lambda_2 = 15323.34$$

The Hessian Matrix: $H = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$, the lower fundamental values are -2 and 8. Therefore, the matrix is semi-defined negative (concave). The point found (X_1, X_2) is a global maximum.

(c) What is the maximum quantity that the firm would be willing to pay for 1 h of overtime production? How much would the daily profit increase per additional COLI unit produced?

The answer is nothing because the shadow price, or Lagrange multiplier, for the production capacity constraint is zero, which indicates that not all the available resources are being put to the best use. Besides, for each additional COLI unit produced, the daily profit would increase by 15323.34\$ (λ_2).

3.7 Production Planning in a Firm in the Automobile Sector

The Operations Department of the FORDASA automobile manufacturing and assembly firm must establish the daily production quantities of two families of cars (four-door saloon and people carrier) which maximise profits according to the production capacity constraints. The firm also wishes to respect the environment, and in the future to benefit its sales, given its commitment to increase the *FFM* (*fleet fuel mileage*) indicator; that is, the mileage efficiency per litre of fuel of produced fleet.

To simplify this study case, a linear relation between the sale price and the quantity to be produced is assumed, given by the following function: $q_i = a_i - b_i p_i$, where a_i and b_i are constants, and are provided in Table 3.2, and p_i

Table 3.2 The problem data

i	Family of cars	a	b	C (thousands of dollars)	R (s)	Kmpl
1	4-door saloons	440	20	17	54	12
2	People carriers	248	12	13	56	20

is the mean unit price per family of cars in thousands of dollars. Parameter C_i is also provided, which represents the cost per unit for the cars of family i. The only productive resource that restricts the system is the assembly line. It is estimated that the firm spends 1 h every day on assembling four-door saloons and people carriers. The time required (R_i) to produce one unit of each family of cars and the remaining former data are shown in Table 3.2.

The *FFM* indicator, which must at least be below 15 Kmpl, is calculated by using the harmonic mean: $\text{FFM} = \dfrac{q_1 + q_2}{\dfrac{q_1}{\text{kmpl}_1} + \dfrac{q_2}{\text{kmpl}_2}}$

(a) Formulate a non-linear programming model to help the Operations Department establish an optimum production plan which meets the existing constraints
(b) Find the solution using the Kuhn-Tucker conditions
(c) Interpret the result

Solution

(a) Formulate a non-linear programming model to help the Operations Department establish an optimum production plan which meets the existing constraints

Decision Variables:

q_i = Weekly quantity to be produced of each family of cars ($i = 1,\ldots, 2$).
p_i = Mean price per family of cars ($i = 1,\ldots, 2$).

Objective Function:

$$\text{Max } z = \sum_i q_i \times (p_i - C_i) \to z = q_1\left(\frac{q_1 - 440}{-20} - 17\right) + q_2\left(\frac{q_2 - 248}{-12} - 13\right) \to$$
$$z = -0.05q_1^2 - 0.083q_2^2 + 5q_1 + 7.6q_2$$

Constraints:

$$\sum_i R_i q_i < = 3600 \to 54q_1 + 56q_2 \leq 3600;$$

(production capacity on the assembly line in seconds)

$$\frac{q_1 + q_2}{\frac{q_1}{12} + \frac{q_2}{20}} \geq 15\text{Kmpl} \quad (\text{FFMindicator})$$
$$p_i, q_i \geq 0 \quad (\text{non-negativity constraint})$$

(b) Find the solution using the Kuhn-Tucker conditions

Canonical form:

$$z = -0.05q_1^2 - 0.083q_2^2 + 5q_1 + 7.6q_2$$
$$54q_1 + 56q_2 - 3600 \leq 0$$
$$0.25q_1 - 0.25q_2 \leq 0$$

The Kuhn-Tucker conditions:
Lagrangian Function:

$$L(\lambda) = -0.05q_1^2 - 0.083q_2^2 + 5q_1 + 7.6q_2 + \lambda_1(54q_1 + 56q_2 - 3600) + \lambda_2(0.25q_1 - 0.25q_2)$$

Gradient Condition:

$$\frac{\partial L}{\partial q1} = -\, 0.1q1 + 5 + 54\lambda_1 + 0.25\lambda_2 = 0$$
$$\frac{\partial L}{\partial p2} = -\, 0.166q2 + 7.6 + 56\lambda_1 - 0.25\lambda_2 = 0$$

Orthogonality Condition:

$$\lambda_1(54q1 + 56q2 - 3600) = 0$$
$$\lambda_2(0.25q1 - 0.25q2) = 0$$

Feasibility Condition:

$$54q_1 + 56q_2 - 3600 \leq 0$$
$$0.25q_1 - 0.25q_2 \leq 0$$
$$\lambda_1, \lambda_2 \leq 0 \quad (\text{for the max. non} - \text{positivity})$$

If we take $\lambda_1 \neq 0$ and $\lambda_2 = 0$, the following results are obtained:

$$q_1 = 31.14;$$
$$q_2 = 34.26;$$
$$p_1 = 20.44;$$
$$p_2 = 17.81;$$
$$\lambda_1 = 0.034;$$
$$\lambda_2 = 0;$$

Determine whether the points obtained are a global or a local maximum:

$$H = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.166 \end{bmatrix}$$

If we multiply this matrix by -1, we see that the lower fundamental values are positive. Thus the point found is a global maximum.

(c) Interpret the result.

It is necessary to manufacture 31.14 cars of the four-door saloon family and 34.26 cars of the people carrier family. If the firm had one additional second of production capacity, the functional objective value would increase by 0.034 dollars; in other words, for each additional hour of production capacity, the profits would increase by 2.04 thousand dollars.

3.8 Planning Materials in the Chemical Industry

A certain physical–chemical process requires two liquid components to obtain the end product. The pursued objective is to maximise the quality of the product obtained, measured as the difference between the Napierian logarithm of the product of the litres used of each component and the cube root of the summation of the first component litres as well as double the second component litres. For the obtained set to be stable, it is necessary that the squared difference between the litres employed of each component and the mean litres utilised of both components is below ten. Moreover, only 80 l fit in the mixture tank.

(a) Model the problem by considering that the units to be obtained cannot be an integer number
(b) Consider the Kuhn-Tucker conditions to solve the problem
(c) Solve the problem
(d) Interpret the result.

Solution

(a) Model the problem by considering that the units to be obtained cannot be an integer number

Decision Variables:

X_1 litres of the first component.

X_2 litres of the second component.

Objective Function:

$$\text{Max} f = \text{Ln}(X_1 \cdot X_2) - \sqrt[3]{X_1 + 2 \cdot X_2}$$

Constraints:

$$\left(X_1 - \frac{X_1 + X_2}{2}\right)^2 \leq 10$$
$$\left(X_2 - \frac{X_1 + X_2}{2}\right)^2 \leq 10$$
$$X_1 + X_2 \leq 80$$
$$X_i \geq 0$$

(b) Consider the Kuhn-Tucker conditions to solve the problem

Lagrangian Function:

$$L = \mathrm{Ln}(X_1 \cdot X_2) - \sqrt[3]{X_1 + 2 \cdot X_2} + \lambda_1 \cdot \left(\left(X_1 - \frac{X_1 + X_2}{2}\right)^2 - 10\right) + \lambda_2 \cdot \left(\left(X_2 - \frac{X_1 + X_2}{2}\right)^2 - 10\right) + \lambda_3 \cdot (X_1 + X_2 - 80)$$

The Kuhn-Tucker conditions:

$$\frac{\partial L}{\partial X_1} = 0 \Rightarrow \frac{1}{X_1 \cdot X_2} \cdot X_2 - \frac{1}{3}(X_1 + 2 \cdot X_2)^{-2/3} \cdot 1 + 2 \cdot \lambda_1 \cdot \left(X_1 - \frac{X_1 + X_2}{2}\right) \cdot \left(1 - \frac{1}{2}\right) + 2 \cdot \lambda_2 \cdot \left(X_2 - \frac{X_1 + X_2}{2}\right) \cdot \left(-\frac{1}{2}\right) + \lambda_3$$

$$\frac{\partial L}{\partial X_2} = 0 \Rightarrow \frac{1}{X_1 \cdot X_2} \cdot X_1 - \frac{1}{3}(X_1 + 2 \cdot X_2)^{-2/3} \cdot 2 + 2 \cdot \lambda_1 \cdot \left(X_1 - \frac{X_1 + X_2}{2}\right) \cdot \left(-\frac{1}{2}\right) + 2 \cdot \lambda_2 \cdot \left(X_2 - \frac{X_1 + X_2}{2}\right) \cdot \left(1 - \frac{1}{2}\right) + \lambda_3$$

$$\lambda_1 \cdot \left(\left(X_1 - \frac{X_1 + X_2}{2}\right)^2 - 10\right) = 0$$

$$\lambda_2 \cdot \left(\left(X_2 - \frac{X_1 + X_2}{2}\right)^2 - 10\right) = 0$$

$$\lambda_3 \cdot (X_1 + X_2 - 80) = 0$$

$$\lambda_i \geq 0$$

(c) Solve the problem

By assuming that the full tank capacity is used: $\lambda_3 \neq 0$; $(X_1 + X_2 - 80) = 0$

$$X_1 = 80 - X_2$$
$$\frac{X_1 + X_2}{2} = 40$$

By assuming that the quadratic difference restriction is strictly met:

$$\lambda_1 \neq 0; \left(X_1 - \frac{X_1 + X_2}{2}\right)^2 - 10 = 0$$
$$X_1 = 40 \pm \sqrt{10}$$

As X_1 provides more profit than X_2: $\rightarrow X_1 = 40 + \sqrt{10} = 46.16$ $\rightarrow X_2 = 80 - \left(40 + \sqrt{10}\right) = 36.84$

The objective function is a defined negative function. Thus the obtained point is a maximum.

(d) Interpret the result

The λ_3 value in the solution: 0.000530119. An increase of 1 l in the tank capacity would equal an increase of λ_3 quality units.

The z value in the solution: 1.554.

3.9 Production Planning

A firm is attempting to optimise its production plan by bearing in mind the following profit function:

$$f = \ln(1 + x_1 + x_2)$$

Besides, the following capacity constraint is also considered:

By taking into account that 1 machining hour is required to produce x_1 and 2 machining hours are needed to produce x_2, it is necessary to consider that only a total of 5 machining hours per day is available.

(a) Formulate a non-linear programming model that helps the firm maximise its profit
(b) Use the Kuhn-Tucker conditions to obtain an optimum solution
(c) Interpret the Lagrange indicator value in economic terms.

Solution

(a) Formulate a non-linear programming model that helps the firm maximise its profit

Objective Function:

$$\text{Max } f = \ln(1 + x_1 + x_2)$$

Constraints:

$$x_1 + 2x_2 \leq 5$$

(b) Use the Kuhn-Tucker conditions to obtain an optimum solution

Objective Function

$$L = \ln(1 + x_1 + x_2) + \lambda(x_1 + 2x_2 - 5)$$

Gradient condition

$$\frac{\partial L}{\partial x_1} = \frac{1}{1 + x_1 + x_2} + \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = \frac{1}{1 + x_1 + x_2} + 2\lambda = 0$$

Feasibility condition: $x_1 + 2x_2 - 5 \leq 0$
Orthogonality condition: $\lambda(x_1 + 2x_2 - 5) = 0$
Non-positivity condition: $\lambda \leq 0$
By taking $\lambda \neq 0$, we obtain:

$$x_1 = 5; x_2 = 0; \lambda = -0.16.$$

(c) Interpret the Lagrange indicator value in economic terms

For each available daily machining hour that is achieved, the objective function increases by 0.16.

3.10 Maximising Utility

The consumer utility function is $U(x, y) = xy$, where x and y are the quantities consumed of products A and B, whose unit prices are 2 and 3 dollars, respectively. The intention is to:

(a) Maximise this consumer utility by bearing in mind that no more than 90 dollars can be spent on acquiring these products
(b) Analyse the variation in the maximum utility if the consumer can spend an extra dollar on acquiring these products.

Solution

(a) Maximise this consumer utility by bearing in mind that no more than 90 dollars can be spent on acquiring these products

Objective Function:
Maximise $z = xy$

Constraints:
$2x + 3y \leq 90$
$x, y \geq 0$

The Lagrangian function of the problem is:

$L(x, y, \lambda) = xy + \lambda(2x + 3y - 90)$

The Kuhn-Tucker conditions are:

$\partial L/\partial x = y + 2\lambda = 0$
$\partial L/\partial y = x + 3\lambda = 0$
$\partial L/\partial \lambda = 2x + 3y - 90 \leq 0$

(b) Analyse the variation in the maximum utility if the consumer can spend an extra dollar on acquiring these products

Critical points

By analysing the possible λ values, we obtain:

• If $\lambda = 0$, then when it is substituted in the first two equations, we obtain, $x = y = 0$. The point (0, 0) with $\lambda = 0$ verifies the Kuhn-Tucker conditions. Therefore it is a critical point.

• If $\lambda \neq 0$, then when it is substituted in the Kuhn-Tucker conditions we obtain, $y = -2\lambda$, $x = -3\lambda$, $2x + 3y - 90 = 2(-3\lambda) + 3(-2\lambda) - 90 = 0$.

Hence, $\lambda = -7.5$; $x = 22.5$; $y = 15$. The point (452, 15) = (22.5, 15) with $\lambda = -7.5$ verifies the Kuhn-Tucker conditions. Therefore it is a critical point.

Sufficient conditions:

Two maximum critical points have been obtained: (0, 0) with $\lambda = 0$, and (22.5, 15) with $\lambda = -7.5$.

When we substitute both points in function $f(x, y) = xy$, we obtain $f(0, 0) = 0 < f(22.5, 15) = 337.5$. Therefore, in point (22.5, 15) with $\lambda = -7.5$, the global maximum is achieved. To verify it, the Hessian Matrix is obtained and it is established that the objective function is convex.

If the customer had another monetary unit, the utility profit would increase by 7.5 monetary units.

3.11 Designing Boxes

A firm must design a box of minimum dimensions to pack three circular objects with the following radii: $R1 = 6$, $R2 = 12$, and $R3 = 16$ cm, as shown in Fig. 3.2.

By bearing in mind that the circles placed inside the box cannot overlap, consider a non-linear programming model that minimises the perimeter of the box.

Solution

Decision Variables:

A = cm of the side of box A
B = cm of the side of box B

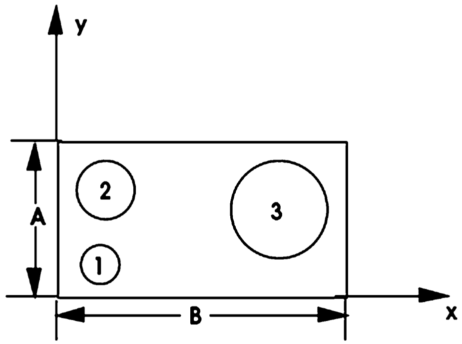


Fig. 3.2 Dimensions of the box and the objects

X_1 = coordinate x_1 of the centre of circle 1
Y_1 = coordinate y_1 of the centre of circle 1
X_2 = coordinate x_2 of the centre of circle 2
Y_2 = coordinate y_2 of the centre of circle 2
X_3 = coordinate x_3 of the centre of circle 3
Y_3 = coordinate y_3 of the centre of circle 3

Objective Function:

Minimise $z = 2\,(A + B)$ (minimise the perimeter)

Constraints:

The circles remain in the box:

$X_1, Y_1 \geq 6$
$X_2, Y_2 \geq 12$
$X_3, Y_3 \geq 16$
$X_1 \leq B - 6$
$Y_1 \leq A - 6$
$X_2 \leq B - 12$
$Y_2 \leq A - 12$
$X_3 \leq B - 16$
$Y_3 \leq A - 16.$

Overlaps between circles are not allowed:

$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 \geq (6 + 12)^2 = 324$
$(X_1 - X_3)^2 + (Y_1 - Y_3)^2 \geq (6 + 16)^2 = 484$
$(X_2 - X_3)^2 + (Y_2 - Y_3)^2 \geq (12 + 16)^2 = 784$

Non-negativity:

$X_1, Y_1, X_2, Y_2, X_3, Y_3, A, B \geq 0$

3.12 Designing and Planning a Project to Launch a Scientific Balloon

A research centre has to design and launch a balloon with an X-ray telescope and other scientific equipment. An approximate measure of performance can be expressed in terms of the height that the balloon reaches and the weight of the equipment raised. Evidently, height itself is given by the volume of the balloon. Based on past experience, it was concluded that to maximise a satisfactory function for performance, then:

$$P = f(V, W) = 100V - 0.3V^2 + 80W - 0.2W^2$$

where V is the volume of the balloon and W is the weight of the equipment. The project budget is limited to 1,040 dollars. The cost associated with volume V is 2 V, and the cost of the equipment is 4 W. To obtain a reasonable balance between performance owing to the height and weight of the scientific equipment, the design must fulfil constraint 80 $W \geq$ 100 V.

Obtain an optimum design in terms of the volume and weight of the equipment, and solve it using the Kuhn-Tucker algorithm.

Solution

Decision Variables:

V = volume of the balloon;
W = weight of the equipment;

Objective Function:

$$\text{Maximise} = f(V, W) = 100V - 0.3V^2 + 80W - 0.2W^2$$

Constraints:

$2V + 4W \leq 1040 \rightarrow 2V + 4W - 1040 \leq 0$
$80W \geq 100V \rightarrow 100V - 80W \leq 0$

Non-negativity:

$V, W \geq 0$

Lagrangian Function:

$$L(V, W, \lambda 1, \lambda 2) = 100V - 0.3V^2 + 80W - 0.2W^2 + \lambda 1(2\text{V} + 4\text{W} - 1040) + \lambda 2(100\text{V} - 80W)$$

Gradient condition:

$\partial \text{L}/\partial V = 100 - 0.6V + 2\lambda_1 + 100\lambda_2 = 0$
$\partial \text{L}/\partial W = 80 - 0.4W + 4\lambda_1 - 80\lambda_2 = 0$

Orthogonality condition:

$\boldsymbol{\lambda}1(2V + 4W - 1040) = 0\boldsymbol{\lambda}2\ (100V{-}80W) = 0$

Feasibility condition:

$\partial L/\partial\lambda_1 = 2V + 4W{-}1040 \leq 0$
$\partial L/\partial\lambda_2 = 100V - 80W \leq 0$

Non-positivity condition:

$$\boldsymbol{\lambda}1, \boldsymbol{\lambda}2 \leq 0$$

By analysing the possible values of λ_1 and λ_2, we obtain:

V = 148.57;
W = 185.71;
$\lambda_1 = -2.57$;
$\lambda_2 = -0.06$;

Based on the Hessian Matrix of *f*, H = [−0.6, 0, 0, −0.4], it is possible to conclude that the objective function is concave. Therefore the critical point obtained is a global maximum.

3.13 Production Planning in an Oil-Packing Firm

An oil-packing firm contemplates the problem of determining how many units of three product types it must pack in a given month by considering various constraints. The products in question are provided in Table 3.3.

"Production cost" includes packing, labelling, labour, power and others.

"Market" refers to the minimum market share that must be covered.

"Rappel" refers to the linear reduction in the sale price by 1 cent for every X litres produced of each product.

The cost to store oil is 2 cents per litre and month, which is increased by "Y" cents multiplied by the percentage of litres of the article in question in relation to all the litres stored.

Table 3.3 The problem data

Product	Packing (l)	Cost of oil (cent/l)	Production cost (cent/unit)	Market (l)	Sale price (cent/l)	Rappel (l)	Y
Virgin olive oil	Glass 1	345	75	175000	445	350000	8
Olive oil, 1°	PVC	270	15	900000	303	750000	4
Olive oil, 0.4°	PVC	280	15	750000	303	750000	4

Packaging speed depends on the packaging material: for the products packed with PVC, the speed reached is 100,000 units/day, while the speed is 35,000 units/day for those packed in glass.

The month in question includes 20 working days, 8-h shifts are worked and there are five employees.

(a) Obtain the non-linear programming model of this problem
(b) Consider the Kuhn-Tucker conditions for this problem
(c) Obtain the value of the decision variables, of the objective function and of the Lagrange multipliers. Interpret the result.

Solution

(a) Obtain the non-linear programming model of this problem

Decision Variables:
X_i: units to be packed of product type i, where
$i = 1$ (Virgin olive oil)
$i = 2$ (Olive oil, 1°)
$i = 3$ (Olive oil 0.4°).

Data:

LU_i	litres per unit of product type i
SP_i	sale price of product type i
OC_i	cost of olive oil of product type i
PC_i	production cost of product type i
SC_i	storage cost of product type i
Y_i	increase in storage cost of product type i
RA_i	rappel of product type i
MK_i	minimum units to be packed of product type i
PS_i	packaging speed of product type i

Objective Function:

$$\text{Max}\, f = \sum_{i=1}^{3} X_i \cdot LU_i \cdot \left(SP_i - \frac{X_i \cdot LU_i}{100 \cdot RA_i} \right) - X_i \cdot LU_i \cdot OC_i - X_i \cdot PC_i - X_i \cdot LU_i \cdot SC_i \cdot \left(1 + Y_I \cdot \frac{X_i \cdot LU_i}{\sum_{j=1}^{3} X_j \cdot LU_j} \right)$$

Constraints:

$$\begin{aligned}
&\sum_{i=1}^{3}\frac{X_i}{PS_i}\leq 20 \rightarrow \sum_{i=1}^{3}\frac{X_i}{PS_i}-20\leq 0\,(g_1)\\
&\frac{X_1}{LU_1}\geq MK_1 \rightarrow MK_1-\frac{X_1}{LU_1}\leq 0\,(g_2)\\
&\frac{X_2}{LU_2}\geq MK_2 \rightarrow MK_2-\frac{X_2}{LU_2}\leq 0\,(g_3)\\
&\frac{X_3}{LU_3}\geq MK_3 \rightarrow MK_3-\frac{X_3}{LU_3}\leq 0\,(g_4)\\
&X_i\geq 0
\end{aligned}$$

(g_1) means that the days spent on packaging must be lower than or equal to 20
(g_2) means that the litres of each type to be packaged must be, at least, those that the market demands.

(b) Consider the Kuhn-Tucker conditions for this problem

Lagrangian Function:

$$L=f+\sum_{k=1}^{2}\lambda_k\cdot g_k$$

The Kuhn-Tucker conditions:

$$\begin{aligned}
\frac{\partial L}{\partial X_i}=0\Rightarrow LU_i\cdot\left(SP_i-\frac{X_i\cdot LU_i}{100\cdot RA_i}\right)+X_i\cdot LU_i\cdot\left(-\frac{LU_i}{100\cdot RA_i}\right)-LU_i\cdot OC_i-PC_i\\
-LU_i\cdot SC_i\cdot\left(1+Y_I\cdot\frac{X_i\cdot LU_i}{\sum_{j=1}^{3}X_j\cdot LU_j}\right)-X_i\cdot LU_i\cdot SC_i\cdot\\
\left(\frac{Y_i\cdot LU_i}{\sum_{j=1}^{3}X_j\cdot LU_j}-\frac{Y_i\cdot LU_i i}{\left(\sum_{j=1}^{3}X_j\cdot LU_j\right)^2}\right)+\lambda_1\cdot\frac{1}{PS_i}-\lambda_2\cdot\frac{1}{LU_i}
\end{aligned}$$

$$\lambda_1\cdot\sum_{i=1}^{3}\frac{X_i}{PS_i}-20=0$$

$$\lambda_2\cdot MK_i-\frac{X_i}{LU_i}=0$$

$$\lambda_i\leq 0$$

(c) Obtain the value of the decision variables, of the objective function and of the Lagrange multipliers. Interpret the result.

The data values are provided in Table 3.4.

Table 3.4 Values to solve this problem

Data	Value $i = 1$	Value $i = 2$	Value $i = 3$
LU_i	0.75	1	1
SP_i	4.45	3.03	3.03
OC_i	3.45	2.7	2.8
PC_i	0.75	0.15	0.15
SC_i	0.02	0.02	0.02
Y_i	0.08	0.04	0.04
RA_i	350.000	750000	750000
MK_i	175000	700000	500000
PS_i	35000	100000	100000

By assuming that condition g_1 of packaging capacity (it is logical to think that all the available capacity is to be employed), that g_2 of the market of product type 1 (because despite it providing more profit, it requires almost three times the bottling capacity as compared with the other two product types), and that g_4 of the market of product type 3 (because product type 2 is more profitable than product type 3, but it employs the same capacity, so the minimum units of product types 1 and 3 are produced to use all the excess capacity of product type 2) are saturated, by clearing the unknown quantities of the previous equations, we obtain:

The interpretation of the result is set out below:

$X_1 = 233,333.33$	$\lambda_1 = -11,126.09$
$X_2 = 843,333.33$	$\lambda_2 = -0.455$
$X_3 = 500,000.00$	$\lambda_3 = 0$
$f = 500,000$	$\lambda_4 = -0.073$

- With the calculated production mix, a profit of 119,695.39 dollars is obtained.
- λ_1: The marginal benefit of 1 extra day of packaging capacity would be 11,126.09 dollars.
- λ_2: If the market of product type 1 is restricted by 1 unit, then profit would increase by 0.45 dollars.
- λ_3: No additional profit is obtained by reducing the restriction of the product type 2 market.
- λ_4: If the market of product type 3 is restricted by 1 unit, then profit would increase by 0.073 dollars.

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Chapter 4
Network Modelling

Abstract This chapter begins with an introduction to the Graphs Theory or Network Modelling. Next, the formulation of a varied set of network modelling problems is proposed with the corresponding solution. Specifically, shortest path problems, maximal flow problems, minimal spanning tree problems and minimal cost flow problems are contemplated. The solution is carried out using Ford and Bellman-Kalaba algorithms for minimum spanning problems, the Ford-Fulkerson algorithm for maximal flow problems and the Kruskal algorithm for the minimal spanning tree problems. Only the modelling of the minimal cost flow problems is presented. Therefore, different formulations for the problems are presented along with their solutions related to Industrial Organisation Engineering and the management setting.

4.1 Introduction

A graph is a diagram made up of a set (finite or not, but numerable) of nodes and a set of arcs which join nodes in two's, and which serve to model real or simulated problems where there is a series of interrelated activities. Some examples of their usefulness can be found in large construction projects which include many subtasks, networks that connect different cities and piping systems, among others. Using graphs can help those in charge of planning, organising and controlling this type of problems by acting as a tool to help them model and solve them more efficiently. Graphs can prove very useful for: (i) expressing the possibility of communication: road map; telephone network; electric circuit; and irrigation network; (ii) translating an ordered relation: preference; hierarchy; organisation; (iii) representing feasible evolution: Markov chain; dynamical problems; sequential decisions; and (iv) schematising an association of a defined nature: algebraic; chemical; mechanic; sociological; semantic.

R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5_4,

Therefore the Graph Theory, or network modelling, allows to schematise and solve many problems in different science and technology fields. The Graph Theory began with the work of Euler (1736). In 1852, Francis Guthrie formulated the four colours problem to colour a cartographic map. This problem can be considered the most famous and productive of network modelling and it led to a large number of mathematicians, such as Cayley, Hamilton, De Morgan, Kempe, Tait or Ramsey, to study and develop this theory. However, more than a century went by before the results were shown (Appel and Haken 1977a, b), and powerful computers were needed to carry it out. Hierholzer (1873) provided a characterisation of the so-called Euler graphs which are convex and all their nodes have even valency. Nonetheless, the term "graph" was introduced for the first time in the work by Sylvester (1878).

Regarding type of problems, they are based, on the one hand, on linear programming, such as the transportation problem, the assignment problem, the network flow problem and the travelling salesman problem, which form part of network flow problems (Taha 2010; Anderson et al. 2009; Winston 2003; Hillier and Lieberman 2002). The different characteristics of customers, demand, warehouses and vehicles and the operative constraints on routes, schedules, etc. imply a very large number of problem variants. In the literature, some authors have attempted to classify and simplify a broad variety of possible problems, such as the criteria proposed by Bodin and Golden (1981) and by Desrochers et al. (1990), who attempted to reflect and order the main characteristics into aspects such as: the warehouse or depot, fleet, demand, service and objectives to fulfil. This problem classification has facilitated not only the development of mathematical models and solution strategies, but also firms' decision making. On the other hand, another characteristic type of network problems is the Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) method. According to Taha (2010), CPM and PERT are methods based on networks and have been designed to help plan, programme and control projects, whose objective is to provide the analytical means to programme activities.

This chapter synthesises the problem types included in network optimisation models. Network problems appear in a good number of situations (transport, electric, communication networks, etc.) and in such diverse areas as production, distribution, locating installations, administering resources, etc. (Hillier and Lieberman 2002). We now go on to propose and model specific problems of the following types:

- Shortest path problem: it seeks the shortest path between the source point and the destination point in a network. This problem also permits us to model other situations like minimising the total cost of a sequence of activities (replacing equipment), minimising the total distance covered (reliable) or establishing the best route. It also takes into account the probabilities of an event happening or not. Here, these problems are solved by the Ford algorithm (1956) and the Bellman-Kalaba algorithm (1960), and the latter is especially indicated when the graph contains backward arcs.

- Maximal flow problem: its objective is to transport the maximal flow quantity from the source point to the destination point. It is important to know the network's capacity to find out how much can be sent from one source node to one destination node, although this can also be done by defining cut-off points; in other words, the cut-off capacity equals the sum of the capacities of the associated arcs and among all the possible cut-offs in the network, the cut-off with the least capacity provides the maximal flow in the network. Many situations can be modelled by a network with a maximal flow index; e.g. flow of traffic, hydraulic systems, electric circuits, transporting merchandise, telematic networks capacities, etc. A variety of different assignment situations, such as maximal flow problems, can also be modelled. These problems can be solved by the Ford-Fulkerson algorithm (1956).
- Minimal cost flow problem: it aims to determine the flows of the various arcs by minimising the total cost while fulfilling the flow constraints in the arcs at the same time, as well as the quantities of the supply and demand in the nodes. When applying these problems to real life, they can be found in the commercialisation of products in a production–distribution network, programming employment, etc. These problems can be solved using the Busacker-Gowen algorithm (1961).
- Minimal spanning tree problem: it aims to link network nodes, either directly or indirectly, by seeking the shortest length in the connecting branches. One of the algorithms employed to solve this type of problems is the Kruskal algorithm (1956), whose objective is to construct a tree formed by minimum weight arcs which are successively selected from a weighted graph in the arcs. This problem is typically applied to designing telecommunication networks with a minimum total cost, although it can also be used to design transport networks, cable TV, distributed systems, to interpret climatological data, etc.

The purpose of this book chapter is to help learn the formulation of network modelling models and to show some of their applications in the Industrial Engineering and Management area. Therefore, management problems are modelled using graphs, while models are solved with shortest path, maximal flow and minimal spanning tree problems.

After reading this chapter, readers should be able to model and solve different prototype shortest path, maximal flow and minimal spanning tree problems and to model minimal cost flow problems.

4.2 Planning Storage in a Library

In one of EPSA's libraries, shelving units for 200 books of 19 cm high, 150 books of 24 cm high, 100 books of 31 cm high and 80 books of 35 cm high are needed. A mean thickness of 3 cm is considered for all the books.

There are several possibilities to store books. For instance, a shelving unit measuring 31 cm high can be built to store all the books of a height of less than or equal to 31 cm; and one measuring 35 cm high for books of 35 cm in height. Otherwise, a single shelving unit measuring 35 cm can be built to store all the books, etc.

It is estimated that the construction of the shelving unit costs $2,500 and that this incurs a cost of $5/cm^2 for the available area to store books. Let us assume that the area required to store one book is the multiplication of shelving unit height by book thickness.

Consider and solve the shortest path problem that could be used to help determine how to store books at a minimum cost.

Solution (Table 4.1)

Cost of the path:

Let us assume nodes 0, 19, 24, 31 and 35; for each arc, c_{ij} is the total cost of storing all the books of height $>i$ and $\leq j$ on a single shelving unit, plus the cost of constructing the shelving unit (Fig. 4.1) (Table 4.2).

Solved by the Ford algorithm (1956):

$$\begin{aligned}
X_0 &= (0, 0) \\
X_{19} &= (595, 000) \\
X_{24} &= (128, 500, 0) \rightarrow (59, 500 + 56, 500, 19) \; = \; (11, 600, 019) \\
&\quad (11, 600, 019) \\
X_{31} &= (2, 117, 500) \rightarrow (116, 000 + 4, 900, 024) \; = \; (16, 500, 024) \\
&\quad (16, 500, 024) \\
&\quad (59, 500 + 11, 875, 019) \; = \; (17, 825, 019) \\
X_{35} &= (2, 807, 500) \rightarrow (165, 000 + 4, 450, 031) \; = \; (20, 950, 031) \\
&\quad (59, 500 + 17, 575, 019) \; = \; (23, 525, 019) \\
&\quad (1, 16, 000 + 9, 700, 024) \; = \; (21, 300, 024) \\
&\quad (165, 000 + 4, 450, 031) \; = \; (20, 950, 031)
\end{aligned}$$

Table 4.1 The shortest paths graph

Feasible paths
Construct 1 shelving unit measuring 35 cm high
Construct 1 shelving unit measuring 31 cm high and 1 that is 35 cm high
Construct 1 shelving unit measuring 24 cm high and 1 that is 35 cm high
Construct 1 shelving unit measuring 24 cm, 1 of 31 cm and 1 of 35 cm high
Construct 1 shelving unit measuring 19 cm high and 1 that is 35 cm high
Construct 1 shelving unit measuring 19 cm, 1 of 24 cm and 1 of 35 cm high
Construct 1 shelving unit measuring 19 cm, 1 of 31 cm and 1 of 35 cm high
Construct 1 shelving unit measuring 19 cm, 1 of 24 cm, 1 of 31 cm and 1 of 35 cm high

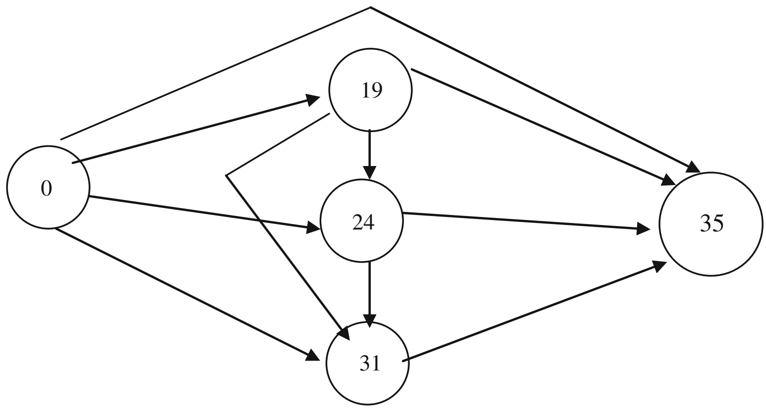

Fig. 4.1 Library shelving units construction graph

Table 4.2 Arc cost graphs

Arc	Cost
C0–19	(200 books·19 cm·3 cm·\$5/cm^2) + \$2,500 = \$59,500
C0–24	(350 books·24 cm·3 cm·\$5/cm^2) + \$2,500 = \$128,500
C0–31	(450 books·31 cm·3 cm·5 \$/cm^2) + \$2,500 = \$211,750
C0–35	(530 books·35 cm·3 cm·5 \$/cm^2) + \$2,500 = \$280,750
C19–24	(150 books·24 cm·3 cm·5 \$/cm^2) + \$2,500 = \$56,500
C19–31	(250 books·31 cm·3 cm·5 \$/cm^2) + \$2,500 = \$118,750
C24–31	(100 books·31 cm·3 cm·5 \$/cm^2) + \$2,500 = \$49,000
C19–35	(330 books·35 cm·3 cm·5 \$/cm^2) + \$2,500 = \$175,750
C24–35	(180 books·35 cm·3 cm·5 \$/cm^2) + \$2,500 = \$97,000
C31–35	(80 books·35 cm·3 cm·5 \$/cm^2) + \$2,500 = \$44,500

Thus the minimum cost is \$209,500 and the shortest path = {0, 19, 24, 31, 35} or, that is, a shelving unit must be constructed for each height.

4.3 Organising Tables for a Charity Gala

Five actors and five actresses will attend a charity gala. The challenge the organiser faces is to sit each actor with an actress in order to maximise the number of people who are compatible. Table 4.3 describes the compatibility of the actors with the actresses.

Draw a network that can represent the problem of maximising the number of compatible pairs as a maximal flow problem.

Solution

All the arcs have a capacity of 1.

Table 4.3 Compatibility of pairs

	Angelina Jolie	Charlice Teeron	Naomi Watts	Penelope Cruz	Sienna Miller
Ewan McGregor	–	1	–	–	–
Brad Pitt	1	–	–	–	–
George Cloney	1	1	–	–	–
Jude Law	1	1	–	–	1
Tom Cruise	–	–	1	1	1

Figure 4.2 shows that there is an arc with a capacity of 1 which joins source node S with each actor, an arc with a capacity of 1 which joins each pair of compatible friends and an arc with a capacity of 1 which joins destiny node T to each actress. The maximal flow in this network is the number of compatible pairs that the organiser can form.

As flow is conserved, each actress will be paired with at least one actor because the arc joining each actress with the destination node has a capacity of 1. Similarly, it is possible to pair each actor with at least one actress because each arc from the source node towards an actor has a capacity of 1. As there are no arcs that pair incompatible people, we can be sure that a flow of k units from the source to the destination represents an assignment of actors to actresses in which k compatible pairs are created.

The solution is a flow of 4 from the source node to the destination node. So if use the Ford-Fulkerson algorithm (1956) to solve this, CT can choose EM or GC (if CT chooses GC, AJ must be joined with BP), AJ can select BP or GC (if AJ

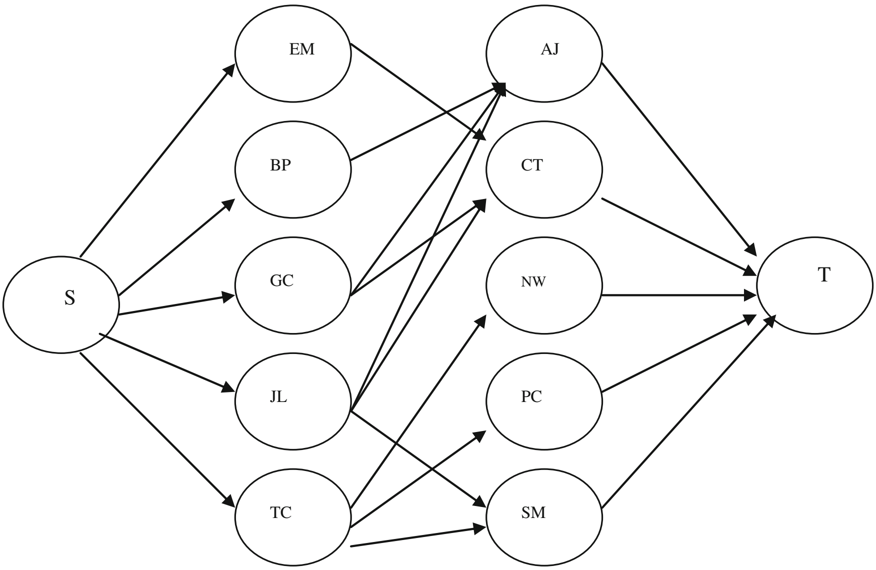

Fig. 4.2 Compatibility of pairs graph

chooses GC, CT must be joined with EM), JL must be paired with SM and TC must select NW or PC. For example, it is possible to join: EM and CT, BP and AJ, JL and SM and TC and PC.

4.4 Planning Loads on Trucks

Five transport trucks have to deliver seven types of packages. There are three packets per type and the capacities on the five trucks are 6, 4, 5, 4 and 3 packages.

Model a maximal flow problem that can be employed to determine if packages can be loaded in such a way that no truck loads two packages of the same type.

Solution

Figure 4.3 shows that there is an arc with a capacity of 3 that joins source node S with each package type, there is an arc with a capacity of 1 that joins each package with all the trucks and an arc with the corresponding capacity which joins each truck with destination node T.

4.5 Designing a Communications Network

A firm has eight warehouses distributed at the coordinates indicated in Table 4.4.

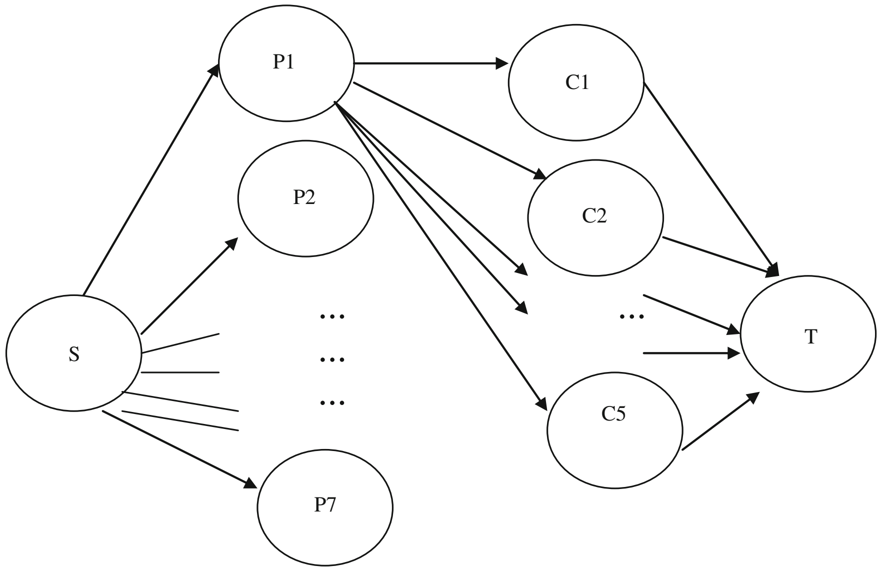

Fig. 4.3 Planning loads graph

Table 4.4 Location of warehouses

Warehouse	Coordinates (km)	Warehouse	Coordinates (km)
A	(50, 40)	E	(90, 50)
B	(90, 10)	F	(30, 80)
C	(50, 70)	G	(50, 20)
D	(50, 80)	H	(10, 60)

All the warehouses are to be connected by means of a telephone network using the lines of the Telefón firm. The cost to contract a line between two points is $50 per year and per km of line. The link between two points is considered a straight line. Sending information from one warehouse to another will be done only by the contracted lines, and this information will travel from one line to another (if required) through switch centres located on the nodes where lines cross.

By assuming that switch centres can be located only in the firm's eight warehouses and that the firm wishes to connect all the points with the shortest possible length of line, what is the telephone network with the minimum cost that the firm should contract? How much would it cost?

Solution

We need to find a minimum spanning tree, and for this purpose, the Kruskal algorithm (1956) is used. There are two possible minimum spanning trees:

$$\text{HFDCAGBE} \quad \text{and} \quad \begin{array}{l}\text{HFDCAEB} \\ \hspace{3.2em}| \\ \hspace{3.1em}\text{G}\end{array}$$

The total cost of the telephone network is:

$$50 \cdot \left[(800)^{0.5} + 20 + 10 + 30 + 20 + (1700)^{0.5} + 40\right] = \$9475.77$$

4.6 Assigning Cases in a Lawyer's Office

A lawyer's office has accepted five new cases, and each can be suitably managed by any of the five more recent partners. Owing to the difference in experience and practice, lawyers will employ different times on their cases. One of the most

Tab 4.5 Estimation of the time requirements per lawyer and case

	Case 1	Case 2	Case 3	Case 4	Case 5
Lawyer 1	145	122	130	95	115
Lawyer 2	80	63	85	48	78
Lawyer 3	121	107	93	69	95
Lawyer 4	118	83	116	80	105
Lawyer 5	97	75	120	80	111

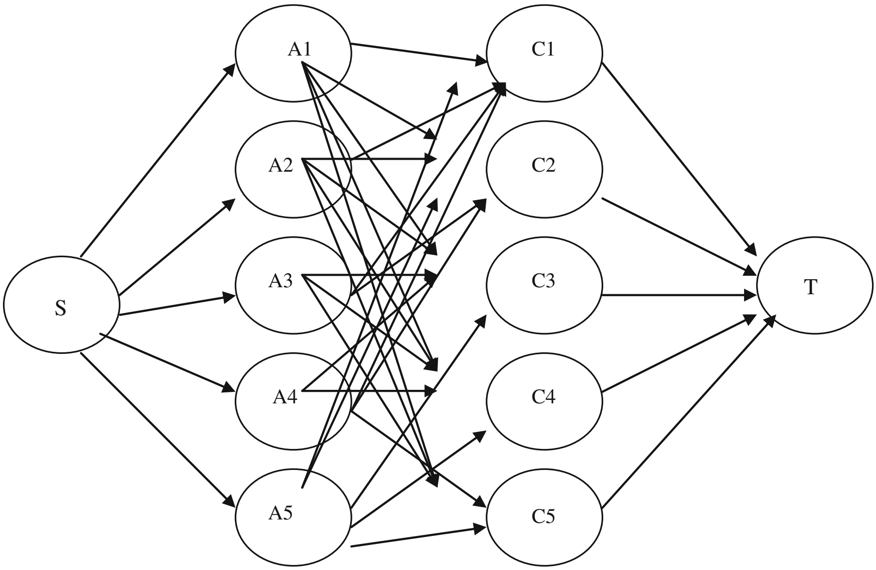

Fig. 4.4 The assigning cases to lawyers graph

experienced partners has estimated time requirements (in hours) which are indicated in Table 4.5.

Formulate this problem as a Graph Theory model to optimally determine the assignment of cases to lawyers so that each one is dedicated to a different case and that the total time spent in hours is minimum.

Solution

This is a maximal flow model at a minimum cost. The structure of the graph is indicated in Fig. 4.4.

Each arc must contain the following information:

A_i = minimum capacity.
B_i = maximum capacity.
C_i = minimum cost.

In Fig. 4.4, the arcs going from source node S to each lawyer have a minimum capacity of 0, a maximum capacity of 1 and a minimum cost of 0; that is: (0, 1, 0).

The arcs going from each lawyer to each case will have a minimum capacity of 0, a maximum capacity of 1 and the cost of the corresponding hours. For example, the arc that goes from lawyer 1 to case 1, will be: (0, 1, 145).

Finally, the arcs going from each case to destination node T will have a minimum capacity of 0, a maximum capacity of 1 and a minimum cost of 0; that is: (0, 1, 0).

4.7 Project Planning

ORGA, S.A., a firm that implements information systems, must finish implementing three projects over the next 4 months. Project 1 (P1) must be finished in 3 months and requires a consultant's work of 8 months. Project 2 (P2) has to finish within 4 months and needs 10 months of a consultant's work. Project 3 (P3) needs to end in 2 months and requires 12 months of a consultant's work. The consultancy firm has eight consultants specialised in information systems available each month but, according to operational rules, no more than six consultants can work on the same project.

(a) Formulate a Graph Theory model to determine whether ORGA, S.A. will be able to complete all three projects on time.
(b) Solve the problem and indicate how the consultancy company must act.

Solution

(a) Formulate a Graph Theory model to determine whether ORGA, S.A. will be able to complete all three projects on time.

It is a maximal flow problem.

Figure 4.5 shows an arc with a capacity of 8 (CONSULTANTS PER MONTH) which links source node *S* with each MONTH, an arc with a capacity of 6 (MONTHS-CONSULTANTS PER PROJECT) which links each available MONTH-CONSULTANT with each PROJECT and an arc with the corresponding MONTH-CONSULTANTS capacity (8, 10 and 12) which links each PROJECT with destination node *T*. The maximal flow in this network is the number of MONTH-CONSULTANTS that the consultancy can assign to each project.

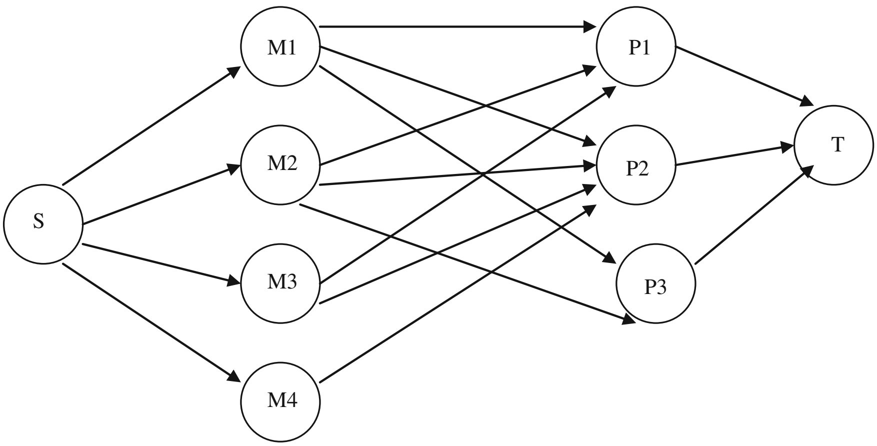

Fig. 4.5 Projects planning graph

(b) Solve the problem and indicate how the consultancy company must act.

Using the Ford-Fulkerson algorithm (1956), the following solution is reached. The maximal flow that would circulate around the corresponding network would be 30. One of the possible solutions is to assign two consultants to Project P2 in month 1 and six consultants to Project P3. In month 2, two consultants are assigned to P1 and six consultants to P3. In month 3, six consultants to P1 and two consultants to P2 are assigned. Finally in month 4, six consultants are assigned to P2.

4.8 Production Planning in a Plastics Firms

PLACASA, a supplier of plastic injection parts for the automobile sector, foresees the demand of an article for the following 4 months of 100, 140, 210 and 180 units, respectively. The firm can maintain only sufficient stock to meet the demand of each month or it can have excess stock to meet the demand of two successive months or more with a storage cost of $1.2 per month and unit of excessive stock. PLACASA calculates that the production cost for the following months, which varies depending on the units produced, is $15, $12, $10 and $14, respectively. A preparation cost of $200 is incurred every time a production order is placed. The firm wishes to develop a production plan that minimises the total costs of production orders, of production and of keeping an article in stock. Formulate and solve a shortest path model to find an optimum production plan.

Solution

To represent the graph in Fig. 4.6, each node corresponding to a monthly period is considered; thus, there will be five nodes that go from period 0 to period 4. The arcs represent the quantity produced to meet the demands in the periods they

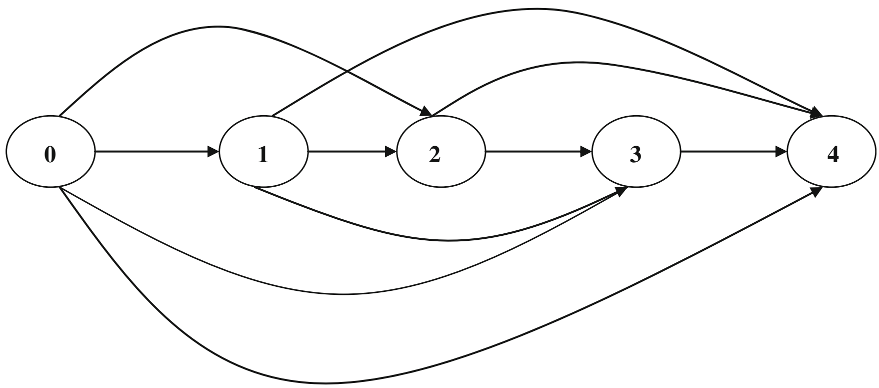

Fig. 4.6 The plastics production planning graph

Table 4.6 Costs of the arcs in the plastics production graphs

Path	Production	K	Cp	Ca	Total ($)
0–1	100	200	15	1.2	1,700
0–2	240	200	15	1.2	3,968
0–3	450	200	15	1.2	7,622
0–4	630	200	15	1.2	10,970
1–2	140	200	12	1.2	1,880
1–3	350	200	12	1.2	4,652
1–4	530	200	12	1.2	7,028
2–3	210	200	10	1.2	2,300
2–4	390	200	10	1.2	4,316
3–4	180	200	14	1.2	2,720

cover; for example, arc 0–1 represents producing the quantity of 100 units, arc 0–2 represents producing 240 units, etc.

Table 4.6 presents the costs associated with each arc.

Use the Ford algorithm (1956) to calculate the shortest path. The optimum solution is to produce 100 units in month 1, 140 units in month 2 and 390 units in month 3, with a total cost of $7,896.

4.9 Overbooking on Airlines

A low-cost airline, OrgaAir, operates four daily flights from Valencia to London at 10:00, 12:00, 14:00 and 16:00 hours. The first two flights hold 100 passengers and the last two can fly up to 150 passengers each. Should overbooking occur, which implies having sold more seats than the plane actually has, the airline can place a passenger on the later flight. Evidently each delayed traveller is compensated by being paid $200 plus $20 for each hour delayed. The firm always places delayed travellers on the flight leaving at 16:00 hours or on one of the other flights at 20:00 hours, which always has available seats (a capacity of 999 passengers is always considered) at no extra charge.

Let us assume that at the beginning of the day we know that OrgaAir has sold 110, 160, 100 and 100 seats on its four daily flights, respectively. Model this problem as a maximal flow model at a minimum cost in order to minimise the airline’s total overbooking cost.

Solution

It is a maximal flow model at a minimum cost. The structure of the graph is shown in Fig. 4.7.

Each arc must contain the following information:

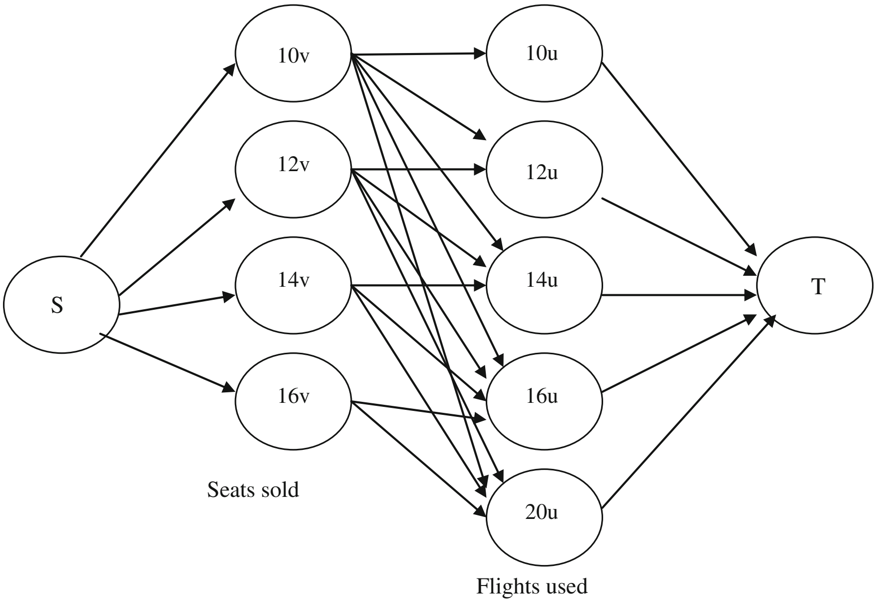

Fig. 4.7 The flights distribution graph

a_i = minimum capacity.
b_i = maximum capacity.
c_i = minimum cost.

In Fig. 4.7, the arcs going from source node S to each flight represent the seats sold and they have a minimum and maximum capacity of 110, 160, 100 and 140, respectively, and a minimum cost of 0; that is: (0, 110, 0); for each node corresponding to the 10:00 h flight. The arcs going from each flight used to destination node T will have a minimum capacity of 0, and a maximum capacity of 100, 100, 150, 150 and 999 (representing infinite capacity), respectively, and a minimum cost of 0; that is: (0, 1, 0).

The arcs going from each flight or sold seat to each flight or sold seat will have a minimum capacity of 0, a maximum capacity corresponding to that left over between the seat sold and that available and the unit cost for the corresponding delay. Table 4.7 provides the values corresponding to each arc.

4.10 Production Sequencing in a Firm of the Metal Sector

A manufacturer of the metal sector in Stafford manufactures four types of products in sequence on two machines. Table 4.8 provides this manufacturer's technical production details.

Table 4.7 Costs per arc in the flights distribution graph

From node	To node	Minimum capacity	Maximum capacity	Cost ($)
S	10v	0	110	0
S	12v	0	160	0
S	14v	0	100	0
S	16v	0	140	0
10v	10u	0	100	0
10v	12u	0	100	240
10v	14u	0	100	280
10v	16u	0	150	320
10v	20u	0	999	400
12v	12u	0	100	0
12v	14u	0	100	240
12v	16u	0	150	280
12v	20u	0	999	360
14v	14u	0	150	0
14v	16u	0	150	240
14v	20u	0	999	320
16v	16u	0	150	0
16v	20u	0	999	280
10u	T	0	100	0
12u	T	0	100	0
14u	T	0	150	0
16u	T	0	150	0
20u	T	0	999	0

Table 4.8 Production data

Production time per unit (minutes)					
Machine	Product 1	Product 2	Product 3	Product 4	Daily production capacity (minutes)
1	2	3	4	2	500
2	3	2	1	2	380

By contemplating only the data on the production times per product and the maximum daily production capacity of each machine, and taking a maximum daily production of 100 units of each product type, consider and solve a maximal flow model in a graph to determine the maximum number of minutes during which machines 1 and 2 can operate on a production-in-sequence basis given the existing capacity constraints.

Solution

This is a maximal flow problem. Figure 4.8 shows an arc with a capacity of 500 (daily production capacity in minutes) that joins source node *S* with M1 (machine 1), four arcs whose capacity corresponds to the production time in minutes of each product on machine 1 (2, 3, 4, 2) which joins M1 with each product (P1, P2, P3,

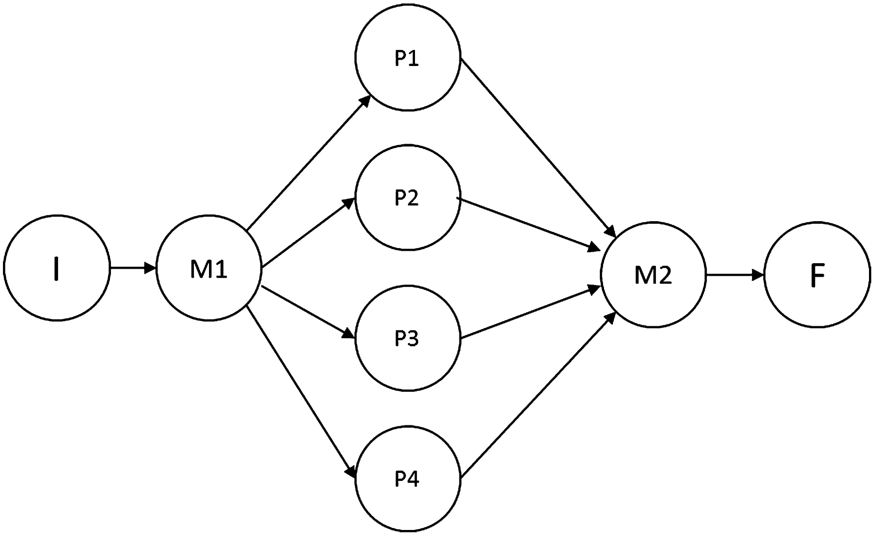

Fig. 4.8 Machines sequencing graph

P4), four arcs whose capacity corresponds to the production time in minutes of each product on machine 2 (3, 2, 1, 2) which joins each product to M2 and an arc with a capacity of 380 which joins M2 with destination node *T*. The maximal flow of this network is the number of minutes that the firm can allocate to sequencing these four products.

Using the Ford-Fulkerson algorithm (1956), a maximal flow of 380 is obtained. One possible solution is to use 180 min of M1 for product P1 and 200 min for P2, and to similarly allocate 180 min of M2 to product P1 and 200 min to P2.

4.11 Fibre Optics Network Planning

EPSA intends to reinforce its internal fibre optics network by placing new fibre between its two buildings, Ferrándiz and Carbonell. For this purpose, it wishes to use existing piping, whose lengths are indicated in Table 4.9.

(a) Bear in mind that the cost is proportional to the wiring run length. Find out the lowest possible cost that connects 1 and 6. Indicate the piping through which one of the optimum runs passes.
(b) If each wired fibre segment has the capacities indicated in Table 4.10, determine the maximum total capacity between connections 1 and 6, as well as the runs required to achieve this maximum capacity.

Table 4.9 Lengths of piping between network nodes

From	1	1	1	2	2	2	3	3	3	4	4	5
To	2	3	5	3	4	6	4	5	6	5	6	6
Length	4	3	5	1	3	5	2	2	5	1	1	3

Table 4.10 Fibre capacities between network nodes

From	1	1	1	2	2	2	3	3	3	4	4	5
To	2	3	5	3	4	6	4	5	6	5	6	6
Length	2	3	2	2	2	2	3	2	2	2	3	2

Solution

(a) Bear in mind that the cost is proportional to the wiring run length. Find out the lowest possible cost that connects 1 and 6. Indicate the piping through which one of the optimum runs passes.

To answer this question, the Ford algorithm (1956) is applied to obtain the run with the minimum cost in a graph based on the distances from 1 to the remaining nodes.

The shortest distance is obtained by following run 1–3–4–6 with a minimum cost of 6.

(a) If each wired fibre segment has the capacities indicated in Table 4.10, determine the maximum total capacity between connections 1 and 6, as well as the runs required to achieve this maximum capacity.

To answer this question, the network capacities have to be considered and it is necessary to seek the run with the maximal flow by applying the Ford-Fulkerson algorithm (1956).

The maximal flow has seven units via runs 1–2–6 (2), 1–3–6 (2), 1–5–6 (2) and 1–3–4–6 (1). An alternative solution is: 1–2–6 (2), 1–3–4–6 (3), 1–5–6 (2).

4.12 Assigning Subject Matters to Teachers

EPSA's Business Organisation Department has four teachers working part-time and it wishes to assign them five new subject matters. The head of the department knows from past experience that it is counterproductive to impose subject matters upon teachers. By bearing this in mind, teachers were asked to make a list of their preferences among the five subject matters, which are provided in Table 4.11.

The challenge the department faces is to assign as many subject matters as possible by respecting teachers' preferences at the same time.

Use a Graph Theory model to determine the maximum number of subject matters that can be assigned and the assignment of the subject matters to the teachers.

Tab 4.11 Teachers' preferences

Teacher	Preferred subject matter
P1	3, 4 or 5
P2	1
P3	1 or 2
P4	1, 2 or 5
P5	2

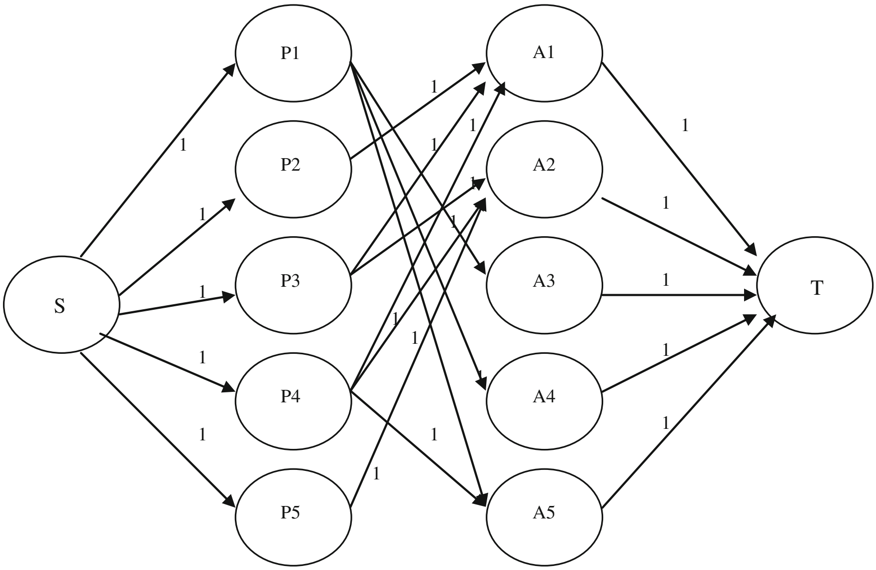

Fig. 4.9 Subject matters assignment graph

Solution

All the arcs have a capacity of 1.

Figure 4.9 shows an arc with a capacity of 1 which joins source node *S* with each teacher, an arc with a capacity of 1 which joins each preferred teacher-subject matter pair and an arc with a capacity of 1, which links each subject matter to destination node *T*. The maximal flow in this network is the number of teacher-subject matter pairs that the head of the department can form.

By conserving flow, each teacher is paired with one subject matter at the most as the arc that links each subject matter to the destination node has a capacity of 1. Similarly, each teacher can be paired with one subject matter at the most because each arc from the source node to a teacher has a capacity of 1.

Using the Ford-Fulkerson algorithm (1956), the solution is a maximal flow of 4 from the source node to the destination node. For example: P1-A4, P3-A1, P4-5 and P5-A2.

4.13 Water Network Planning

Table 4.12 shows the existing water connections between a given firm's different warehouses. Each arc indicates the maximum water transport capacity (m^3/h) of the pipe or pipes connecting the warehouses.

(a) What is the maximum quantity of water that can be held in warehouse F?
(b) Determine which flow should circulate through each pipe to obtain the maximum flow.
(c) Formulate a linear programming model to help solve this problem.

Solution

(a) What is the maximum quantity of water that can be held in warehouse F?

This is a maximal flow problem. The Ford-Fulkerson algorithm (1956) will help obtain the maximum flow of 9 m^3/h.

(b) Determine which flow should circulate through each pipe to obtain the maximum flow.

From pipe A to pipe D, 4 m^3/h will circulate, 1 m^3/h from B to C, 3 m^3/h from B to E, 2 m^3/h from C to E, 4 m^3/h from D to F and 5 m^3/h from E to F.

(c) Formulate a linear programming model to help solve this problem.

Decision variables:

X_{IA}: flow that circulates between node I and A
X_{IB}: flow that circulates between node I and B
X_{IC}: flow that circulates between node I and C
X_{AD}: flow that circulates between node A and D
X_{BD}: flow that circulates between node B and D
X_{BC}: flow that circulates between node B and C
X_{BE}: flow that circulates between node B and E
X_{CE}: flow that circulates between node C and E
X_{DF}: flow that circulates between node D and F
X_{EF}: flow that circulates between node E and F

Objective function:

Maximise $z = X_{DF} + X_{EF}$ ó

Tab 4.12 Capacities of the water connections between warehouses

Arc	Capacity	Arc	Capacity	Arc	Capacity
I–A	6	B–D	1	D–F	4
I–B	4	B–C	3	E–F	9
I–	1	B–E	3		
A–D	4	C–E	4		

Maximise $z = X_{IA} + X_{IB} + X_{IC}$

Constraints:

The maximum capacity constraints in each arc:

$X_{IA} \leq 6$
$X_{IB} \leq 4$
$X_{IC} \leq 1$
$X_{AD} \leq 4$
$X_{BD} \leq 1$
$X_{BC} \leq 3$
$X_{BE} \leq 3$
$X_{CE} \leq 4$
$X_{DF} \leq 4$
$X_{EF} \leq 9$

The flow conservation constraints in each node:

$X_{IA} = X_{AD}$
$X_{IB} = X_{BD} + X_{BE} + X_{BC}$
$X_{IC} + X_{BC} = X_{CE}$
$X_{AD} + X_{BD} = X_{DF}$
$X_{BE} + X_{CE} = X_{EF}$

4.14 Production Planning in a Firm of the Textile Sector

The star product of a baby clothes firm, Babidá, is babygros, whose size varies from 0 to 12 months. Monthly demand (in thousands of units) and variable cost (in dollars) of producing each babygro are provided in Table 4.13. A fixed cost of $1,000 is incurred to produce any babygro type on a monthly basis. If the firm wishes, demand corresponding to a given size can be met with a larger babygro size.

Formulate and solve this problem with a Graph Theory model. The solution must minimise the cost of meeting the monthly demand of babygros.

Tab 4.13 Monthly demand and production costs

	Size (months)						
	0	1	3	4	6	9	12
Variable cost ($/unit)	4	3	2.,6	2.4	1.9	1.8	1.7
Monthly demand (thousands of units)	400	300	500	700	400	200	700

Tab 4.14 Arcs and costs of the production planning graph

Arc	Cost (thousands of dollars)	Arc	Cost (thousands of dollars)
I–0	1,601	I–1	2,101
0–1	901	0–3	2,081
1–3	1,301	1–4	2,881
3–4	1,681	3–6	2,091
4–6	761	4–9	1,081
6–9	361	6–12	1,531
9–12	1,191		

Solution

This problem is modelled using a shortest path graph. The resulting graph is formed by the arcs and costs indicated in Table 4.14.

The solution of this graph can be obtained by applying the Ford algorithm (1956), and it consists in serving babygros in sizes 1, 3, 6 and 12 months at a total cost of $7,024.000.

4.15 Project Planning and Designing an Air Conditioning Network

The firm CONDUAIR, a supplier of air conditioning pipes, previously cut and prepared ready for installing at the construction site, has been contracted by an air conditioning fitter to provide it with an optimum design and ready-cut pipes for a large-scale fitting. The Projects Director is a former Industrial Organisation Engineering student who tends to apply the Graphs Theory when designing projects to optimise the installations supplied because this is the main competitive advantage for fitters over the traditional cutting of pipes on the work site. One of the project conditions is that a multi-split system is installed, which consists in a single external unit (with the compressor) and several internal units (with an evaporator and fan). To fit this system, the pipes through which refrigerated liquid circulates has to go through all the rooms of the home where an internal unit is to be fitted. The Projects Director has taken the measurements shown in Fig. 4.10, where the nodes represent the rooms where an interior unit must go, while the arcs represent the distances in metres between these rooms. The exterior unit is to be placed on node f.

Provide an optimum solution based on the Graph Theory for CONDUAIR's Projects Director.

Solution

This problem is known as minimal spanning tree and, for this reason, the Kruskal algorithm (1956) is applied to obtain the optimum solution, which employs 32 m of piping to connect rooms, as indicated in Fig. 4.11.

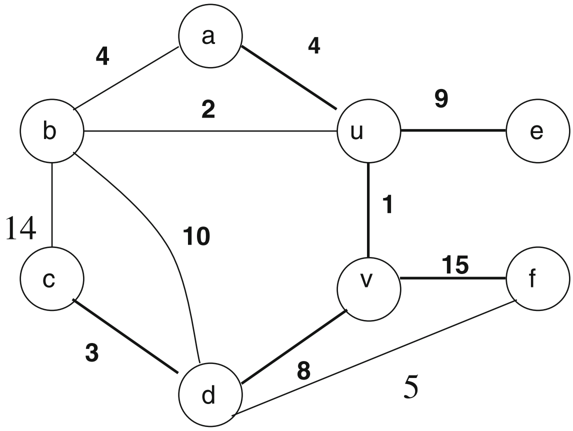

Fig. 4.10 Structure of the air conditioning network

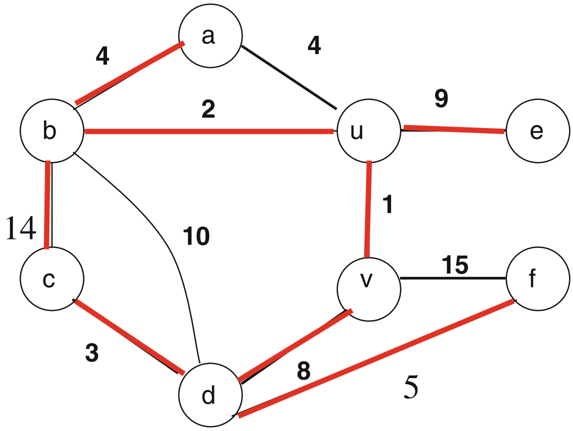

Fig. 4.11 Solution according to the Kruskal algorithm

4.16 Machinery Transport Planning

A firm must transport machines from production plants A, B and C to warehouses X, Y and Z. Five machines are required in X, four in Y and three in Z and there are eight machines available in A, five in B and three in C.

(a) Contemplate and solve a Graph Theory model to determine the maximal flow of the machines that can be transported and the run of this flow.
(b) By assuming a mean cost of $45 per transported machine, what would the total cost be of transporting the machines obtained in the former section?

Solution

(a) Contemplate and solve a Graph Theory model to determine the maximal flow of the machines that could be transported and the run of this flow

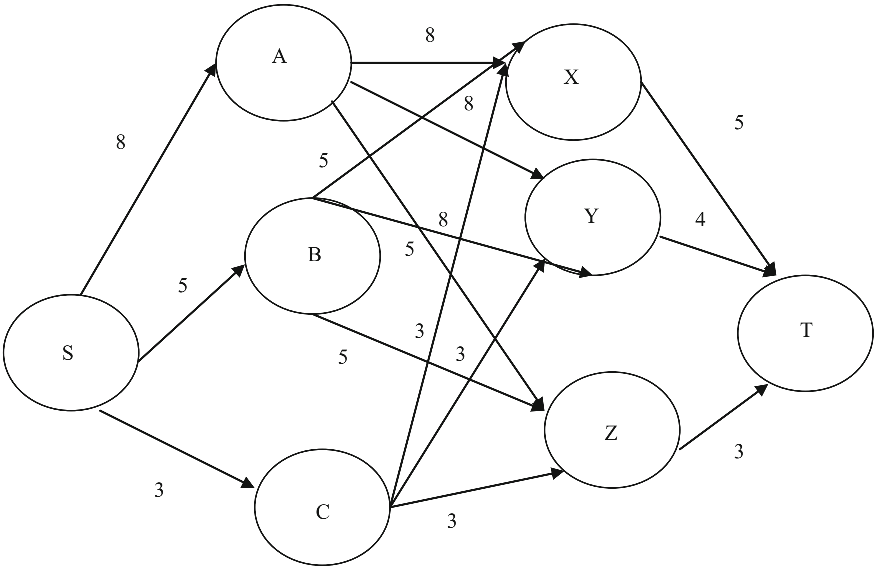

Fig. 4.12 Machine transport planning graph

By applying the Ford-Fulkerson algorithm (1956) to the maximal flow graph indicated in Fig. 4.12, the maximum number obtained of machines that can be transported is 12 with several possible flow distributions (Table 4.15).

(b) By assuming a mean cost of $45 per transported machine, what would the total cost be of transporting the machines obtained in the former section?

$540

4.17 Emergency Routes Should a Fire Break Out

In Table 4.16, node A represents a fire station and node E is a nature reserve. The values associated with the arcs represent the time required to go from one node to another during the rush hour.

Using an adequate algorithm, determine the quickest route to go from node A to node E during the rush hour. Clearly show your work for each iteration of the algorithm.

Solution

Given that this is a shortest path problem with backward arcs, the Bellmann-Kalaba algorithm (1960) is applied to solve it.

Iteration 0:

Table 4.15 Possible flow distributions to transport machinery

S–A	4	S–A	5	S–A	6
S–B	5	S–B	4	S–B	3
S–C	3	S–C	3	S–C	3
A–Y	4	A–X	5	A–X	3
B–X	5	B–Y	4	B–Y	1
C–Z	3	C–Z	3	C–Z	1
X–T	5	X–T	5	X–T	5
Y–T	4	Y–T	4	Y–T	4
Z–T	3	Z–T	3	Z–T	3

Tab 4.16 Times to travel between nodes

From/to	A	B	C	D	E
A	0	100	∞	∞	∞
B	∞	0	220	300	100
C	100	80	0	∞	90
D	∞	∞	∞	0	230
E	∞	130	110	120	0

$vA(0) = \infty$
$vB(0) = 100$
$vC(0) = 90$
$vD(0) = 230$
$vE(0) = 0$

Iteration 1:

$vA(1) = 200$
$vB(0) = 100$
$vC(0) = 90$
$vD(0) = 230$
$vE(0) = 0$

Iteration 2:

$vA(1) = 200$
$vB(0) = 100$
$vC(0) = 90$
$vD(0) = 230$
$vE(0) = 0$

The total time is 200 and the quickest route is: A–B–E.

Tab 4.17 Production costs

Order/machine	M1	M2	M3	M4	M5
P1	16	4	9	5	6
P2	2	14	7	5	13
P3	8	10	3	12	11
P4	3	7	6	10	5
P5	3	6	8	11	7

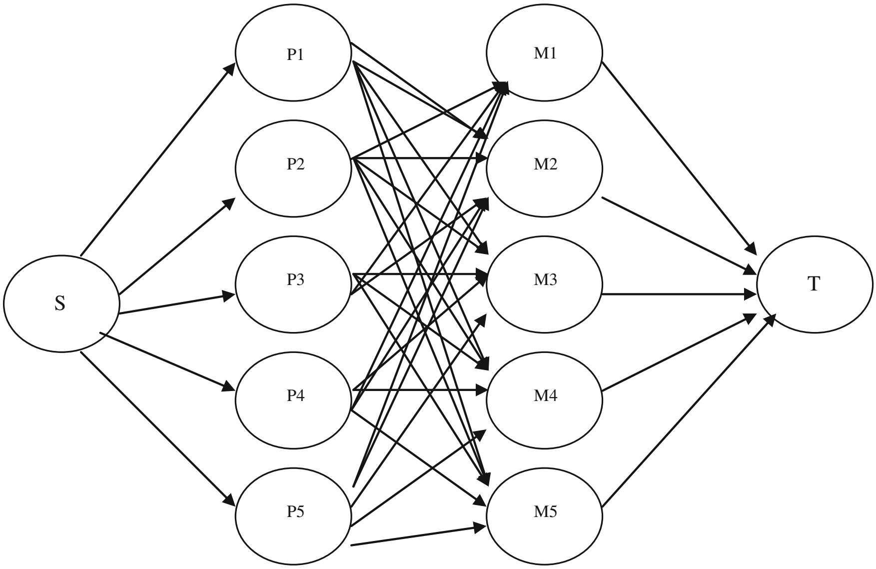

Fig. 4.13 Orders planning graph

4.18 Planning Customers Orders

The firm Orgasa has received five orders (P1, P2, P3, P4, P5) which must be served. For this purpose, there are five machines available (M1, M2, M3, M4, M5). Each machine can deal with each order at the cost shown in Table 4.17.

To be able to determine the optimum assignment that minimises the total cost of serving these orders, by assuming that each machine can deal with one order and that all the orders must be served, contemplate a Graph Theory model that represents the problem set out.

Solution

This is a maximal flow model at a minimum cost. The structure of the graph is shown in Fig. 4.13.

Each arc must contain the following information:

A_i = minimum capacity.
B_i = maximum capacity.
C_i = minimum cost.

Figure 4.12 shows the arcs going from source node S to each order. They have a minimum capacity of 0, a maximum capacity of 1 and a minimum cost of 0; that is: (0, 1, 0). The arcs going from each order to each machine have a minimum capacity of 0, a maximum capacity of 1 and the corresponding cost in hours. For example, the arc going from P1 to M1 is as follows: (0, 1, 16). Finally, the arcs going from each case to destination node T have a minimum capacity of 0, a maximum capacity of 1 and a minimum cost of 0: that is: (0, 1, 0).

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Chapter 5
Inventory Theory

Abstract This chapter begins with an introduction to the inventory theory. Next, it proposes the formulation of a varied set of theoretical inventory-related problems with their corresponding solutions. Specifically, basic EOQ (economic order quantity) problems, EOQ problems with non-null lead times, EOQ problems with discount for volume and EOQ problems with backorders are contemplated. The solution is carried out using the corresponding analytical formulae. Thus, different formulations of the problems are proposed along with their solutions in relation to industrial organisation engineering and in management domain.

5.1 Introduction

The object of the inventory theory is to determine rules that can be applied to reduce the costs of maintaining stocks as much as possible and to meeting customer demand.

One important factor in the formulation and solution of an inventory model is that the demand (per time unit) of an article can be deterministic (known with a certain degree of certainty) or probabilistic (represented by probability distribution).

Another important factor is the dependent versus the independent demand concept. The dependent demand concept was introduced by Orlicky (1975) and is key to MRP (Material Requirement Planning) systems. It recognises the role that the bill of materials plays to generate demand profiles instead of the statistical procedures of the reorder point which trigger production once the inventory levels of the components fall below the specified level. MRP refers to the demand of components and sub-assemblies directly from the production programme for end products. This is known as dependent demand, unlike the demand which stems directly from customers orders and is called independent demand. The quantitative

R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5_5,

methods to manage inventories included herein contemplate independent demand models.

The inventory problem consists in making and receiving orders of certain volumes repeatedly and at given intervals. An inventory policy must answer these two questions: How much must be ordered? When must orders be made? The answer to the first question if demand is considered deterministic is established by an economic lot or EOQ (economic order quantity). This is the most elemental model of all inventory management, and the classical theory is based on it. It was developed by F. W. Harris of the Westinghouse Corporation in 1915, and was extended by a consultant called Wilson. The answer to the second question depends on whether the system needs to undergo a periodical review in accordance with the period of time or if it continues according to the reorder point. The basic EOQ model offers several variants and this chapter considers the following: the EOQ model with a different zero delivery time; the EOQ model with discounts for volume; the EOQ model with demands that can be re-ordered, or backorders Some works on deterministic inventory models are those by Silver and Peterson (1985), Tersine (1988) and Waters (1992).

Probabilistic models are generally classified in continuous and periodical analysis situations. Some methods are based on a probabilistic version of the deterministic EOQ, such as the fixed amount model with a specific service level, and the fixed period model with a specific service level.

In EOQ models, the requested decision is repetitive; that is, it is repeated regularly. Demand is assumed to occur at a constant, known speed. The time that elapses between making the order and it arriving is a known constant. An order can be done on any occasion; i.e. continuously reviewed models.

In the basic EOQ model, the initial hypotheses are: the demand of an article is constant in time; the storage cost is proportional to the quantity stored and the inventory time; replenishment is done so that deliveries arrive exactly when the inventory level is zero, thus there is never any excess or scarcity; the delivery time for each order is constant; finally, only one article is considered, therefore orders of different articles are independent of the rest.

EOQ inventory models mainly consider the following costs:

The supply cost is the total cost which emerges every time an order is made and it consists of two subcosts: the order cost, which is the article's unit value per the number of articles included in the order; the cost of sending off or dispatching an order, which is a fixed cost associated with every order.

The storage cost is the cost of maintaining one inventory unit during a certain time period. It considers the obsolescence cost of products, thefts and damage, insurance and taxes on stocks, the warehouse and capital opportunity, among others.

The cost of unmet or scarce demand is the cost incurred when demand cannot be met because, when it emerges, there are no stocks in the warehouse (stockout). It can be considered a deferred cost of unmet demand or backorder when customer orders arrive when there are not stocks available, so they are delayed to be covered no sooner than warehouse stocks become available. Should customers not accept delayed deliveries, the lost unmet demand cost or sales loss is considered.

In the basic EOQ model, the total cost is:

$$\begin{aligned} \text{CT } (Q) &= \text{ total order cost } (\text{C}p) + \text{ total storage cost } (\text{C}a) + \text{ total acquisition cost } (\text{C}a) \\ &= K\text{D}/\text{Q} + \text{ hQ}/2 + \text{p}D. \end{aligned}$$

where Q is the quantity to order, K is the order cost, h is the storage cost and D is the total demand for the considered planning horizon.

The minimum total cost value corresponds to the minimum of the curve's equation. The ordered quantity is always the same and it is known as economic lot, Q^*, which minimises the costs generated in inventory management.

$$Q^* = \sqrt{\frac{2KD}{h}} = \text{ EOQ}$$

The maximum inventory level coincides with the order quantity. The order is dispatched when the warehouse reaches a certain stock level, known as the reorder point (P_p) (Fig. 5.1).

When the delivery time is different to zero, two cases can arise depending on the relation between T_r (replenishment time) and T_s (supply time). If $T_r \geq T_s$, $P_p = T_s \cdot D$. If $T_r < T_s$, $P_p = (T_s - E \cdot T_r) \cdot D$, where E represents the integer part of T_s/T_r. In both cases, demand D must be expressed according to time units T_s and T_r.

In the basic model, and in the assumed case of discount for volume, the acquisition and storage costs are affected, and the order cost does not change. The total acquisition cost varies depending on the quantity of ordered units.

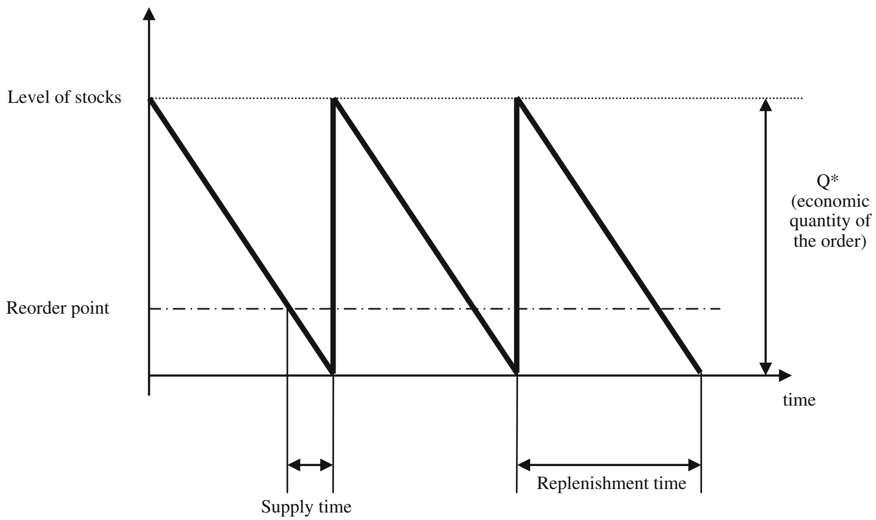


Fig. 5.1 A basic EOQ model

The steps to follow are: the optimum order quantity is calculated for all the lots prices. The total storage cost is assumed to be proportional to the acquisition cost. Verifications are made to see if the optimum lot is within the price margins currently in force. If this were not the case, the lot size that provides the lowest cost within the margins between which the price is currently in force is selected. The total cost is calculated with the lot size that minimises each lot and that with the lowest cost is selected.

In order to allow backorders, it is necessary to modify the basic EOQ model. It is necessary to bear some assumptions in mind, such as: it is assumed that all demand accumulates, and that no sales are lost; the level of stocks at the end of the planning horizon is zero, so demand is met. The total cost, the economic lot and the maximum inventory level in the EOQ model with backorders are as so:

$$CT = \frac{M^2 h}{2Q} + \frac{(Q-M)^2 s}{2Q} + K\frac{D}{Q} + pD$$

$$Q^* = \text{EOQ} \cdot \sqrt{\frac{h+s}{s}}$$

$$M^* = \text{EOQ} \cdot \sqrt{\frac{s}{h+s}}$$

where s is the unit cost of scarcity and M is the maximum inventory level.

We refer readers to Taha (2010) and Winston (2003) for further theories of these models.

The objective of this chapter is to help learn to formulate and solve deterministic models based on the inventory theory in the independent demand context and to show some of their applications in the industrial engineering and management area. Thus, management problems are modelled by analytical EOQ formulation along with some of its variants.

After reading this chapter, readers should be able to model and solve different inventory problems by means of the basic EOQ model and its variants with discounts for volume, delivery times other than zero and backorders.

5.2 Minced Meat Inventory

McBurger orders minced meat at the beginning of each week to cover its weekly demand of 300 pounds. The fixed order cost is 20$. Refrigerating and storing meat costs approximately 0.03$ per pound and day. The meat purchasing cost is not considered.

It is necessary to:

(a) Determine the weekly inventory cost of the current orders policy

(b) Determine the optimum inventory policy that McBurger must use by assuming a zero delivery time between the time an order is made and the time it is received
(c) Determine the difference between the weekly cost between McBurger's current orders policy and its optimum order policy
(d) Graphic representation of an optimum model.

Solution

(a) Determine the weekly inventory cost of the current orders policy

$$CT = Ca + Cp = \frac{Q}{2}h + K\frac{D}{Q}$$

$Q = 300$ pounds/week
$K = \$20$/order
$h = 0.03 \cdot 7 = \$0.21$/pound and week (McBurger works 7 days a week)

$$CT = Ca + Cp = \frac{Q}{2}h + K\frac{D}{Q} = 300/2 \cdot 0.21 + 20 = \$51.5/\text{week}$$

(b) Determine the optimum inventory policy that McBurger must use by assuming a zero delivery time between the time an order is made and the time it is received

$$Q^* = \text{EOQ} = \sqrt{\frac{2\text{KD}}{h}} = \sqrt{\frac{2 \cdot 20 \cdot 300}{0.21}} = 239{,}045 \text{ pounds/order}$$

Number of orders = 300·52/239,045 = 65.26 orders/year = 1.25 orders/week (1 year = 52 weeks).

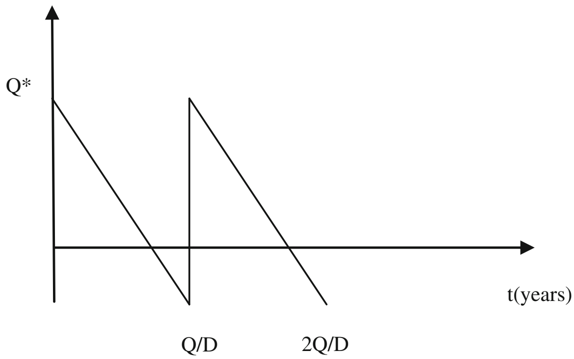
An order of 239,045 pounds will be made every time that the inventory is zero.

(c) Determine the difference between the weekly cost between McBurger's current orders policy and its optimum order policy

$$\text{CT } (Q = Q^*) = \frac{Q}{2}h + K\frac{D}{Q} = (239{,}045 \text{ pounds/order})/2 \cdot \$10.92/\text{pound year} + \$20/\text{order} \\ \cdot (300 \cdot 52/239{,}045 \text{ orders/year}) = \$2610{,}4/\text{year} = \$50.2/\text{week}$$

The difference is 51.5 – 50.2 = \$1.3/week

(d) Graphic representation of an optimum model



where $Q^* = 239,045$ pounds/order.
if $Q = Q^*$:
Q/D = (239,045 pounds/order)/(300 pounds/week · 52 weeks/1 year) = 0.015/year
No. of orders = D/Q = 65.26 orders/year
This graph could also be expressed in the weeks time unit, where:

Q/D = (239,045 pounds/order)/(300 pounds/week) = 0.79/week
No. of orders = D/Q = 1.25 orders/week

5.3 Soft Drinks Orders in a Cafeteria

Assume that the university cafeteria sells a soft drink with a constant annual demand rate of 3,600 boxes. One box of soft drinks costs the cafeteria \$3. The order costs are \$20 per order and the preserving costs are 5 % of the inventory value. The cafeteria operates 250 working days a year and the delivery time is 5 days.

(a) Calculate the economic order quantity, the reorder point and graphically represent the inventory model
(b) The cafeteria has negotiated the prices of the soft drinks with its supplier in accordance with the number of boxes ordered (see Table 5.1). Determine the optimum quantity of orders and the number of orders to be made every year.

Solution

(a) Calculate the economic order quantity, the reorder point and graphically represent the inventory model.

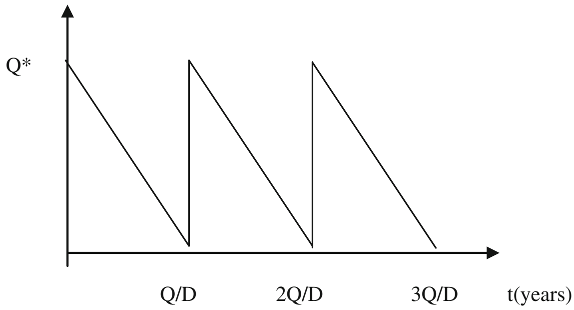
Economic order quantity:

$$Q^* = \text{EOQ} = \sqrt{\frac{2\text{KD}}{h}} = \sqrt{\frac{2 \cdot 20 \cdot 3600}{3 \cdot 0.05}} = 979.7 = 979 \text{ or } 980 \text{ boxes}$$

Reorder point:

$$T_s = 250 \cdot 980/3600 = 68.06 \text{ days}; \quad T_s = 5 \text{ days}$$
$$T_r > T_s \rightarrow P_p = D \cdot T_s = 3600/250 \cdot 5 = 72 \text{ boxes}$$

Graphic representation of the inventory model:



where $Q^* = 979.7$ boxes/order.
if $Q = Q^*$:

$Q/D = 0.27$/year
No. of orders $= D/Q = 3.67$ orders/year

(b) The cafeteria has negotiated the prices of the soft drinks with its supplier in accordance with the number of boxes ordered (see Table 5.1). Determine the optimum quantity of orders and the number of orders to be made every year.

$$Q(_{\text{Precio}=2}) = \sqrt{\frac{2 \cdot 20 \cdot 3600}{2 \cdot 0.05}} = 1200 \,(\text{unacceptable})$$
$$Q(_{\text{Precio}=2.5}) = \sqrt{\frac{2 \cdot 20 \cdot 3600}{2.5 \cdot 0.05}} = 1073.3 \,(\text{acceptable})$$

Therefore, we have to verify the total cost for $Q = 2000$ and $Q = 1073.3$:

Table 5.1 Price of the soft drinks depending on the order volume

No. of boxes ordered	Price per box ($)
$0 \leq q < 1000$	3
$1000 \leq q < 2000$	2.5
$q \geq 2000$	2

$$\mathrm{CT}(Q=2000)=\mathrm{Ca}+\mathrm{Cp}+\ \mathrm{Cc}=\frac{Q}{2}h+K\frac{D}{Q}+pD=\$7,336$$

$$\mathrm{CT}(Q=1073.3)=\ \$9,134.16$$

Thus, we have to order 2,000 boxes 1.8 times per year.

5.4 Orders Management in the Footwear Industry

A footwear firm acquires sole number 6,440 from an external supplier which it uses to produce one pair of shoes. The footwear firm expects to produce approximately 100,000 pairs of the shoes which require this sole over 1 year. Demand remains relatively constant all year round. The orders cost is \$25 per order. The inventory cost policy that the footwear firm has traditionally followed involves including 20 % of the purchasing cost as an annual inventory maintenance cost for any article. The price that this footwear firm pays for each pair of sole number 6,440 is \$6.25.

(a) Determine the optimum order quantity that the footwear firm must use in order to minimise its costs
(b) What is the total cost associated with the optimum order quantity?
(c) How many orders will the footwear firm make in 1 year?
(d) What considerations have you made to solve (a), (b) and (c)?
(e) Assume that the footwear firm works 50 weeks a year and 6 days a week. Calculate the reorder point associated with the optimum orders policy by assuming that the time to supply an order is 4 days
(f) Calculate the reorder point if the supply time is 8 days; do the same if it is 10 days.
(g) Assume that the footwear firm has decided to order an order quantity of 4,000 pairs fortnightly. Calculate the costs penalisation that the footwear firm would incur using this policy
(h) Assume that the footwear firm allows stockouts and that the associated cost is \$0.25 per unit per year. Determine the optimum order quantity
(i) What is the maximum inventory level associated with the optimum inventory policy?
(j) Calculate the total cost associated with the optimum policy
(k) How much time is there between orders for the order quantity found in (*h*)?
(l) How many orders will be required according to the optimum inventory policy?

Solution

(a) Determine the optimum order quantity that the footwear firm must use in order to minimise its costs

$$Q^* = \text{EOQ} = \sqrt{\frac{2\text{KD}}{h}} = \sqrt{\frac{2 \cdot 25 \cdot 100000}{0.2 \cdot 6.25}} = 2{,}000 \text{ pairs}$$

(b) What is the total cost associated with the optimum order quantity?

$$\begin{aligned}\text{CT}(Q = Q^*) = \text{Ca} + \text{Cp} + \text{Cc} &= \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 1.25 \cdot 2000/2 + 25 \cdot 100000/2000 + 6.25 \cdot 100000 \\ &= 1250 + 1250 + 625000 = \$627{,}500\end{aligned}$$

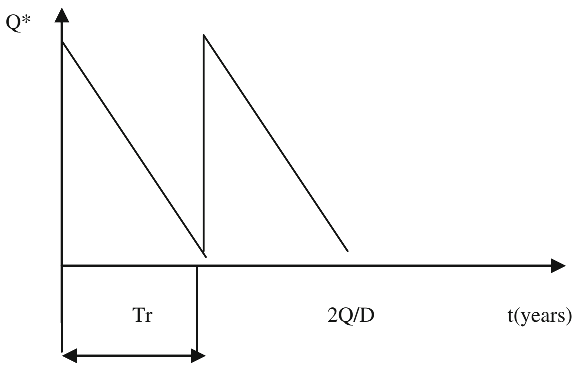
(c) How many orders with the footwear firm make in 1 year?

Frequency of orders = D/Q = 100000/2000 = 50 orders/year

(d) What considerations have you made to solve (a), (b) and (c)?

- Repetitive order: the decision made to order is repetitive in that it is repeated on a regular basis.
- Constant demand: assume that demand occurs with a constant, known speed.
- The delivery time for each order is zero.
- Scarce or excess inventory is not allowed.
- If an order of any size is made, order cost K is incurred.
- The annual unit cost of maintaining a stock is h.

(e) Assume that the footwear firm works 50 weeks a year and 6 days a week. Calculate the reorder point associated with the optimum orders policy by assuming that the time to supply an order is 4 days



where Q^* = 2000 pairs/order
if $Q = Q^*$:

$T_r = Q/D =$ (2,000 pairs/order)/(100,000 pairs/year) $=$ 0.02/year $=$ 6 days
$T_s =$ 4 days; $T_r > T_s \rightarrow$ Reorder point $= T_s \cdot D =$ 4 days (100,000 pairs/year $\cdot$ 1/50 year/week$\cdot$ 1/6 week/day) $=$ 1,333.33 pairs.

(f) Calculate the reorder point if the supply time is 8 days; do the same if it is 10 days.

$$T_r = 6 \text{ days}; T_s = 8 \text{ days}; T_r < T_s \rightarrow \text{Reorder point} = (T_s - E \cdot T_r) \cdot D =$$
$$\text{Reorder point} = (8 - 6) \cdot 333.33 = 666.66 \text{ pairs}$$

$$T_r = 6 \text{ days}; T_s = 10 \text{ days};$$
$$\text{Reorder point} = (10 - 6) \cdot 333.33 = 1333.33 \text{ pairs}$$

(g) Assume that the footwear firm has decided to order an order quantity of 4,000 pairs fortnightly. Calculate the costs penalisation that the footwear firm would incur using this policy

$$\begin{aligned} \text{CT}(Q = 4000) &= \text{Ca} + \text{Cp} + \text{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 1.25 \cdot 4000/2 + 25 \cdot 100000/4000 + 6.25 \cdot 100000 \\ &= 2500 + 625 + 625000 = \$628,125 \end{aligned}$$

$$\text{Difference} = 628125 - 627500 = \$625$$

(h) Assume that the footwear firm allows stockouts and that the associated cost is \$0.25 per unit per year. Determine the optimum order quantity

$$Q^* = \text{EOQ} \cdot \sqrt{\frac{h+s}{s}} = 2000 \cdot \sqrt{\frac{1.25 + 0.25}{0.25}} = 4{,}898.9 \text{ pairs}$$

(i) What is the maximum inventory level associated with the optimum inventory policy?

$$M^* = \text{EOQ} \cdot \sqrt{\frac{s}{h+s}} = 2000 \cdot \sqrt{\frac{0.25}{1.25 + 0.25}} = 816.5 \text{ pairs}$$

(j) Calculate the total cost associated with the optimum policy

$$\text{CT}(Q = Q^*) = \frac{M^2 h}{2Q} + \frac{(Q - M)^2 s}{2Q} + K\frac{D}{Q} + pD = \$626,020$$

(k) How much time is there between orders for the order quantity found in (h)?

if $Q = Q^*$:

$$T = Q/D = 4898.9/100000 = 0.048989/\text{year} = 14.6 \text{ days}$$

(l) How many orders will be required in accordance with the optimum inventory policy?

Frequency of orders = $1/T$ = 20.4 orders/year.

5.5 Components Order Management in an Industrial Firm

An industrial firm orders 500 units per month of a component it needs for its production from its supplier. The purchasing price of each component is \$20. The firm estimates that its internal costs for making an order with its supplier are \$180. The annual costs of maintaining one component unit in stock is estimated to be 10 % of the component unit cost. (Note: A working year is considered to be 48 weeks with 4 weeks to 1 month).

(a) If the firm sends its orders to its supplier on a weekly basis, what is the total cost associated with this orders policy?
(b) Can the firm adopt a better orders policy than the one it has now? If so, what would its annual associated cost be?

The firm has negotiated a discount for volume with its supplier. The component costs will depend on the quantity ordered in each order:

- A 10 % discount will be applied to all the orders with 750 units or more, but less than 1,000 units
- A 20 % discount will be applied to all the orders with 1,000 units or more.

(c) What will be the economic quantity order and what savings (if any) will be achieved annually if we compare the best orders policy previously calculated without applying discounts?
Solution

(a) If the firm sends its orders to its supplier on a weekly basis, what is the total cost associated with this orders policy?

$$\begin{aligned}\mathrm{CT}(Q = 125) &= \mathrm{Ca} + \mathrm{Cp} + \ \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 20 \cdot 0.1 \cdot 125/2 + 180 \cdot (500 \cdot 12)/125 + 20 \cdot (500 \cdot 12) \\ &= \$128.765/\text{year}\end{aligned}$$

(b) Can the firm adopt a better orders policy than the one it has now? If so, what would its annual associated cost be?

$$Q^* = \text{ EOQ} = \sqrt{\frac{2\text{KD}}{h}} = \sqrt{\frac{2 \cdot 180 \cdot 500 \cdot 12}{20 \cdot 0.1}} = 1{,}039.23 \text{ units}$$

$$\begin{aligned} \text{CT}(Q = Q^*) &= \text{Ca} + \text{Cp} + \text{ Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 20 \cdot 0.1 \cdot 1039.23/2 + 180 \cdot (500 \cdot 12/1039.23) + 20 \cdot 500 \cdot 12 \\ &= \$122{,}079/\text{year} \end{aligned}$$

$$\begin{aligned} Q^* &= 1039.23 \text{ units};\ f = \ 6000/1039.23 = 5.77\,\text{orders/year};\ \text{CT} \\ &= \$122{,}079/\text{year} \end{aligned}$$

The firm has negotiated a discount for volume with its supplier. The component costs will depend on the quantity ordered in each order:

- A 10 % discount will be applied to all the orders with 750 units or more, but less than 1,000 units
- A 20 % discount will be applied to all the orders with 1,000 units or more.

(c) What will be the economic quantity order and what savings (if any) will be achieved annually if we compare the best orders policy previously calculated without applying discounts?

$$\begin{aligned} Q_2 &= Q_{(\text{Descuento}=10\,\%)} = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \cdot 180 \cdot 500 \cdot 12}{(20 - 20 \cdot 0.1) \cdot 0.1}} \\ &= 1{,}095.45 \text{ units (unacceptable)} \\ Q_3 &= \ Q_{(\text{Descuento}=20\,\%)} = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \cdot 180 \cdot 500 \cdot 12}{(20 - 20 \cdot 0.2) \cdot 0.1}} \\ &= \ 1{,}161.89 \text{ units (acceptable)} \\ \text{CT} &= \ 180 \cdot 500 \cdot 12/1161.89 + \ 20 \cdot 0.8 \cdot 0{,}1 \cdot 1161.89/2 + \ 500 \cdot 12 \cdot 20 \cdot 0.8 \\ &= \ \$97{,}859.83 \end{aligned}$$

Q_3 = 1n161.89 units (acceptable); CT (Q_3) = \$97,859.83/year. Thus, 1,161.89 units would be ordered per order at a price of \$16/unit, which means approximately \$24,801/year in savings.

5.6 Training Planning for Drivers

The transport firm ORGASA has a training programme for the truck drivers who are to start working for the firm. Provided that the number of drivers per class is lower than or equal to 35, a 6-week training programme costs ORGASA $22,000, which includes driving instructors, equipment, etc. ORGASA's training programme must provide the firm approximately five new drivers each month. Once the training programme has finished, new drivers are paid $1,600/month, but they do not work until a full-time post becomes available. ORGASA considers the $1,600/month it pays each new idle driver to be a necessary possession cost in order to maintain a source of new available drivers to render an immediate service. By considering new drivers to be inventory-type units, establish:

(a) How many students must the training classes include to minimise the new drivers' annual training costs and the annual idle time costs?
(b) How many training classes should the firm give each year?
(c) What are the total annual costs associated with the recommendations made to the firm in (a) and (b)?
(d) Represent the inventory model employed graphically.

Solution

(a) How many students must the training classes include to minimise the new drivers' annual training costs and annual idle time costs?

$$Q^* = \text{EOQ} = \sqrt{\frac{2\text{KD}}{h}} = \sqrt{\frac{2 \cdot 22000 \cdot 512}{1600 \cdot 12}} = 11.73 \cdot 12$$

$Q* = 12$ drivers/class

(b) How many training classes should the firm give each year?

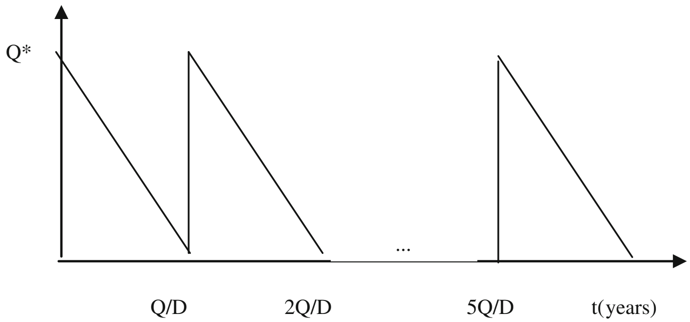
Frequency of Orders = 60/12 = 5 classes/year

(c) What are the total annual costs associated with the recommendations made to the firm in (a) and (b)?

$$\text{CT}(Q = Q^*) = \text{Ca} + \text{Cp} + \ \text{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD$$
$$= \ 1600 \cdot 12 \cdot 12/2 \ + \ 22000 \cdot (5 \cdot 12)/12 = \ \$225{,}200/\text{year}$$

The purchasing cost of the total cost determined has not been considered.

(d) Represent the inventory model employed graphically.



where $Q* = 12$ drivers/class
if $Q = Q^*$:

$$Q/D = 0.2/\text{year}$$
$$\text{No. of classes} = D/Q = 5\,\text{classes/year}$$

5.7 Inventory Management Without Stockouts

A firm's organisation engineer is designing an inventory system and the characteristics are as follows: it is estimated that annual demand is 1,100 product units for the forthcoming year with a constant rate and after considering a 365-day operation. The cost of storing one inventory unit for 1 year is $1. The fixed cost of making an order is $0.5. Every unit cost is $5. Each order received is complete, but with a delay of 1 day from the time the order is made. The firms' policies do not permit stockouts; that is, demand that cannot be met. Based on these conditions, answer the following questions:

(a) What is the lot size to be ordered that minimises the firm's costs?
(b) How many units leave the inventory during a period when the order is being made?
(c) What is the total cost of the system's annual operation?
(d) How long does it take between the lot arriving and when the next lot is ordered?

Solution

(a) What is the lot size to be ordered that minimises the firm's costs?

$$Q^* = \text{EOQ} = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \cdot 0.5 \cdot 1100}{1}} = 33.17\,\text{units/order}$$

(b) How many units leave the inventory during a period when the order is being made?

If $Q = Q^*$:

$$T_r = Q/D = 33.17/1100 = 0.03 \text{ years} = 11 \text{ days}$$
$$T_s = 1 \text{ day} \rightarrow T_s < T_r \rightarrow Pp = D \cdot T_s = (1100/365) \cdot 1 = 3.01 \text{ units}$$

Therefore, $Q^* - Pp = 30.15$ units will have left the inventory

(c) What is the total cost of the system's annual operation?

$$\begin{aligned} \mathrm{CT}(Q = Q^*) &= \mathrm{Ca} + \mathrm{Cp} + \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 1 \cdot 33.17/2 + 0.5 \cdot 1100/33.17 + 1100 \cdot 5 \\ &= \$5,533.17/\text{year} \end{aligned}$$

(d) How long does it take between the lot arriving and when the next lot is ordered?

$T_r - T_s = 10$ days

5.8 Inventory Management of Two Products

Consider the following inventory management situation. The demand of two products, A and B, is constant throughout the 365-day horizon. The products storage cost h is measured in dollars per day and tonne. Furthermore, there is a fixed cost of preparing an order (regardless of size) of K dollars. The demand of the products for the considered horizon D is measured in tons. The parameters for each product are those provided in Table 5.2.

Figure 5.2 depicts the evolution of reserving the two products by assuming that the warehouse is empty at the beginning of the period, and that an order has been made in the initial instant. Mean reservation *Qmed* is calculated as *Q/2,* where Q is the quantity ordered.

Let us assume that the warehouse has a maximum storage capacity of 60 tons.

(a) Formulate a non-linear programming model to decide the order size of each product, Q_A and Q_B, in order to minimise the inventory management policy cost

Table 5.2 Storage cost, order cost and demand of both products

Product	h ($)	K ($)	D (Tm)
A	1	18	365
B	2	25	365

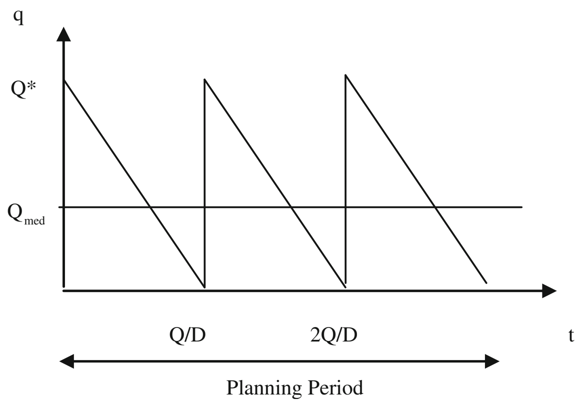


Fig. 5.2 Evolution of reserving time for products A and B

(b) Solve the problem by applying the Kuhn-Tucker conditions
(c) Is it necessary to consider warehouse capacity? Why? Do you think that the warehouse is overlarge for the calculated inventory policy?
(d) Assume that the maximum storage capacity constraint does not exist. Solve the problem using the basic EOQ model for inventory management. What differences do you find when comparing the results provided by the non-linear programming model? What is the reason for this?

Solution

(a) Formulate a non-linear programming model to decide the order size of each product, Q_A and Q_B, in order to minimise the inventory management policy cost

Decision variables

Q_A = Quantity to order of product A
Q_B = Quantity to order of product B

Objective function:

$$\text{Min } z = 18 \cdot \frac{365}{Q_A} + 365 \cdot \frac{Q_A}{2} + 25 \cdot \frac{365}{Q_B} + 730 \cdot \frac{Q_B}{2}$$

Constraints:

$$\frac{Q_A}{2} + \frac{Q_B}{2} \leq 60 \quad \text{(maximum storage capacity)}$$
$$Q_A, Q_B \geq 0$$

(b) Solve the problem by applying the Kuhn-Tucker conditions

$$L(Q, \lambda) = 18\frac{365}{Q_A} + 365\frac{Q_A}{2} + 25\frac{365}{Q_B} + 730\frac{Q_B}{2} + \lambda\left(\frac{Q_A}{2} + \frac{Q_B}{2} - 60\right)$$

Gradient condition:

$$\frac{\partial L}{\partial Q_A} = -\frac{6570}{Q_A^2} + 182.5 + \frac{\lambda}{2} = 0$$
$$\frac{\partial L}{\partial Q_B} = -\frac{9125}{Q_B^2} + 365 + \frac{\lambda}{2} = 0$$

Feasibility condition:

$$\frac{\partial L}{\partial \lambda} = \frac{Q_A}{2} + \frac{Q_B}{2} - 60 \leq 0$$

Orthogonality condition:

$$\lambda\left(\frac{Q_A}{2} + \frac{Q_B}{2} - 60\right) = 0$$

Non-negativity condition:

$$\lambda \geq 0$$

By taking $\lambda = 0 \rightarrow Q_A = 6$ tons/order; $Q_B = 5$ tons/order; $z = \$5{,}840$/year

In order to verify that the solution found is optimum, we should check whether the minimised objective function is convex by using a Hessian matrix (H):

$$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$$

We can see that H is defined as positive, therefore the objective function is convex, which implies that the point found is a global minimum.

(c) Is it necessary to consider warehouse capacity? Why? Do you think that the warehouse is overlarge for the calculated inventory policy?

No, because a shadow price of $\lambda = 0$ is obtained, which indicates that any additional tonne that increases the warehouse capacity does not reduce the current cost at all. Moreover, if we actually consider the warehouse capacity used with the calculated policy, we obtain:

Of the maximum warehouse storage capacity constraint:

$6/2 + 7.07/2 = 6.535 \leq 60$, can be taken as the warehouse being overlarge.

(d) Assume that the maximum storage capacity constraint does not exist. Solve the problem using the basic EOQ model for inventory management. What differences do you find when comparing the results provided by the non-linear programming model? What is the reason for this?

$$Q_A = \sqrt{\frac{2 \cdot 18 \cdot 365}{365}} = 6 \text{ Tm/order}$$

$$Q_B = \sqrt{\frac{2 \cdot 25 \cdot 365}{365 \cdot 2}} = 5 \text{ Tm/order}$$

The same results as with the non-linear programming model are obtained because the maximum warehouse capacity constraint does not actually influence the results.

5.9 Inventory Management of Cold Water Dispensers

BAC is the local distributor of the sources of RFG cold water. An RFG cold water dispenser costs BAC 320$, and (without considering inventory costs) BAC estimates that it would obtain a gross profit of 80$ for each dispenser sold. The cost of making an order of RFG dispensers is 100$, and BAC employs a 20 % annual inventory maintenance cost rate in relation to the purchasing price (and not its sales price). If a stockout should occur in BAC's inventory, BAC estimates a cost of 50$ per unmet dispenser throughout the planning horizon. The weekly mean demand is 75 units, and the supply time is considered null. By considering a 52-week year, work out:

(a) The optimum inventory policy for RFG dispensers
(b) The number of weeks between two consecutive orders. Graphically draw the inventory model employed
(c) The percentage of customers whose order will be delayed or rescheduled
(d) The maximum number of weeks that an unsatisfied customer must wait
(e) BAC's annual profit from selling RFG dispensers.

Solution

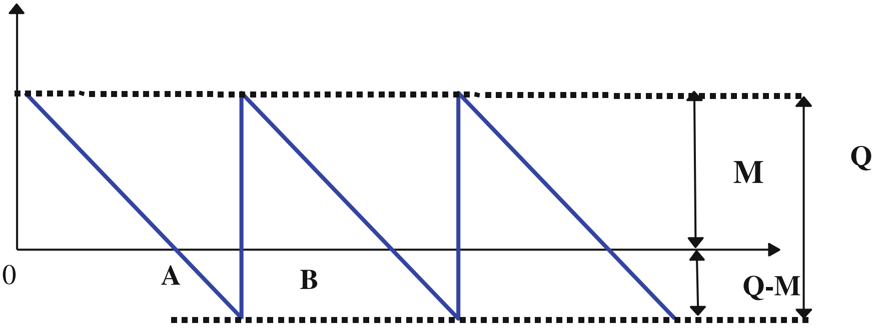
(a) The optimum inventory policy for RFG dispensers

$$\text{EOQ} = \sqrt{\frac{2\text{KD}}{h}} = \sqrt{\frac{2 \cdot 100 \cdot 75 \cdot 52}{320 \cdot 0.2}} = 110.40 \, \text{units/order}$$

$$Q^* = \text{EOQ} \cdot \sqrt{\frac{h+s}{s}} = 110.40 \cdot \sqrt{\frac{320 \cdot 0.2 + 50}{50}} = 166.70 \, \text{units/order}$$

$F = D/Q^* = 23.40$ orders a year

(b) The number of weeks between two consecutive orders. Graphically draw the inventory model employed



If $Q = Q^*$:
$0B = Q/D = 2.22$ weeks

(c) The percentage of customers whose order will be delayed or rescheduled

$$M^* = \text{EOQ} \cdot \sqrt{\frac{s}{h+s}} = 110.40 \cdot \sqrt{\frac{50}{320 \cdot 0.2 + 50 + 0.25}} = 73.11 \text{ units}$$

If $Q = Q^*$:

$$(Q - M)/Q = (166.70 - 73.11)/166.70 = 56.14\ \%$$

(d) The maximum number of weeks that an unsatisfied customer must wait

If $Q = Q^*$:

$$(Q/D) \cdot (Q - M)/Q = Q - M/D = (166.70 - 73.11)/(75 \cdot 52) \cdot 52$$
$$= 1.25 \text{ weeks}$$

(e) BAC's annual profit from selling RFG dispensers

$$\text{CT}(Q = Q^*) = \text{Ca} + \text{Cs} + \text{Cp} + \text{Cc} = \frac{M^2 h}{2Q} + \frac{(Q - M)^2 s}{2Q} + K\frac{D}{Q} + pD$$
$$= (320 \cdot 0.2) \cdot 73.112/(2 \cdot 166.70) + 50 \cdot (166.70 - 73.11)$$
$$2/2 \cdot 166.70 + 100 \cdot 75 \cdot 52/166.70 + 320 \cdot 75 \cdot 52$$
$$= 1,252,679.18 \text{ dollars/year}$$

Total income = 75 · 25 · (320 + 80) = 1,560,000 \$/year

Annual profit = 307,320.82\$/year.

5.10 Orders Management in the Footwear Industry

A footwear firm is considering changing its packaging supplier. Its current supplier charges 10 centimes of a dollar per packaging unit and a minimum order quantity of 490 packing boxes is required. Annual demand is 12,000 packing boxes, the order cost is 20 centimes of a dollar, and the annual inventory maintenance unit cost is 20 % of the purchasing unit cost. A new supplier offers the price of 9 centimes of a dollar in lots of 4,000 packing boxes. What decision do you recommend the footwear firm to make? Give your reasons.

Solution

Calculate the EOQ when the cost is 9 centimes of a dollar:

$$\mathrm{EOQ}(p=9)=\sqrt{\frac{2\cdot 12{,}000\cdot 20}{1.8}}=516\,\text{boxes/order}$$

As this $EOQ = 516$ packing boxes is not feasible because it is lower than 4,000, so calculate the total cost for each price to make a decision.

$$\mathrm{CT}(p=10)=\frac{12{,}000}{490}(20)+\frac{490}{2}(2)+10(12{,}000)=120{,}980\text{ centimes of a dollar}$$
$$\mathrm{CT}(p=9)=\frac{12{,}000}{4{,}000}(20)+\frac{4000}{2}(1.8)+9(12{,}000)=101{,}660\text{ centimes of a dollar}$$

Lots of 4,000 units of packing boxes should be purchased as they offer an annual saving of 19,320 centimes of a dollar.

5.11 Orders Management of X-Ray Plaques in a Hospital

A organisation engineer in a hospital did a costs study and established that the most important costs are X-ray plates. This hospital employs 250 plates every month. The order cost (including transport) is \$10. As the X-ray plates must remain at a controlled temperature and must be thoroughly cleaned, monthly storage costs are very high. These costs and the discounts for quantities are provided in Table 5.3.

Determine the most economic order quantity and the frequency of the orders to be made.

Table 5.3 Costs and discounts per order size

Quantity to order	Costs per unit (\$)	
	Purchase (\$/unit)	Storage (\$/unit and month)
0–199	14	10
200–999	13	9.50
1,000 or more	12	9

Solution

The EOQ value is calculated of \$9/unit and month for the storage cost:

$$Q_3 = \sqrt{\frac{2 \cdot 10 \cdot 250}{9}} = 23.57 \rightarrow \text{It is unacceptable because } Q_3 < 1000$$

For a storage cost of \$9.5/unit and month:

$$Q_2 = \sqrt{\frac{2 \cdot 10 \cdot 250}{9,5}} = 22.94 \rightarrow \text{It is unacceptable because } Q_2 < 200$$

For a storage cost of \$10/unit and month:

$$Q_1 = \sqrt{\frac{2 \cdot 10 \cdot 250}{10}} = 22.36 \rightarrow \text{it is acceptable}$$

Therefore, we must calculate the total cost for $Q = 22.36$, $Q = 200$ and $Q = 1000$.

$$\begin{aligned} \mathrm{CT}(Q = 22.36) &= \mathrm{Ca} + \mathrm{Cp} + \ \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 10 \cdot 22.36/2 \ + \ 10 \cdot 250/22.36 + 14 \cdot 250 \\ &= 3723.61 \,\text{dollars/month} \end{aligned}$$

$$\begin{aligned} \mathrm{CT}(Q = 200) &= \ 10 \cdot 250/200 + 9.5 \cdot 200/2 + 13 \cdot 250 \ = \ 4212.5 \,\text{dollars/month} \\ \mathrm{CT}(Q = 1000) &= 10 \cdot 250/1000 + 9 \cdot 1000/2 + 12 \cdot 250 = \ 7502.5 \,\text{dollars/month} \end{aligned}$$

The optimum quantity to order is 22.36 units with a total of 11.18 orders/month.

5.12 Orders Management with Discount

A firm employs 20 units of an article every day. This article costs the firm \$25/unit, but it is offered a 10 % discount for lots of 150 units or more. The preparation cost to order one lot is \$50 and the storage cost per unit per day is \$0.30.

(a) Should the firm accept this discount?
(b) Determine the percentage range of the discount on the price so that when it is offered for lots of 150 units or more, it will not become a financial advantage for the firm.

Solution

(a) Should the firm accept this discount?

$d = 20$ units/day

$p = \$25/\text{unit if } 0 \geq q > 150$
$p = \$22.5/\text{unit if q} \geq 150$
$K = \$50$
$h = \$0.3/\text{unit and day}$

$$Q^* = \sqrt{\frac{2Kd}{h}} = \sqrt{\frac{2 \cdot 50 \cdot 20}{0.3}} = 81.65\,\text{units/order}$$

$$\text{CT}(Q = Q^*) = \frac{K \cdot d}{Q} + h\frac{Q}{2} + pd = \frac{50 \cdot 20}{81.65} + 0.3 \cdot \frac{81.65}{2} + 25 \cdot 20 = \$524.49/\text{day}$$

$$\text{CT}(Q = 150) = \frac{K \cdot d}{Q} + h\frac{Q}{2} + pd = \frac{50 \cdot 20}{150} + 0.3 \cdot \frac{150}{2} + 22.5 \cdot 20 = \$479.17/\text{day}$$

Yes, and 150 units/order should be ordered.

(b) Determine the percentage range of the discount on the price so that when it is offered for lots of 150 units or more, it will not become a financial advantage for the firm.

$$\text{CT}(Q = 150) = \frac{K \cdot d}{Q} + h\frac{Q}{2} + pd = \frac{50 \cdot 20}{150} + 0.3 \cdot \frac{150}{2} + p \cdot 20 = \$524.49/\text{day}$$
$$p = 24.77$$

As from a discount of over 0.94 %.

5.13 Paper Inventory Management

The Planetas Publishing House, which is currently placing a monthly order of paper, has studied the behaviour of 70-gram book paper in the last 12 months and found that its demand was 10, 11, 10, 9, 10, 11, 9, 10.5, 10, 9, 9 and 11.5 tonnes each month. It is estimated that the purchasing price is to remain at 2,300.000 dollars per tonne, its order cost at 500,000 dollars, with an annual charge policy of 15 % of the unit cost on inventory management, plus 55,000 extra dollars to store every tonne.

(a) Calculate the inventory management model under these conditions.
(b) If the supplier offers a 10 % discount for purchases over 30 tonnes and a discount of 11 % for purchases of 60 tonnes or more, how would the inventories policy change?

(c) If apart from the discount a time which allows the storage unit cost to lower, but only for storage per tonne, is achieved, how would the inventories policy change?

Solution

(a) Calculate the inventory management model under these conditions.

The first point that we should observe is how demand behaves, which is relatively constant. Thus it can be assumed that the model behaves in accordance with the EQO model parameters with the following input data:

$D = 120$ tons/year
$K = \$500{,}000$
$p = \$2{,}300{,}000$/tonne
$h = 2300000 \cdot 0.15 + 55000 = \$400{,}000$ ton/year

Therefore:

$$Q = \sqrt{\frac{2\mathrm{KD}}{h}} = \sqrt{\frac{2 \cdot 120 \cdot 500,000}{400{,}000}} = 17.32 \text{ tonnes per order}$$

N = 120/17.32 = 6.93 orders in 1 year

$$\begin{aligned}\mathrm{CT}(Q = 17.32) &= \mathrm{Ca} + \mathrm{Cp} \ + \ \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 400{,}000 \cdot 17.32/2 + 500{,}000 \cdot 120/17.32 + 120 \cdot 2{,}300{,}000 \\ &= \ \$282{,}928{,}203/\mathrm{year}\end{aligned}$$

$$\begin{aligned}\mathrm{CT}(Q = 10) &= \mathrm{Ca} + \mathrm{Cp} \ + \ \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 400{,}000 \cdot 10/2 + 500{,}000 \cdot 120/10 + 120 \cdot 2{,}300{,}000 \\ &= \$284{,}000{,}000/\mathrm{year}\end{aligned}$$

With this inventory policy, we observe the firm saves \$1,071,797/year in inventory costs.

(b) If the supplier offers a 10 % discount for purchases over 30 tonnes and a discount of 11 % for purchases of 60 tonnes or more, how would the inventories policy change?

For $Q = 30$:

$$\begin{aligned}\mathrm{CT}_1 &= 120 \cdot 2{,}300{,}000 \cdot 0.90 + 120 \cdot 500{,}000/30 + 30 \cdot 400{,}000/2 \\ &= \ \$256{,}400{,}000\end{aligned}$$

For $Q = 60$:

$$\begin{aligned}\mathrm{CT}_2 &= 120 \cdot 2{,}300{,}000 \cdot 0.89 + 120 \cdot 500{,}000/60 + 60 \cdot 400{,}000/2\\ &= \$258{,}000{,}000\end{aligned}$$

Therefore, the firm should accept the 10 % discount because if it decides on the scale offering the 11 % discount, the additional inventory management costs are higher than the profits that would be obtained with the lower purchasing value.

(c) If apart from the discount a time which allows the storage unit cost to lower, but only for storage per tonne, is achieved, how would the inventories policy change?

The question contains an aphorism, which is often valid: "it is not the price that is important, but time"; for this particular case, drastically changing the preserving cost implies recalculating the whole model with the preserving cost of 55,000 dollars, which would give the following results:

$$Q = \sqrt{\frac{2 \cdot 120 \cdot 500{,}000}{55{,}000}} = 46.71 \text{ tonnes per order}$$

$N = 120/46.71 = 2.57$ orders in 1 year

$$\begin{aligned}\mathrm{CT}(Q = 46.71) &= \mathrm{Ca} + \mathrm{Cp} + \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD\\ &= 55{,}000 \cdot 46.71/2 + 500{,}000 \cdot 120/46.71 + 2{,}300{,}000 \cdot 0.9 \cdot 120\\ &= \$250{,}969{,}047/\text{year}.\end{aligned}$$

5.14 Inventory Planning

The firm ORGASA is in the process of improving its industrial organisation practices with a view to increasing its productivity. ORGASA maintains several thousands of inventory articles, and it leaves the decision of how much inventory it should maintain to the operations personnel. Reducing inventory costs is one of the key components of the overall strategy. In order to estimate the reduction in the potential cost, a representative product was selected which offers the following data:

- Annual demand: 1,600 units
- Order cost: 25 \$/order
- Unit cost of maintaining inventories per year: 25 % of the article cost
- Article cost: 8\$

Currently, the total order cost and the inventory maintenance cost for this article for one year is 500\$.

(a) The first work hypothesis was to use the EOQ model to minimise costs. How much money would ORGASA save if it employed the EOQ model?
(b) ORGASA is also considering making efforts to reduce the inventory maintenance costs from 22 to 25 %. How much more would it save by making this effort?
(c) The purchasing department has been negotiating with the supplier. The supplier of this product has offered ORGASA a discount of 2, 5 and 10 % if the order quantity is at least 400, 800 or 1,600, respectively. What level of discount should ORGASA select and how much would it save?
(d) Assume that ORGASA can allow stockouts at a cost of 4$/year for the considered article. What total cost would this new inventory model imply for the firm?

Solution

(a) The first work hypothesis was to use the EOQ model to minimise costs. How much money would ORGASA save if it employed the EOQ model?

$$Q^* = \mathrm{EOQ} = \sqrt{\frac{2\mathrm{KD}}{h}} = \sqrt{\frac{2 \cdot 25 \cdot 1600}{8 \cdot 0.25}} = 200\,\text{articles/order}$$

If the total purchasing cost is not considered, then:

$$\begin{aligned}\mathrm{CT}(Q = Q^*) &= \mathrm{Ca} + \mathrm{Cp} = \frac{Q}{2}h + K\frac{D}{Q} = 8 \cdot 0.25 \cdot 200/2 + 25 \cdot 1600/200 \\ &= \$400/\text{year}\end{aligned}$$

This would imply a saving of \$100/year as compared to the \$500 of the current inventory policy.

(b) ORGASA is also considering making efforts to reduce the inventory maintenance costs from 22 to 25 %. How much more would it save by making this effort?

$$\mathrm{CT}(Q = Q^*) = 8 \cdot 0.22 \cdot 200/2 + 25 \cdot 1600/200 = \$376/\text{year}$$

Difference = 400 – 376 = \$24/year

(c) The purchasing department has been negotiating with the supplier. The supplier of this product has offered ORGASA a discount of 2 %, 5 % and 10 % if the order quantity is at least 400, 800 or 1,600, respectively. What level of discount should ORGASA select and how much would it save?

$$\begin{aligned}
\mathrm{CT}(Q=Q^{*}) &= 8\cdot 0.25\cdot 200/2+25\cdot 1600/200+1600\cdot 8=\$13{,}200\\
\mathrm{CT}(Q=400) &= (8-0.02\cdot 8)\cdot 0.25\cdot 400/2+25\cdot 1600/200\\
&\quad +1600\cdot(8-0.02\cdot 8)=\$13{,}136\\
\mathrm{CT}(Q=800) &= (8-0.05\cdot 8)\cdot 0.25\cdot 800/2+25\cdot 1600/400+1600\cdot\\
&\quad (8-0.05\cdot 8)=\$13{,}020\\
\mathrm{CT}(Q=1600) &= (8-0.1\cdot 8)\cdot 0.25\cdot 1600/2+25\cdot 1600/1600+1600\cdot\\
&\quad (8-0.1\cdot 8)=\$12{,}985
\end{aligned}$$

A saving of $= 13200 - 12985 = \$215$

The best decision that ORGASA can make is to accept a 10 % discount and send its supplier an order of 800 units each time. This would imply an annual saving of \$215 in annual costs.

(d) Assume that ORGASA can allow stockouts at a cost of 4 \$/year for the considered article. What total cost would this new inventory model imply for the firm?

$$Q^{*}=\mathrm{EOQ}\cdot\sqrt{\frac{h+s}{s}}=200\cdot\sqrt{\frac{8\cdot 0.25+4}{4}}=244.95\ \text{articles/order}$$

$$M^{*}=\mathrm{EOQ}\cdot\sqrt{\frac{s}{h+s}}=200\cdot\sqrt{\frac{4}{8\cdot 0.25+4}}=163.30\ \text{articles}$$

$$\begin{aligned}
\mathrm{CT}(Q=Q^{*}) &= \mathrm{Ca}+\mathrm{Cs}+\mathrm{Cp}+\mathrm{Cc}=\frac{M^2h}{2Q}+\frac{(Q-M)^2s}{2Q}+K\frac{D}{Q}+pD\\
&= 8\cdot 0.25\cdot(163.30)^2\big/(2\cdot 244.95)+4(244.95-163.3)^2\big/(2\cdot 244.95)\\
&\quad +25\cdot 1600/244.95+1600\cdot 8=\$13{,}126{,}60/\text{year}
\end{aligned}$$

5.15 Inventory Management in a Textile Firm

A textile company in Zaragoza purchases an approximate value of articles of 81,000 dollars from a supplier each year. It is estimated that the cost of making orders is 25 dollars per purchasing order and that the inventory maintenance cost ratio is 20 %

(a) How many dollars in the value of articles should be purchased in each order?
(b) If the supplier offers the firm a 2 % discount if it purchases at least 9,000 dollars in the value of articles in each order, should the firm in Zaragoza accept this offer?

Solution

(a) How many dollars in the value of articles should be purchased in each order?

In this case, demand units are considered monetary units.

$$Q^* = \sqrt{\frac{2 \cdot 25 \cdot 81000}{0.2}} = 4{,}500\,\text{dollars/order}$$

(b) If the supplier offers the firm a 2 % discount if it purchases at least 9,000 dollars in the value of articles in each order, should the firm in Zaragoza accept this offer?

It is calculated that Q* for a discount of 2 %:

$$Q^* = \sqrt{\frac{2 \cdot 25 \cdot (81000 - 0.02 \cdot 81000)}{0.2}} = 4,454.72\,\text{dollars/order}$$

As this is unacceptable, the total cost is established for $Q = 4500$ and $Q = 9000$ by bearing in mind that the purchasing value of articles for the second case would be 79,380 dollars (2 % discount).

$$\begin{aligned}\text{CT}(Q = 4500) &= \text{Ca} + \text{Cp} + \text{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD = 0.2 \cdot 4500/2 \\ &\quad + 25 \cdot 81000/4500 + \ 81000 = \$81{,}900 \\ \text{CT}(Q = 9000) &= \text{Ca} + \text{Cp} + \text{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD = 0.2 \cdot 9000/2 \\ &\quad + 25 \cdot 79830/9000 + 79830 = \$80{,}500.5\end{aligned}$$

Therefore, it is worth accepting the offer made by the supplier.

5.16 Inventory Management in a Supermarket

Supercur sells some 200,000 litres of milk a year. Milk is purchased at 4 Cm of a dollar per litre. Inventory maintenance costs are estimated to be 1.4 Cm of a dollar per litre and year. Making an order implies a management cost of 35 centimes of a dollar. By bearing in mind that milk expires in only 5 days:

(a) How many litres should be requested in each order?
(b) How many orders should be placed yearly?
(c) What would the annual saving be in inventory management costs if milk was not a perishable product?

Solution

(a) How many litres should be requested in each order?

D = 200,000 L/year
p = 4 centimes of a \$/L
h = 1.40 centimes of a \$/L/year
K = 35 centimes of a \$/order
Expiry date = 5 días

$$Q^* = \mathrm{EOQ} = \sqrt{\frac{2\mathrm{KD}}{h}} = \sqrt{\frac{2 \cdot 200{,}000 \cdot 35}{1.40}} = 3{,}162.28\ \mathrm{L/order}$$

$$T_r = 365 \cdot Q^*/D = 365 \cdot 3162.28/200{,}000 = 5.77\ \text{days}$$

With a Q^* of 3,162.28 litres, an order should be placed every 5.77 days. However, milk expires in 5 days, so a total of 2,739.73 litres (200,000/365·5 = 2,739.73 litres) should be ordered every 5 days.

(b) How many orders should be placed yearly?

$$f = \frac{200{,}000}{2{,}739.73} = 73\ \text{orders}$$

(c) What would the annual saving be in inventory management costs if milk was not a perishable product?

$$\begin{aligned}\mathrm{CT}(Q = 3162.28) &= \mathrm{Ca} + \mathrm{Cp} + \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 3{,}162.28 \cdot 1.4/2 + 35 \cdot 200{,}000/3162.28 + 800{,}000 \\ &= 804{,}427.189\ \text{centimes of a \$}\end{aligned}$$

$$\begin{aligned}\mathrm{CT}(Q = 2739.73) &= \mathrm{Ca} + \mathrm{Cp} + \mathrm{Cc} = \frac{Q}{2}h + K\frac{D}{Q} + pD \\ &= 2{,}739.73 \cdot 1.4/2 + 35 \cdot 200{,}000/2739.73 + 800{,}000 \\ &= 804{,}472.807\ \text{centimes of a \$}\end{aligned}$$

Therefore, there would be an annual saving of 45.62 centimes of a dollar if we do not consider that the product is perishable.

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Chapter 6
Queueing Theory

Abstract This chapter begins with an introduction to the Queueing Theory. Then it proposes the formulation of a varied set of Queueing Theory problems with their corresponding solutions. Here problems relating to steady-state queueing models performance measures with, for example, one queue, one serve and an infinite population, one queue, one server and a finite population, one queue, multiple parallel servers and an infinite population, one queue, multiple parallel servers and a finite population, one queue and multiple serial servers, are put forward. The solution is carried out by means of the corresponding analytical formulae. Different problems relating to Industrial Organisation Engineering and the management domain are set out and their solutions are provided.

6.1 Introduction

The Queueing Theory began with the work of Erlang, the Danish engineer (1909, 1917), in the telephone industry at the beginning of the twentieth century. He did detailed studies of the most usual models in which the distribution of deliveries to the system and the service time are both known, and they belong to well-established categories, and this study is totally characterised. Later, Karmarkar (1987) used the Queueing Theory to determine delivery times according to the available capacity and the balance between current inventory costs and time and preparation costs.

A queue is formed by the aleatory arrival of customers who reach somewhere to receive a service provided by a server. The object of the Queueing Theory is to quantitatively and qualitatively characterise a queue or a waiting line by means of a mathematical analysis.

There are two types of costs in any queueing system: a social cost, which refers to the waiting time for a service, and a service cost, which is associated with the consumption of the resources that the service requires. Then it is a matter of

R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5_6,

determining the suitable levels of certain queueing systems parameters that balance the social waiting cost with the cost associated with the resources consumed.

The quantification of the waiting line can be done, on the one hand, by a mathematical analysis which provides optimum results, despite requiring some very strict assumptions as far as the nature of the arrivals of customers, number of servers and the system structure are concerned. On the other hand, it is feasible to resort to simulation, which is a more general application, although it does not provide optimum values and is more costly. In this way, the mathematical-type technical resources needed to carry out these analyses mean that simulation tends to be the usual study method followed when queueing processes involve a degree of complexity.

The main elements in a mathematical queueing model are: customers, who can be finite or infinite; number of servers (one or more). Customers are generated from a source which produces the finite or infinite population.

The arrival of customers is represented by the time between arrivals. This is a constant or an independent aleatory variable with a known or unknown probability distribution. The mathematical analysis, which is what this book chapter focuses on, considers a Poisson arrival distribution. Dependent arrivals or other distributions can be considered for simulation.

The service is represented by the service time. It is a constant or an aleatory variable that can be dependent or independent, with a known or unknown probability distribution. The mathematical focus is used when the service time is constant and has a negative exponential or Erlang distribution (Taha 2010).

The waiting form can be of one or several queues, with or without the option of changing queue. The queue discipline can be FIFO (first in, first out), priorities, LIFO (last in, last out) or aleatory. The service stations structure can be serial, parallel or mixed. Finally, system stability can be steady (birth and death) or transitory.

Some considered steady-state queueing performance measures are:

λ	Average number of arrivals at the system per time unit
μ	Average number of services per time unit
$\rho = (\lambda/\mu) \leq 1$	The system utilisation factor with a server
If $\rho > 1$	more customers arrive than can be attended and an infinite queue forms. More servers have to be added to the system
S	Number of servers in the system
$\rho = \lambda/(S \cdot \mu)$	System utilisation factor with multiple servers
T_s	Expected value of the waiting time in the queue before receiving the service
Tw	Expected value of the time to remain in the system
$1/\lambda$	Average time that passes between two consecutive arrivals
$1/\mu$	A customer's average service time

L	Expected value of the number of people in the queue
W	Expected value of the number of people in the system
$P_m(t)$	Probability when arriving at the queue at time t of there being m people in system S receiving the service, and m-S people in the queue
$P_o(t)$	Probability when arriving at the queue at time t of the system being empty
$P(W > Z)$	Probability that the number of people in system W is greater than Z
$P(T_s > g)$	Probability that total queueing time T_s is longer than the g time units
$P(T_w > h)$	Probability that total waiting time T_w in the system is longer than the h time units.

This book chapter considers the following steady-state queueing systems (Feller 1965):

- One queue, one server and an infinite population:

$$P_{0(t)} = \frac{\mu - \lambda}{\mu}$$
$$P_{m(t)} = \rho^m(1 - \rho)$$
$$W = \frac{\lambda}{\mu - \lambda}$$
$$L = \frac{\lambda^2}{\mu(\mu - \lambda)}$$
$$T_s = \frac{\lambda}{\mu(\mu - \lambda)}$$
$$T_w = T_s + \frac{1}{\mu}$$
$$P(W > Z) = \rho^{(Z+1)}$$
$$P(Ts > g) = \rho \cdot e^{-\mu(1-\rho)\cdot g}$$
$$P(Tw > h) = e^{-\mu(1-\rho)\cdot h}$$

- One queue, one server and a finite population:

$$
\begin{aligned}
&\frac{P_n(t)}{P_0(t)} = \frac{m!}{(m-n)!} \cdot \left(\frac{\lambda}{\mu}\right)^n \\
&P_0(t) = \frac{1}{\sum\limits_{n=0}^{m} \frac{P_n(t)}{P_0(t)}} \\
&L = m - \left(\frac{\mu+\lambda}{\lambda}\right) \cdot (1 - P_0(t)) \\
&W = L + (1 - P_0(t)) \\
&T_s = \frac{L}{\mu(1 - P_0(t))} \\
&T_w = T_s + \frac{1}{\mu} \\
&P_n(t) = P_o(t) \frac{m!}{(m-n)!} \cdot \left(\frac{\lambda}{\mu}\right)^n
\end{aligned}
$$

where m represents the population which requires a given service and n ($n < m$) denotes the customers of the population requesting the service.

- One queue, multiple servers in parallel and an infinite population:

$$
\begin{aligned}
&\frac{\lambda}{S\mu} < 1 \\
&P_0(t) = \frac{1}{\sum\limits_{m=0}^{S-1} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{(S\mu-\lambda)}} \\
&L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^S}{(S-1)!(S\mu-\lambda)^2} P_o(t) \\
&W = L + \frac{\lambda}{\mu};\ T_s = \frac{L}{\lambda}; \qquad T_w = T_s + \frac{1}{\mu} \\
&P_m(t) = \frac{P_0(t)}{m!}\left(\frac{\lambda}{\mu}\right)^m \qquad m < S \\
&P_m(t) = \left(\frac{P_0(t)}{S!S^{(m-S)}}\right)\left(\frac{\lambda}{\mu}\right)^m \qquad m > S
\end{aligned}
$$

where m represents the number of people in the system and S is the number of servers.

- One queue, multiple serial servers and an infinite population:

$$P(L_1 = Z_1, L_2 = Z_2, \ldots, L_n = Z_n) = (1 - \rho_1) \cdot \rho_1^{z_1} \cdot (1 - \rho_2) \cdot \rho_2^{Z_2} \ldots \left(1 - \rho_n^{Z_n}\right)$$

$$\rho_i = \frac{\lambda}{\mu_i} < 1$$

$$W = W_1 + W_2 + \ldots + W_n = \sum_{i=1}^{n} \left(\frac{\rho_i}{1 - \rho_i}\right)$$

$$Ts = Ts_1 + Ts_2 + \ldots + Ts_n = \sum_{i=1}^{n} \left(\frac{1}{1 - \rho_i}\right)\left(\frac{1}{\mu_i}\right)$$

where n represents the number of serial servers. In this case, the capacity of the space for the wait between service stations is considered unlimited.

The objective of this book chapter is to help to learn how to formulate and solve analytical Queueing Theory models and to show some of their applications in the industrial engineering and management domain. Therefore, some management problems are modelled by the analytical formulation of various steady-state queueing systems.

After reading this chapter, readers should be able to quantitatively and qualitatively characterise a queue by a mathematical analysis, and determine the suitable levels of certain queueing system parameters which balance the social waiting cost with the cost associated with the resources consumed.

Selected books for further reading can be found in the References section.

6.2 System with Three Stations

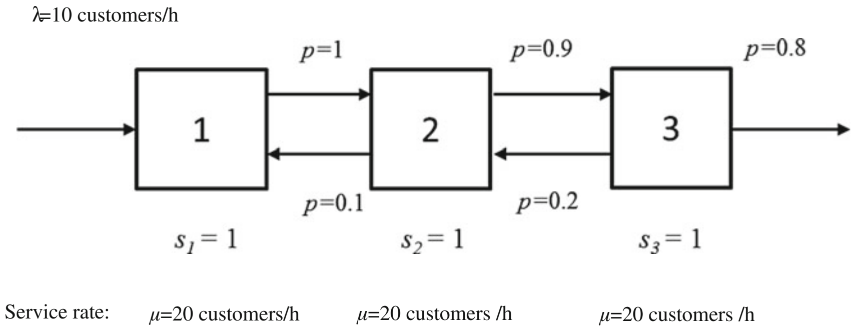
A queueing system consists of three serial stations. Each station has a single server which can process an average of 20 jobs per hour. The processing times in each station are exponential. On average, ten jobs per hour arrive at station 1. The times between arrivals are exponential. When a job completes its service in station 2, there is a probability of 0.1 that it returns to station 1 and a probability of 0.9 that it moves to station 3. When a job completes its service in station 3, there are probabilities of 0.20 that it returns to station 2 and of 0.80 that it leaves the system. All the jobs that complete their service in station 1 immediately move to station 2.

(a) Calculate the λ values for each station.
(b) Determine the fraction of time that each service is occupied (utilisation factor).
(c) Determine the expected number of jobs in the system.
(d) Estimate the average time that a job takes in the system.

Solution

(a) Calculate the λ values per station.

The considered system can be modelled as shown below:



According to the above problem, we have an open queue with three serial stations.

The data that they provide us with to solve the problem are the following:

- On average, 10 jobs arrive at the queue hourly; that is, $\lambda = 10$ jobs/h.
- The number of servers is: $s_1 = 1$; $s_2 = 1$; $s_3 = 1$
- The average jobs covered, that is, parameter μ is:
- $\mu_1 = 20$ jobs/h; $\mu_2 = 20$ jobs/h; $\mu_3 = 20$ jobs/h
- The probability that one job moves from station 1 to station 2 is 1.
- The probability that one job moves from station 2 to station 1 is 0.1.
- The probability that one job moves from station 2 to station 3 is 0.9.
- The probability that one job returns to station 2 from station 3 is 0.2.
- The probability that one job leaves the station is 0.8.

First, it is necessary to calculate the input rate to each station because the problem provides us with the rate of input into the queue. For this purpose, we consider the following equations system:

$$\lambda_1 = 10 + 0.1\lambda_2$$
$$\lambda_2 = \lambda_1 + 0.2\lambda_3$$
$$\lambda_3 = 0.9\lambda_2$$

where
$\lambda_1 = 11.39$; $\lambda_2 = 13.89$; $\lambda_3 = 12.50$

(b) Determine the fraction of time that each service is occupied (utilisation factor).

By applying the formulae corresponding to this system type, we obtain:

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{11.39}{20} = 0.5695$$

Thus, 56.95 % of the first server time is occupied

$$\rho_2 = \frac{\lambda_2}{\mu_2} = \frac{13.89}{20} = 0.6945$$

Thus, 69.45 % of the second server time is occupied

$$\rho_3 = \frac{\lambda_3}{\mu_3} = \frac{12.50}{20} = 0.625$$

Thus, 62.5 % of the third server time is occupied

(c) Determine the expected number of jobs in the system.

Similarly:

$$W_1 = \frac{\rho_1}{1-\rho_1} = \frac{0.5695}{1-0.5695} = 1.323$$
$$W_2 = \frac{\rho_2}{1-\rho_2} = \frac{0.6945}{1-0.6945} = 2.273$$
$$W_3 = \frac{\rho_3}{1-\rho_3} = \frac{0.625}{1-0.625} = 1.667$$
$$W = W_1 + W_2 + W_3 = 5.26$$

Therefore, the mean number of jobs in the system is 5.26 jobs.

(d) Estimate the average time that a job takes in the system.

$$TW_1 = \frac{1}{\mu_1 - \lambda_1} = \frac{1}{20 - 11.39} = 0.1161$$
$$TW_2 = \frac{1}{\mu_2 - \lambda_2} = \frac{1}{20 - 13.89} = 0.1637$$
$$TW_3 = \frac{1}{\mu_3 - \lambda_3} = \frac{1}{20 - 12.50} = 0.1333$$
$$TW_1 + TW_2 + TW_3 = 0.4131$$

The average time that a job takes in the system is 0.4131 h.

6.3 Planning Bank Cashiers

The manager of a bank must decide how many cashiers must work on Fridays. For every minute a customer must wait in a queue, the manager assumes an incurred cost of 5 euros. On average, two customers arrive at the bank every minute and an average cashier's transaction takes 2 min. One hour of a cashier's work costs the bank 12 euros. To cut the sum of delay and service costs as much as possible, how many cashiers should work on Fridays? It is assumed that customers arrive following a Poisson distribution, and that service times follow a negative exponential distribution.

Solution

This is a queueing problem with one queue, one multiple server and a finite population.

We calculate the parameters:

$$\lambda = 2\,\text{customers/min}$$
$$\mu = 1/2 = 0.5\,\text{customers/min}$$

Let's see what number of cashiers is necessary:

$$\frac{\lambda}{S\mu} < 1 \Rightarrow S > \frac{\lambda}{\mu} = \frac{2}{0.5} = 4$$

So the minimum number of cashiers is five. However, we still do not know the number that provides the minimum cost.

Let's see how much the delay cost (DC) is worth

$$\frac{\text{DC}}{\text{customer}} = 5(\lambda \times T_s) = (10 \times T_s)\ \text{euros/min}$$

Let's look at the service costs (SC) if we pay each cashier 12 euros/h, which means 0.20 euros/min. Therefore:

$$\frac{\text{SC}}{\text{min}} = (20 \times S)\ \text{euros/min}$$

with S being the number of contracted cashiers. Now we have to calculate $T_s = L/\lambda$.

We need to calculate L (average number of people in the queue).

$$L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^S}{(S-1)!(S\mu-\lambda)^2} P_o(t)$$

Everything in the previous formula is known, except *Po(t)*, so we have to calculate:

$$P_0(t) = \frac{1}{\sum_{m=0}^{S-1} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{(S\mu-\lambda)}}$$

$\lambda/\mu = 4$, then *Po(t)* $= 0.013$. Thus, the probability that there is no-one in the system is 2 %.

$L =$ an average of 2.22 people in the queue.

$T_s =$ an average queueing time of 1.11 min.

The total cost of five cashiers is: $CT = 111.1$ euros/min.

It is now necessary to verify the total cost for six cashiers:

The service cost equals $20 \cdot 6 = 120$ euros/min, which exceeds the former case without bearing in mind the delay cost. So the number of cashiers to work on Fridays is five.

6.4 Hospital Pharmacy Planning

The arrival of nurses at the LaSalud hospital pharmacy can be described with a Poisson distribution. Service times use an exponential distribution. The arrival rate is 45 nurses every hour, while each pharmacist takes 72 s to attend to a nurse. The cost to attend to each nurse is \$15/h, whereas each pharmacist earns \$10/h. Calculate the optimum number of pharmacists to be contracted.

Solution

By applying the formulae corresponding to the systems with one queue, one server and an infinite population, and that with one queue, multiple parallel servers and an infinite population, we obtain:

Arrival rate (λ)	45	45	45
Service rate (μ)	50	50	50
Number of servers (S)	1	2	3
Utilisation factor (ρ)	0.90	0.45	0.3
Probability of 0 in the system ($Po(t)$)	0.10	0.38	0.40
Mean number in the queue (L)	8.1	0.23	0.03
Mean queueing time (T_s)	0.18	0.01	0.01
Mean number in the system (W)	9.0	1.13	0.93
Mean time in the system (T_w)	0.2	0.03	0.02
Waiting cost	135	20.5	13.5
Cost of the pharmacists	10	20	30
Total cost	145	40.5	43.5

The waiting cost per hour is calculated as follows:

$$\text{Waiting cost per hour (\$ /h)} = T_w \cdot \text{Cost per hour (\$ /h)}.$$

The total cost is obtained by calculating the waiting cost per combination and by summing the relative cost of the pharmacists, and the lowest total cost is achieved for two pharmacists. Therefore, two pharmacists is the optimum number to be contracted.

6.5 Luggage Control in Airports

In one US airport, all passengers and their luggage must be checked to see if they have any arms. Let's assume that, on average, 10 passengers arrive at this airport every minute. The times between arrivals are exponential. To check all the passengers, the airport has two stations which consist in a metal detector and an X-ray machine for luggage. When the stations are operating, two employees are required. On average, it is possible to check 12 passengers per minute. The time needed to check one passenger is exponential. Determine:

(a) What is the probability that one passenger finds three people in the queue?
(b) On average, how many passengers wait in the queue?
(c) On average, how much time does a passenger spend in the checking station?
(d) If one employee costs \$25 per hour and it is estimated that the waiting time incurs a cost of \$15/h for the airport, what is the total cost of the system?

Solution

(a) What is the probability that one passenger finds three people in the queue?

This is a system with one queue, two servers and an infinite population. Based on the corresponding formula that the probability of the system being empty when arriving at the queue in t, we obtain:

$$P_0(t) = \frac{1}{\sum_{m=0}^{S-1} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{(S\mu-\lambda)}} = 0.41$$

and given that m is the number of people in the system, the probability that at arrival time t there are five people in the system ($m = 5$), P_5(t), with two receiving the service and three waiting in the queue, is:

$$P_5(\mathrm{t}) = 0.0103$$

(b) On average, how many passengers wait in the queue?

$$L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^S}{(S-1)!(S\mu-\lambda)^2} P_o(t) = 0.18$$

Thus, on average, 0.18 passengers wait in a queue.

(c) On average, how much time does a passenger spend in the checking station?

$$T_s = \frac{L}{\lambda} = \frac{0.18}{10} = 0.02\,\mathrm{min}$$

$$T_w = T_s + \frac{1}{\mu} = 0.02 + \frac{1}{12} = 0.10\,\mathrm{min}$$

On average, the passenger spends 0.10 min in the checking station.

(d) If one employee costs $25/h and it is estimated that the waiting time incurs a cost of $15/h for the airport, what is the total cost of the system?

The waiting time per hour is calculated as follows:

$$\text{Waiting time per hour (dollars/hour)} = T_w \cdot \text{Cost per hour (dollars/hour)}.$$

Hence:

$$\text{Waiting cost per hour} = 0.1 \cdot \$15/\text{h} \cdot 10 = \$15/\text{h}$$
$$\text{Total cost of the system} = 25 \cdot 2 + 15 = \$65/\text{h}$$

The total cost of the system is $65/h

6.6 System with a Processing Centre

Jobs arrive at a processing centre in accordance with a Poisson process at a mean rate of two/day, and the operation time has an average exponential distribution of 0.25 days. This centre has enough space for the material to be processed to take on three jobs besides that it is already processing. Additional works are kept temporarily in a less convenient place. What proportion of time is suitable for the space that the processing centre has to keep all the jobs that arrive?

Solution

It is a system with one queue, one server and an infinite population, where:

$$1 - P(W > 3) = \rho^{(3+1)} = 0.9375$$

Thus for 93.75 % of the time, there is sufficient space to store the jobs that arrive.

6.7 Processing Jobs in a Calculation Centre

The calculation centre in Inland Revenue's premises in Nice is equipped with four identical central computers which process the jobs. There are 25 users connected from their own PC at any given time. Each user is capable of processing one job via his/her PC every 15 min, on average, but the real time between sending jobs is exponential. The jobs that arrive automatically reach the first available central computer. The execution time per job is exponential and averages 2 min. Calculate the following:

(a) The percentage of time that the entire calculating centre is not operating
(b) The average number of jobs waiting to be executed

(c) The probability that one job is not immediately executed in the central computer
(d) The average time before one job is returned to the user
(e) The average number of central computers not in use.

Solution

(a) The percentage of time that the entire calculating centre is not operating

$$P_0(t) = \frac{1}{\sum_{m=0}^{S-1} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{(S\mu-\lambda)}} = 2.1\,\%$$

The percentage of idle time is 2.1 %

(b) The average number of jobs waiting to be executed

$$L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^S}{(S-1)!(S\mu-\lambda)^2} P_o(t) = 3.29 \text{ jobs}$$

So on average, 3.29 jobs wait to be processed.

(c) The probability that one job is not immediately executed in the central computer

The system has one queue, multiple parallel servers and an infinite population. Given that $\lambda = 25/15 = 1.67$ jobs/min and $\mu = 4/2 = 2$, then $\rho = \lambda/\mu = 0.83 < 1$. Therefore, the probability of there being more than three jobs in the system and, therefore, the next job to arrive is not immediately executed, is:

$$\begin{aligned} P(m > 3) &= 1 - Po(t) - P_1(t) - P_2(t) - P_3(t) \\ &= 1 - 0.022 - 0.072 - 0.119 - 0.133 = 0.65 \end{aligned}$$

(d) The average time before one job is returned to the user

$$T_s = \frac{L}{\lambda}$$

$$T_w = T_s + \frac{1}{\mu} = 3.98\,\text{min} = 0.662\,\text{h}$$

Hence, the average time before one job is returned to the user is 0.662 h.

(e) The average number of central computers not in use

$\rho = 0.835 \rightarrow$ No. of central computers operating $= 0.835 \cdot 4 = 3.34 \rightarrow$ Not operating $= 0.667$

Therefore, the average number of computers not operating is 0.667.

6.8 Customer Counselling Service

A firm located in Alicante which sells and supplies building materials offers its customers a decoration counselling service for bathrooms and kitchens. In a normal operation, 2.5 customers arrive on average every hour. A design consultant is available to answer customers' questions and to give recommendations about the product. The consultant takes an average time of 10 min per customer.

(a) The service targets indicate that one customer who arrives must not wait more than 5 min before being attended on average. Is this target met? If not, what action do you recommend?
(b) If the consultant cuts the average time spent per customer to 8 min, is the service target met?
(c) The Alicante-based firm wishes to evaluate two alternatives:

- Using one consultant with an average service time of 8 min per customer
- Using two consultants, each spending an average service time of 10 min per customer

If the firm pays the consultants \$16/h and the expected waiting time before a service for customers is valued at \$25/h, should the firm in Alicante have two consultants? Provide an explanation.

Solution

(a) The service targets indicate that one customer who arrives must not wait more than 5 min before being attended on average. Is this target met? If not, what action do you recommend?

This is a system with one queue, one server and an infinite population, where:

$$\lambda = 2.5\,\text{customers/h}$$
$$\mu = 60/10 = 6\,\text{customers/h}$$

$$L = \frac{\lambda^2}{\mu(\mu-\lambda)} = 0.2976\,\text{jobs}$$
$$W = \frac{\lambda}{\mu-\lambda} = 0.7143\,\text{jobs}$$
$$Ts = \frac{\lambda}{\mu(\mu-\lambda)} = 0.1190\,\text{h} \;\; = 7.14\,\text{min} > 5\,\text{min}$$

So, the firm must increase the mean service rate (μ) for the consultant because, if not, it must contract a second consultant.

(b) If the consultant cuts the average time spent per customer to 8 min, is the service target met?

In this case:

$$\mu = 7.5 \text{ customers/hour}$$

$$L = \frac{\lambda^2}{\mu(\mu - \lambda)} = 0.1667 \text{ jobs}$$

$$Ts = \frac{\lambda}{\mu(\mu - \lambda)} = 0.0667 \text{ h} = 4 \text{ min} < 5 \text{ min}$$

Thus, the service target is being met.

(c) The Alicante-based firm wishes to evaluate two alternatives:

- Using one consultant with an average service time of 8 min per customer
- Using two consultants, each spending an average service time of 10 min per customer.

If the firm pays the consultants \$16/h and the expected waiting time before a service for customers is valued at \$25/h, should the firm in Alicante have two consultants? Provide an explanation.

For one consultant with a service time of 8 min:

$$W = \frac{\lambda}{\mu - \lambda} = 0.5 \text{ jobs}$$

$$\text{Total cost} = \$25 \cdot \text{W} + \$16 = \$28.5$$

For two consultants, one system with one queue, two servers and an infinite populatin is evaluated:

$$\lambda = 2.5 \text{ customers/h}$$
$$\mu = 60/10 = 6 \text{ customers/h}$$

$$P_0(t) = \frac{1}{\sum_{m=0}^{S-1} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{(S\mu - \lambda)}} = 0.6552$$

$$L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^S}{(S-1)!(S\mu - \lambda)^2} P_o(t) = 0.0189 \text{ jobs}$$

$$W = L + \frac{\lambda}{\mu} = 0.4356 \text{ jobs}$$

$$\text{Total cost} = \$25 \cdot \text{W} + 16 \cdot \$2 = \$42.89$$

Therefore, it is recommended to have one consultant working an 8 min service time.

6.9 Acquiring Machinery for an Assembly Line

The Production Director of an industrial firm has to decide on whether to buy, or not, a new machine (MAQ2) for the assembly line, which apparently produces one bottle neck in a given operation performed by MAQ1. The aim is that machines MAQ1 and MAQ2 do the same operation in parallel according to the jobs that arrive at this assembly line point without stopping it.

The Production Director has resorted to a group of Industrial Organisation Engineering students at the EPSA who have assured the Director that applying the Queueing Theory can help to optimally solve this problem.

With the help of a simulation problem and the Production Director's experience, the students have devised Table 6.1, which reflects the simulated operation of both machines in parallel.

(a) How many servers/machines must be used to simulate the problem? Why?
(b) Using analytical Queueing Theory calculations, confirm the conclusion you reach in question (a)

Solution

(a) How many servers/machines must be used to simulate the problem? Why?
Two servers because two jobs can arrive at the same time, which implies the assembly line stopping if there is only one machine/server operating.
(b) Using analytical Queueing Theory calculations, confirm the conclusion you reach in question (a)

Given that:

$$\lambda = \frac{\text{no. customers}}{\text{arrival times}} = \frac{9}{7} \approx 1.3$$

$$\mu = \frac{\text{no. customers}}{\text{total time}} = \frac{9}{13} \approx 0.69$$

Table 6.1 Machines operating data

Job number	Minute of arrival	Minute when leaving	Waiting time in minutes	Service time in minutes	Server
1	0	9	0	9	MAQ1
2	1	9	0	8	MAQ2
3	2	10	7	1	MAQ1
4	3	11	6	2	MAQ2
5	3	12	7	2	MAQ1
6	5	12	6	1	MAQ2
7	5	13	7	1	MAQ1
8	6	13	6	1	MAQ2
9	7	13	6	0	MAQ1

Then the utilisation factor for a single server is greater than 1, which would lead to an infinite queue.

$$\rho = \frac{\lambda}{\mu} = \frac{1.3}{0.69} \approx 1.88 > 1 \text{ (For } S = 1)$$

The number of servers S is calculated, and it is advisable that the utilisation factor is below 1.

$$\rho = \frac{\lambda}{S\mu} = \frac{1.3}{2 \cdot 0.69} \approx 0.94 < 1 \text{ (For } S = 2)$$

With the utilisation factor value it is possible to conclude that two servers or machines are required to be able to execute all the jobs that arrive without the assembly line stopping.

6.10 Receiving Orders Via a Telephone Communications Service

ParaElla, a firm that sells women's clothing by catalogue, has a CSR (Customer Service Representative) communications service which is in charge of receiving orders from customers who ring an 805 telephone number. If the CSR is engaged, the call is held. To simplify, it is assumed that any number of incoming calls can be held, and that no-one hangs up from frustration after waiting a long time. All the customers placed on hold or talking with a specialised agent occupy one telephone line (let's assume that there enough telephone lines so that no caller hears an engaged tone). A phone call is expected to last a mean time of 5 min and the CSR uses a mean time of 3 min to attend each customer call. The CSR receives \$20/h, but the telephone company charges \$5/h and line for the time the line is being used. It is estimated that for every minute a customer call is held, it costs the firm \$2 owing to lack of customer satisfaction and loss of future sales.

(a) Estimate the mean time a customer call is placed on hold, the mean total time a customer is on the phone, the mean number of customer calls on hold and the mean number of customers on the phone.
(b) Estimate the total cost per hour.
(c) Assume that ParaElla is considering hiring another CSR to cut the waiting time. Is it a good idea? Explain your answer.

Solution

(a) Estimate the mean time a customer call is placed on hold, the mean total time a customer is on the phone, the mean number of customer calls on hold and the mean number of customers on the phone.

This is a system with one queue, one server and an infinite population:

$$\lambda = 1 \text{ call/5 min} = 12 \text{ calls/h}$$
$$\mu = 1 \text{ call/3 min} = 20 \text{ calls/h}$$
$$S = 1$$
$$\rho = 12/20 = 0.6$$

The mean time that a customer call is placed on hold: $Ts = \frac{\lambda}{\mu(\mu-\lambda)} = 0.075 \text{ h} = 4.5 \text{ min}$;

The total mean time that a customer is on the phone: $Tw = Ts + \frac{1}{\mu} = 0.125 \text{ h} = 7.5 \text{ min}$;

The mean number of customer calls placed on hold: $L = \frac{\lambda^2}{\mu(\mu-\lambda)} = 0.9$;

The mean number of customers on the phone: $W = \frac{\lambda}{\mu-\lambda} = 1.5$;

(b) Estimate the total cost per hour.

The CSR per hour = \$20
Telephone charges per hour = W · 5 = $\lambda \cdot T_w \cdot 5$ = \$7.5
Charges to place a customer call on hold per hour = $L \cdot 120$ \$ = $\lambda \cdot T_s \cdot 2$ = \$108
The total cost per hour = CSR per hour + Telephone charges per hour + Charges of customer calls placed on hold per hour = \$135.5
Therefore, the total cost per hour is \$135.5

(c) Assume that ParaElla is considering hiring another CSR to cut the waiting time. Is it a good idea? Explain your answer.

In this case, a system with one queue, two servers and an infinite population is evaluated:

$$\lambda = 1 \text{ call/5 min} = 12 \text{ calls/h}$$
$$\mu = 1 \text{ call/3 min} = 20 \text{ calls/h}$$
$$S = 2$$
$$\rho = 12/2 \cdot 20 = 0.3$$

The probability of the system being empty: $P_0(t) = \frac{1}{\sum_{m=0}^{S-1} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{(S\mu-\lambda)}} = 0.96$;

Mean number of customer calls placed on hold: $L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^S}{(S-1)!(S\mu-\lambda)^2} P_o(t) = 0.06$;

Mean number of customers on the phone: $W = L + \frac{\lambda}{\mu} = 0.66$;

CSR per hour = \$40;
Telephone charges per hour = $W.5$ = \$3.3;
Customer waiting charges per hour = L.\$120 = \$7.2;
Total cost per hour = \$50.5;
Therefore this is the best alternative as it saves \$85/h.

6.11 Customer Services in an Agency

An agency has three people in charge of customer services; each takes a mean time of 10 min to see to a customer.

(a) Let's assume that customers arrive at a rate of 15/h. What probability is there of a customer waiting to be attended to? What is the mean number of customers in the queue? What is the mean waiting time in the agency?
(b) Let's assume that the agency is divided into three services: one manages purchases/sales; one works with documents (ID cards, passports, driving licences, etc.) and one does all the other services. The rate at which customers arrive at each service division is 5 per hour. Besides, the three employees are assigned to one service each. What probability is there of a customer having to wait to be attended to? What is the mean number of customers in the queue? What is the mean time in the agency per customer?
(c) Which of the two previous alternatives (a) and (b) do you think is more suitable? Give your reasons.

Solution

(a) Let's assume that customers arrive at a rate of 15/h. What probability is there of a customer waiting to be attended to? What is the mean number of customers in the queue? What is the mean waiting time in the agency?

The system in the first case is one queue with an infinite population and multiple servers ($S = 3$), where $\lambda = 15$ and $\mu = 6$. The utilisation factor is $\rho = \frac{\lambda}{S\mu} = \frac{5}{6} = 0.83 < 1$ and, therefore, there is a steady state.

The probability of one customer having to wait in the first case is given by:

$$1 - (P_0 + P_1 + P_2) = 1 - P_o\left(1 + \left(\frac{\lambda}{\mu}\right) + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\right)$$

$$P_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{\left(\frac{\lambda}{\mu}\right)^3}{3!(1-\rho)}} = \frac{4}{89}$$

Therefore:

$$1 - (P_0 + P_1 + P_2) = 1 - P_o\left(1 + \left(\frac{\lambda}{\mu}\right) + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\right) = \frac{125}{178} \approx 0.70$$

There is a 70 % probability of a customer having to wait to be attended to.

The mean number of customers in the queue is:

$$L = \frac{\left(\frac{\lambda}{\mu}\right)^3 P_o \rho}{3!(1-\rho)^2} = \frac{625}{178} \approx 3.51$$

The waiting time is:

$$Ts = \frac{L}{\lambda} = \frac{3.51}{15}\text{ h} \approx 14\text{min}$$
$$Tw = Ts + \frac{1}{\lambda} = 0.3\,\text{h} \approx 18\text{min}$$

(b) Let's assume that the agency is divided into three services: one manages purchases/sales; one works with documents (ID cards, passports, driving licences, etc.) and one does all the other services. The rate at which customers arrive at each service division is 5 per hour. Besides, the three employees are assigned to one service each. What probability is there of a customer having to wait to be attended to? What is the mean number of customers in the queue? What is the mean time in the agency per customer?

This section has three independent queues, all with $\lambda = 5$ and $\mu = 6$. Now $\rho = 5/6 < 1$.

The probability of a customer having to wait is:

$$1 - P_0 = \rho = \frac{5}{6} = 0.83$$

The mean number of customers in each queue is:

$$L = \frac{\rho^2}{1-\rho} = \frac{25}{6} = 4.16 \text{ customers}$$

Among the three queues, a mean total of $3L = 12.5$ customers is waiting.

$$Ts = \frac{3L}{\lambda} = 0.83\text{ h} = 50\,\text{min}$$
$$Tw = Ts + \frac{1}{\lambda} = 1\text{ h} = 60\,\text{min}$$

Therefore, the mean waiting time is 60 min.

(c) Which of the two previous alternatives (a) and (b) do you think is more suitable? Give your reasons.

The proposed restructuring increases not only the probability of a customer having to wait, but also the time the customer spends in the agency. Besides, the mean number of customers waiting to be attended to is also larger. Therefore, the proposed restructuring is not advisable as far as queue system performance is concerned.

6.12 Orders from Cars in a Fast Food Restaurant

McAuto, a fast food restaurant, has a window where it receives orders from cars. The arrivals of customers follow a Poisson probability distribution, and the mean arrival rate is 10 cars per hour. Service times follow an exponential probability distribution with a mean service rate of 12 cars per hour.

(a) What is the probability of there being no customer waiting in the system?
(b) If you arrive at the restaurant, how many cars do you expect to see waiting and being served?
(c) What is the probability of a least one car waiting?
(d) What is the average time to wait in a queue to be attended to?
(e) As a potential customer of this system, would you be satisfied with the system? Why?
(f) To improve the service, the restaurant's administrators wish to investigate the effect that a second window to receive orders from cars would have. Let's assume a mean arrival rate of 12 cars per hour for each window. What effect would adding a new window have on the system? Would this system be acceptable?

Solution

(a) What is the probability of there being no customer waiting in the system?

This is a system with one queue, one server and an infinite population where $\lambda = 10$ cars/h and $\mu = 12$ cars/h. The probability of the system being empty in t is:

$$P0(t) = \frac{\mu - \lambda}{\mu} = 16.7\,\%$$

(b) If you arrive at the restaurant, how many cars do you expect to see waiting and being served?

$$W = \frac{\lambda}{\mu - \lambda} = 5$$

So, there would be five cars waiting and being served in the system.

(c) What is the probability of a least one car waiting?

The probability of there being two cars in the system, one waiting and another being served, is calculated. In this case $m = 2$:

$$P_2(t) = \rho^2(1-\rho) = \left(\frac{10}{12}\right)^2 \cdot \left(1 - \frac{10}{12}\right) = 11.6\,\%$$

The probability of there being at least one car waiting is 11.6 %

(d) What is the average time to wait in a queue to be attended to?

$$Ts = \frac{\lambda}{\mu(\mu-\lambda)} = 0.4167\,\text{h} \;\; = 25\,\text{min}$$

Therefore, the average time to wait in a queue is 25 min

(e) As a potential customer of this system, would you be satisfied with the system? Why?
No, because waiting 25 min in a queue is a long time.

(f) To improve the service, the restaurant's administrators wish to investigate the effect that a second window to receive orders from cars would have. Let's assume a mean arrival rate of 12 cars per hour for each window. What effect would adding a new window have on the system? Would this system be acceptable?

In this case, a system with one queue, two parallel servers and one infinite population is evaluated:

$$P_0(t) = \frac{1}{\sum_{m=0}^{S-1} \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{(S\mu-\lambda)}} = 41.18\,\%$$

$$L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^S}{(S-1)!(S\mu-\lambda)^2} P_o(t) = 0.175 \text{ people in the queue}$$

$$W = L + \frac{\lambda}{\mu} = \;1 \text{ person in the system}$$

$$T_s = \frac{L}{\lambda} = 1.05 \text{ min in the queue}$$

$$T_w = T_s + \frac{1}{\mu} = 6.05 \text{ min in the system}$$

By considering only the calculated system parameters, and not considering the costs of the new window, this system seems acceptable for a fast food restaurant.

6.13 Buying a Photocopier

Beatrice is in charge of the reprography service at EPSA and is considering buying a new photocopier from among the models whose technical features are detailed in Table 6.2:

After measuring the several times in which the tasks to be photocopied arrive, it can be assumed that these arrive following a Poisson distribution with an average of 24 jobs/day. It is assumed that the time taken to do the photocopies in accordance with size follows an exponential distribution.

Beatrice estimates that the cost to wait for jobs to be photocopied can be considered \$80 per job and per day, based on loss of customers who end up taking jobs to other reprography services near the university centre.

If we consider that Beatrice hopes that the cost per operation plus the cost of a delayed delivery is as low as possible, what photocopying machine model should she buy?

Solution

For photocopier $i = 1, 2, 3, 4$, we obtain:

$$TC_i = SC_i + WC_i = OC_i \cdot 24 + 80 \cdot L_i$$

where TC is the total cost, SC is the service cost, WC is the waiting cost, OC is the operation cost and L is the value expected from the number of jobs in the system.

$\lambda = 4\,\text{jobs/day}$ for all the photocopiers.

The average time per job for photocopier i is $\frac{1}{v_i \cdot 60}$ hours, where v represents speed. The service rate for each photocopier is $\mu_i = 24 \cdot v_i \cdot 60$

From the corresponding formulation of a system with one queue, one server and an infinite population, the result offered in Table 6.3 are obtained:

Hence, the firm should buy the third photocopier machine.

Table 6.2 Technical details of photocopiers

Model	Cost of each operation (\$/h)	Speed (tasks/min)
1	15	30/10000
2	20	36/10000
3	24	50/10000
4	27	66/10000

Table 6.3 The results obtained for each machine

	M1	M2	M3	M4
	4	4	4	4
	4.32	5.18	7.2	9.5
L	12.5	3.39	1.25	0.73
TC	1360	751.2	676	706.4

6.14 A Printer Shared in a Computer Network

A computer network shares a printer. Jobs arrive at a mean rate of two jobs per minute and follow a Poisson process. The printer prints 10 pages per minute and the mean number of pages per job is four. There is a 3-second idle time between one job and the next. The service time follows an exponential distribution. Calculate:

(a) The percentage of time that the printer is available.
(b) The mean queue length.
(c) The service rate required for the mean time in the system to be under 3 min.

Solution

(a) The percentage of time that the printer is available.

The system can be modelled as one queue, one server and an infinite population, where $\lambda = 2$ jobs/minute.

As each job has a mean of four pages, the printer takes a mean of 4/10 min to print it, plus the 3-second idle time (1/20 min) that each job has. Therefore, $\frac{1}{\mu} = \frac{4}{10} + \frac{1}{20} = \frac{9}{20}$ min and $\mu = \frac{20}{9}$ jobs/min. $\rho = \frac{2}{20/9} = 0.9 < 1$.

The probability of there being 0 jobs in the system is:

$$p_0 = 1 - \rho = 1/10$$

Thus, the printer is free for 10 % of the time.

(b) The mean queue length.

$$L = \frac{\lambda^2}{\mu(\mu - \lambda)} = 8.1 \text{ jobs}$$

Hence, there is an average of 8.1 jobs in the queue.

(c) The service rate required for the mean time in the system to be under 3 min.

By leaving the $\lambda = 2$ constant, obtain the μ value for $T_w < 3$. It is necessary to express $T_w < 3$ depending on λ and μ and to solve the inequation. A services rate μ of over 2.33 jobs/min is necessary for the time to remain in the system to fall below 3 min.

6.15 Income from Temporary Work

A year five Business Studies student studying at EPSA does some temporary substitute work in the banking sector every 30 days on average, and this time follows an exponential distribution. After acquiring one of these works, its duration is also aleatory, exponentially distributed and its mean duration is 60 days.

The money earned depends on the total number of days worked in each contract, paid at \$100 daily. For unemployment periods, the student has taken out an insurance for which \$50 is paid for each day on the dole.

Build a queueing model to establish the mean annual income (1 year = 365 -days) that this student earns.

Solution

An approach with one queue, one server and a finite population is adopted:

$$\lambda = \frac{1}{30} = 0.0333\,\text{jobs/day}$$
$$\mu = \frac{1}{60} = 0.0167\,\text{jobs/day}$$

In this model, the ρ value does not have to be below 1 because the arrivals at the system are controlled by the N limit of the system which, in this case, is one single work at one time.

$$\text{If}\, m = n = 1$$

$$\frac{P_n(t)}{P_o(t)} = \frac{m!}{(m-n)!}\left(\frac{\lambda}{\mu}\right)^n = \frac{1!}{(1-1)!}\left(\frac{0.0333}{0.0167}\right)^1 = 2$$

The probability of being unemployed is:

$$P_0(t) = \frac{1}{\sum_{n=0}^{m}\frac{P_n(t)}{P_0(t)}} = \frac{1}{1+2} = \frac{1}{3} = 0.3333$$

The probability of being employed is:

$$P_1(t) = P_o(t)\frac{m!}{(m-n)!}\left(\frac{\lambda}{\mu}\right)^n = \frac{1}{3}\frac{1}{0!}\left(\frac{0.0333}{0.0167}\right)^1 = \frac{2}{3} = 0.6667$$

Thus, the student's mean annual income is:

$$50 \cdot 0.3333 + 100 \cdot 0.6667 = 83.335\,\$ \text{ /day} = 30{,}417.275\$/\text{year}$$

6.16 Fast Food Restaurants

This problem intends to compare a McDonolds and a Burger-Kong. It is assumed that McDonolds has two queues in operation, one for each cashier, and that customers aleatorily choose the queue when they enter, without considering the length of the queue and without being able to change queue. Burger-Kong also has two cashiers, but a single queue where the customers at the front of the queue go to whichever cashier is available. Consider that one person arrives at both premises every minute and that the mean service time is 30 s.

(a) Which method implies a longer waiting time and a longer service time for the customer?
(b) Let's assume that it is more realistic to allow the McDonolds customers to change queues whenever they wish, therefore the times in both premises are the same. What advantages and disadvantages do you find in each method?

Solution

(a) Which method implies a longer waiting time and a longer service time for the customer?

For Burger-Kong:

$$\lambda = 1\,\text{customer/min}$$
$$\mu = 2\,\text{customers/min};$$

By applying the corresponding formulae for the model with one queue, multiple parallel servers and an infinite population:

$$W = \frac{\lambda}{\mu - \lambda} = 0.53333\,\text{min} = 32\,\text{s}$$

Therefore, the average waiting and service time is 32 s.

For McDonolds:

If we assume that each customer chooses a queue aleatorily and cannot change queue, then each queue can be modelled individually. Therefore, this is a model with one queue, one server and an infinite population with:

$$\lambda = 0.5\,\text{customer/min}$$
$$\mu = 2\,\text{customers/min}\,;$$

By applying the formulae for a model with one queue, one server and an infinite population, we obtain:

$$W = \frac{\lambda}{\mu - \lambda} = 0.66666\,\text{min} = 40\,\text{s}$$

Therefore the average waiting and service time is 40 s.

So the McDonolds method implies a longer waiting and service time.

(b) Let's assume that it is more realistic to allow the McDonolds customers to change queues whenever they wish, therefore the times in both premises are the same. What advantages and disadvantages do you find in each method?

The Burger-Kong method (one queue for two servers) promotes FIFO priorities, whereas the McDonolds method (two queues for two servers) does not. Besides, although the mean time on both premises is the same, the variation in waiting time is greater in McDonolds than in Burger-Kong.

6.17 Loading Vans to Deliver Orders

An automatic coffee and snacks dispensing firm is based in Valencia and its warehouse only has one loading bay which serves all the customer firms in the region, and there is only one worker to look for the products in the orders on each van and to load them (or distributor). Every now and again, transport vans accumulate at the loading bay and form a queue, while the worker is idle at other times. After examining van arrivals for several weeks, a mean arrival rate of 4 vans/h is established, and the service rate is 6 vans/h. The firm's Industrial Organisation Engineer is thinking of contracting another worker, or even two, to increase the service rate. The problem lies in evaluating these options by taking into account that the operation costs per van per hour are $12 and that workers are paid $11 for each hour they work, and they work 8 h/day. It is assumed that the work capacity is proportional to the number of workers.

Solution

If one or two worker(s) is/are added, the system will still have one queue, one server and an infinite population as only one van can be loaded at the same time. If two employees work, the service rate will be 12, and it will be 18 if there are three workers. The system efficiency measures are shown below:

	Workers		
	1	2	3
Mean number of vans in the queue (L)	1.333	0.167	0.063
Mean number of vans in the system (W)	2.000	0.500	0.286
Mean time a van is in the queue (Ts)	0.333	0.042	0.016
Mean time a van is in the system (Tw)	0.500	0.125	0.071
Service occupation (ρ)	0.667	0.333	0.222

The associated costs are:

Workers	Operation cost of van per day ($)	Labour cost per day ($)	Total costs per day ($)
1	192	88	280
2	48	176	224
3	27.46	264	291.46

The firm's Industrial Organisation Engineer will have to add another new worker to the system because this reduces the total costs, although the utilisation factor becomes 33 %. That is to say, the two workers will have 5 h and 20 min to do other tasks within the firm.

6.18 Contracting Shop Assisitants in an Ice-Cream Parlour

A shop manager wants to open an ice-cream parlour and intends to contract one employee. It is assumed that customers are served as they arrive. The earning per ice-cream sold is \$0.90. After several interviews, the manager is finding it hard to decide between two people: Alan and Alexander. Alan's mean service time is 10 s, and that of Alexander is 30 s, but Alan is paid \$9/ h, whereas Alexander is paid \$6/ h. Market research has revealed that the expected ice-cream sales demand per customer every minute and the cost per shop of having customers waiting is, on average, \$9/h per customer. It is assumed that both the time between arrivals and the service time are exponential.

(a) Help the shop manager decide to contract one employee by taking into account the worker who generates more mean profits per hour.
(b) As Alexander really wants the job, he tells the shop manager that he is willing to be paid a lower salary. How much can his salary lower for him to be competitive with Alan?
(c) Alexander wants to work, but is not willing to earn a lower salary. So he does a Master's degree on ice-cream sales to be more competitive; by how much time must he cut his service time to be competitive with Alan?

Solution

(a) Help the shop manager decide to contract one employee by taking into account the worker who generates more mean profits per hour.

This is a model with one queue, one server and an infinite population.
Alternative 1: Contracting Alan

$$\lambda = 1 \text{ customer/min}$$

$$\mu = 6 \text{ customers/min}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1}{6} = 0.16$$

$$W = \frac{\lambda}{\mu - \lambda} = \frac{1}{6-1} = 0.2\,\text{customers}$$

$$Ts = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1}{6(6-1)} = 0.033\,\text{min}$$

Average income/h = 60·0.9 = \$54/h
System cost = Server cost + Waiting cost
Server cost = \$9/h

$$\text{Waiting cost} = \lambda \cdot T_s \cdot \text{Delay cost/h} = 60 \cdot 0.00055556 \cdot 9 = \$0.3/\text{h}$$

$$\text{Profit per hour} = 54 - 9 - 0.3 = \$44.7/\text{h}$$

Alternative 2: Contracting Alexander

$$\lambda = 1\,\text{customer/min}$$
$$\mu = 2\,\text{customers/min}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1}{2} = 0.5$$

$$W = \frac{\lambda}{\mu - \lambda} = \frac{1}{2-1} = 1_{\text{customer}}$$

$$Ts = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{1}{2(2-1)} = 0.5_{\text{min}}$$

Average income/h = 60·0.9 = \$54/h
System cost = Server cost + Waiting cost
Server cost = \$6/h

$$\text{Waiting cost} = \lambda \cdot Ts \cdot \text{Delayed cost/h} \ = 60 \cdot 0.00833333 \cdot 9 = \$4.5/\text{h}$$

$$\text{Profit per hour} = 54 - 6 - 4.5 = \$43.5/\text{h}$$

Therefore, we recommend that the firm contracts Alan.

(b) As Alexander really wants the job, he tells the shop manager that he is willing to be paid a lower salary. How much can his salary lower for him to be competitive with Alan?

The difference between the profit per hour of alternative 1 and alternative 2 is 44.7−43.5 = \$1.2/h. So, Alexander's salary will have to lower by \$1.2 from \$6/h to \$4.8/h.

(c) Alexander wants to work, but is not willing to earn a lower salary. So he does a Master's degree on ice-cream sales to be more competitive; by how much time must he cut his service time to be competitive with Alan?

The difference obtained between the waiting time for alternative 2 is \$4.5/h, and the difference between the profits of both alternatives is \$1.2/h. This involves a reduction in the waiting cost in alternative 2 of 4.5−1.2 = \$3.3/h. To achieve this waiting cost value, Ts = 0.37 min; therefore, $\mu = 2.22$ customers/min.

6.19 Traffic in Routers

There are two routers and it is known that:

- Router A sends eight packets per second, on average, to router B.
- The average packet size is 400 bytes (distributed exponentially).
- The line speed is 64 Kbytes/s.

(a) How many packets, on average, in router A are expected to be transmitted or are being transmitted?
(b) What is the probability of the number being 10 or more?

Solution

(a) How many packets, on average, in router A are expected to be transmitted or are being transmitted?

This is a model with one queue, one server and an infinite population:

$$\lambda = 8\,\text{packets/s};$$
$$\mu = 160\,\text{packets/s};$$
$$\rho = 0.05 < 1\,;$$

$$W = \frac{\lambda}{\mu - \lambda} = 0.053\,\text{packets.}$$

So on average, there are 0.053 packets waiting to be transmitted or being transmitted.

(b) What is the probability of the number being 10 or more?

Given that $Z = 9$

$$\mathrm{P}(W > Z) = \rho^{(z+1)} = 0.05^{10} = 9.76 \cdot 10^{-14}$$

The probability of there being 10 packets or more waiting to be transmitted or being transmitted is $9.76 \cdot 10^{-14}$.

6.20 A Taxi Rank at an Airport

At the El Altet airport, taxis and customers have arrival rates of one and two per minute, respectively. The times between arrivals are considered exponential. Irrespectively of how many taxis are present, a taxi has to wait. Finally, if a customer who arrives at the system does not find a taxi, he/she leaves immediately.

(a) Determine the average quantity of taxis waiting for a customer.
(b) Assume that all the customers who travel by taxi pay $20. During any given hour, how much income do taxis receive?

Solution

(a) Determine the average quantity of taxis waiting for a customer.

This problem considers that taxis are the customers which arrive at the system and that the customers execute the service of using each taxi. Thus, a system is modelled with one queue, one server and an infinite population, where:

$$\lambda = 1\,\text{taxi/min}$$
$$\mu = 2\,\text{taxis/min}$$

$$W = \frac{\lambda}{\mu - \lambda} = 1\,\text{taxi}$$

Thus on average, there will be one taxi waiting.

(b) Assume that all the customers who travel by taxi pay \$20. During any given hour, how much income do taxis receive?

In 1 h, 60 taxis would arrive. So the income they would receive is $60 \cdot 20 = \$1200/\text{h}$.

6.21 Attending Telephone Calls

During the registration period, EPSA has a technician in its service centre to answer students' questions. The number of telephone calls arriving at this centre follows a Poisson distribution with an approximate average rate of 10/h. The time required to answer one call follows an exponential distribution with an average of 4 min. Answer the following questions:

(a) What is the average time between incoming calls?
(b) What is the average number of calls that the technician can attend in 1 h?
(c) What is the probability of there being exactly four calls on hold at a given time?
(d) What is the probability of the number of calls in the system exceeding 10?

Solution

(a) What is the average time between incoming calls?

$$\lambda = 10\,\text{calls/h}$$
$$1/\lambda = 1/10 \text{ of } 1\,\text{h} = 6\,\text{min/call}$$

The average time between calls is 6 min.

(b) What is the average number of calls that the technician can attend in 1 h?

$$\mu = 0.25\,\text{calls/min} = 15\,\text{calls/h}$$

In 1 h, the technician attends an average of 15 calls.

(c) What is the probability of there being exactly four calls on hold at a given time?

This is a model with one queue, one server and an infinite population, so:

$$P(m = 5) = \rho^m(1 - \rho) = (10/15)^5(1 - 10/15) = 0.04389575$$

Therefore, the probability of there being exactly four calls on hold at a given time is 0.04389575.

(d) What is the probability of the number of calls in the system exceeding 10?

$$P(W > 10) = \rho^{z+1} = (10/15)^{11} = 0.01156102$$

Hence, the probability of the number of calls in the system exceeding 10 is 0.01156102.

6.22 Warehouse Management in a Firm in the Automobile Sector

Trucks arrive at a warehouse which stores the finished products of a firm that supplies headrests for car seats according to a Poisson process with a mean ratio of 4/h. Only one truck can be loaded at one time. The time required to load one truck follows an exponential distribution in a mean time of 10/n minutes, where n is the number of loaders (n = 1, 2, 3,…). The costs associated with the loading process are: (a) \$18/h for each loader and (b) \$20/h for each truck being loaded or waiting in the queue to be loaded. Calculate the number of loaders needed to minimise the cost expected per hour.

Solution

This is a system with one queue, one server and an infinite population.

- With 1 loader:

$$\lambda = 4 \text{ trucks/h}$$
$$10/1 \text{ min/load} \rightarrow \mu = 6 \text{ trucks/h}$$

- With 2 loaders:

$$\lambda = 4 \text{ trucks/h}$$
$$10/2 \text{ min/load} \rightarrow \mu = 12 \text{ trucks/h}$$

- With 3 loaders:

$$\lambda = 4\,\text{trucks/h}$$
$$10/3\,\text{min/load} \;\rightarrow \mu = 18\,\text{trucks/h}$$

In all cases, the total cost is the cost of the loaders per hour, plus the cost per hour that the trucks waiting to be loaded and are being loaded imply.

Total cost per hour (\$/h) = Cost of loaders (\$/h) + Waiting and loading cost (\$/h).

The cost to wait to be loaded is the time that each truck is in the warehouse waiting to be loaded, but also the loading time (h/truck) by the money that this hour costs (\$/h) by the number of trucks that arrive at the warehouse every hour (trucks/h).

Delay cost per hour (\$/h) = Tw · Cost per hour (\$/h) . λ

Case 1: 1 loader.

$$\rho = \chi/\mu = 4/6 < 1$$

$$Tw = Ts + \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} = 0.41 + 0.08 = 0.5\,\text{h/truck}$$

$$\text{Total cost per hour} = 18\frac{\$}{\text{h}} + 20\frac{\$}{\text{h}} \cdot 0.5\frac{\text{h}}{\text{truck}} \cdot 4\frac{\text{trucks}}{\text{h}} = 58\frac{\$}{\text{h}}$$

Case 2: with 2 loaders

$$\rho = \chi\mu = 4/12 < 1$$

$$Tw = Ts + \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} = 0.125\,\text{h/truck}$$

$$\text{Total cost per hour} = 36\frac{\$}{\text{h}} + 20\frac{\$}{\text{h}} \cdot 0.125\frac{\text{h}}{\text{truck}} \cdot 4\frac{\text{trucks}}{\text{h}} = 46\frac{\$}{\text{h}}$$

Case 3: with 3 loaders

$$\rho = \chi\mu = 4/18 < 1$$

$$Tw = Ts + \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} = 0.071\,\text{h/truck}$$

$$\text{Total cost per hour} = 54\frac{\$}{\text{h}} + 20\frac{\$}{\text{h}} \cdot 0.071\frac{\text{h}}{\text{truck}} \cdot 4\frac{\text{trucks}}{\text{h}} = 59.71\frac{\$}{\text{h}}$$

Therefore, two loaders is the recommended number for the warehouse.

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Chapter 7
Decision Theory

Abstract This chapter starts with an introduction to Decision Theory. Then, a varied set of Decision Theory problems is proposed with their corresponding solutions. This chapter aims to provide a better understanding of modelling decision problems by means of decision trees. Thus, low-risk decision problems are set out in which the decision maker can acquire further information to amend a priori probabilities by Bayes Theorem. The utility concept and its use instead of monetary values are also introduced, which are the most usual kind in Organisation Engineering problems.

7.1 Introduction

Decision Theory (Lehmann 1950) encompasses different problem-solving techniques where the decision maker must select among various alternatives, which yield different profits or costs. Occasionally, lack of information means having to make certain estimations.

In a decision problem, the following elements can be identified: (i) alternatives, (ii) states of nature and (iii) performances. Alternatives are decision variables, which are controllable and depend on the decision maker's decision. States of nature are external variables, and are incontrollable, so they cannot be defined by the decision maker, but he/she has to assume them. Performances are output variables (profits or costs).

Depending on the degree of knowledge about the states of nature that the decision maker may possess, the decision situations may be classified into: (i) certainty, (ii) risk and (iii) uncertainty. Decision problems in a situation of certainty can be solved by simple techniques, such as complete enumeration, if the problem is not complex; if it is complex, for example, a combinatory problem, specific techniques must be used to solve it, which limit the search area. In decision problems in a situation of risk, one part of the information is unknown to the decision maker, but he/she can make certain estimations of it; the most usual

R. Poler et al., *Operations Research Problems*,

DOI: 10.1007/978-1-4471-5577-5_7,

case is not knowing the states of nature to occur, but the probabilities of these states of nature occurring instead. For decision problems in a situation of uncertainty, there is more unknown information; for example, the probabilities of states of nature taking place not being known; for such problems, methods can be employed to estimate the lacking information so that they become low-risk problems, or solution procedures for an optimistic decision maker (who assumes that a better state of nature will occur) or a pessimistic decision maker (who assumes that a worse state of nature will occur) can be utilised.

In practice, the more usual decision problems are those which occur in a situation of risk (Raiffa and Schlaifer 1961). The solution procedure is based on the expected value calculation, a concept which appeared in the seventeenth century which Blaise Pascal announced in his famous *Pensées* (Pascal 1671). When a decision maker is faced with a series of decisions, each with a number of results relating to a different probability, the rational procedure involves identifying the possible results of the actions, determining their values (positive or negative) and their associated probabilities resulting from each action, and the expected value is obtained by multiplying the two values.

In low-risk decision problems, probabilities of states of nature occurring appear, which can be revised if the decision maker accesses further information (normally by paying a price) using Bayes Theorem (Bayes et al. 1763). These problems can be modelled by using decision tables and can be solved by calculating the expected monetary value (EMV) of each alternative, this being the long-term profit made, and by selecting the alternative with the highest EMV. Nonetheless, low-risk decision problems into which the revisions of the probabilities of states of nature are introduced, or multi-stage problems (several linked decisions), obtain a more suitable solution and modelling technique in decision trees. Further information may have an acquisition cost, so the decision maker must assess if buying it before knowing the information is a good idea. Calculating the limit value of this cost, by comparing the result with and without the information, allows the decision maker to know up to what suitable amount should be paid for further information.

In the majority of cases, the results of alternatives (payments or costs) are measured in monetary units. However, these can present certain distortions when the range is wide or when the decision maker is dealing with another type of result (non-monetary), which he/she is interested in combining with the monetary type. To solve this problem, a utility concept emerges. This concept appeared in 1738 when Daniel Bernoulli published a new risk measurement theory (Bernoulli 1954), in which he employed the St. Petersburg paradox to demonstrate that the EMV is a mistaken measure. When attempting to solve this problem, he used the utility function, for the first time, and calculated the expected result instead of EMV. By constructing a utility function which transforms monetary values, or results of other kinds, into utilities, the decision maker can solve the problem in a balanced manner and can make whatever decision offers more utility. Other authors (Stigler 1968) (Bernstein 1996) have continued to develop the Utility Theory in low-risk decision making.

In 1939, Abraham Wald contributed statistical hypothesis tests and the Statistical Estimation Theory to Decision Theory, which imply special aspects for the general decision problem. Wald combined many current ingredients from the modern Decision Theory, including loss functions, risk functions, admissible decision rules a priori distributions, Bayes Decision Theory and minimax rules for decision making.

A standard example of a decision tree is provided in Fig. 7.1. Node (1) is the initial node in which the decision maker must choose between continuing with decision making without further information to that he/she already possesses now or acquiring further information. Node (2) is located in the former case in which the decision maker must decide between choosing alternative a_1, with associated cost ca_1, or a_2, with associated cost ca_2. If the decision maker chooses alternative a_1, he/she is located in Node (3), which is a random node, and must then check if

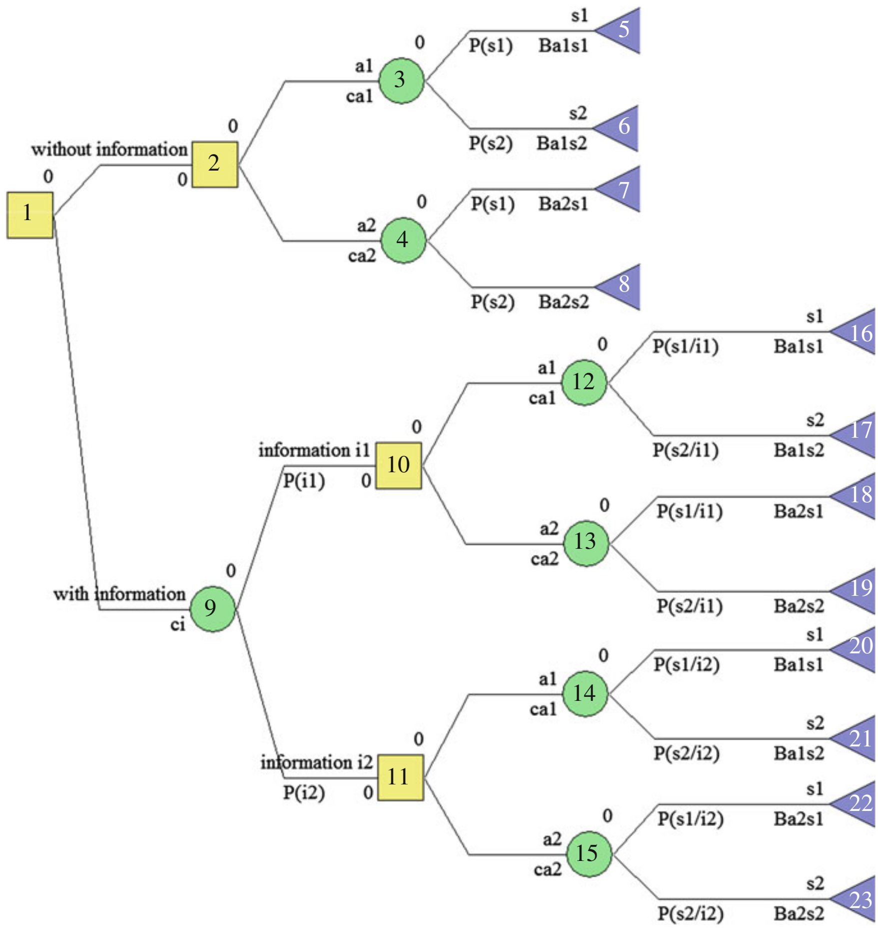


Fig. 7.1 Decision tree

state s_1 occurs or state s_2, with the probabilities of occurrence $P(s_1)$ and $P(s_2)$, respectively. If state s_1 occurs, he/she will be located in Node (5) and will obtain profit Ba_1s_1. If state s_2 occurs, he/she will be located in Node (6) and will obtain profit Ba_1s_2. Should alternative a_2 be selected when located at Node (2), he/she will be located at random Node (4) and the results could be Node (7) or Node (8). In the upper part of the tree, where the decision maker has received no new information, it is possible to calculate the expected value of Node (2) as so:

$$V_2 = \max\{V_3 - ca_1; V_4 - ca_2\}$$
$$V_3 = P(s_1) \cdot Ba_1s_1 + P(s_2) \cdot Ba_1s_2$$
$$V_4 = P(s_1) \cdot Ba_2s_1 + P(s_2) \cdot Ba_2s_2$$

Note that the nodes are calculated from right to left starting by the result nodes, which already have a known value. In general terms, to calculate a given node, all the branch nodes of the given node must take a calculated value.

The decision maker must calculate if he/she is interested in acquiring new information at cost ci. For this purpose, it is necessary to calculate if the value of Node (9), minus cost ci, is higher than the Node (2) value. Note that the structure of the branches and nodes of Nodes (10) and (11) is the same as that of Node (2), and only the probabilities of occurrence of states s_1 and s_2 change, which become revised (a posteriori) probabilities, according to information ci of the initial (a priori) probabilities. When information i_1 is available, the revised probabilities are calculated by applying Bayes Theorem in the following manner:

$$P(s_1/i_1) = \frac{P(s_1) \cdot P(i_1/s_1)}{P(s_1) \cdot P(i_1/s_1) + P(s_2) \cdot P(i_1/s_2)}$$

$$P(s_2/i_1) = \frac{P(s_2) \cdot P(i_1/s_2)}{P(s_1) \cdot P(i_1/s_1) + P(s_2) \cdot P(i_1/s_2)}$$

$$P(i_1) = P(s_1) \cdot P(i_1/s_1) + P(s_2) \cdot P(i_1/s_2)$$

Conditional probabilities $P(i_1/s_1)$ and $P(i_1/s_2)$ refer to the probability that information i_1 occurs should states s_1 and s_2 occur, respectively. Note that $P(i_1)$ is the probability that the information which occurs is i_1 (and not i_2, which would be the complementary information in this example). This probability is calculated by the complete probability theorem, and it coincides with the denominator of Bayes formula. The decision maker wishes to calculate this probability if he/she still has to make a decision about whether to acquire further information or not, so he/she still does not know what the result of this information will be (whether it will be i_1 or i_2).

Revised probabilities are similarly calculated when information i_2 occurs:

$$P(s_1/i_2) = \frac{P(s_1) \cdot P(i_2/s_1)}{P(s_1) \cdot P(i_2/s_1) + P(s_2) \cdot P(i_2/s_2)}$$

$$P(s_2/i_2) = \frac{P(s_2) \cdot P(i_2/s_2)}{P(s_1) \cdot P(i_2/s_1) + P(s_2) \cdot P(i_2/s_2)}$$

$$P(i_2) = P(s_1) \cdot P(i_2/s_1) + P(s_2) \cdot P(i_2/s_2)$$

After calculating all the revised probabilities, the values of Nodes (10) and (11) can be calculated:

$$\begin{aligned}
V_{10} &= \max\{V_{12} - ca_1; V_{13} - ca_2\} \\
V_{12} &= P(s_1/i_1) \cdot Ba_1s_1 + P(s_2/i_1) \cdot Ba_1s_2 \\
V_{13} &= P(s_1/i_1) \cdot Ba_2s_1 + P(s_2/i_1) \cdot Ba_2s_2 \\
V_{11} &= \max\{V_{14} - ca_1; V_{15} - ca_2\} \\
V_{14} &= P(s_1/i_2) \cdot Ba_1s_1 + P(s_2/i_2) \cdot Ba_1s_2 \\
V_{15} &= P(s_1/i_2) \cdot Ba_2s_1 + P(s_2/i_2) \cdot Ba_2s_2
\end{aligned}$$

With the occurrence probabilities of the information, Node (9) can be calculated:

$$V_9 = P(i_1) \cdot V_{10} + P(i_2) \cdot V_{11}$$

Finally, Node (1) can be calculated:

$$V_1 = \max\{V_2; V_9 - ci\}$$

If $V_2 \geq V_9 - ci$, the decision maker is not interested in acquiring new information because the additional profit obtained from employing this information ($V_9 - V_2$) is lower than the cost of information c_i. In the reverse situation, the decision maker pays this cost for further information, checks if this information is s_1 or s_2, then goes to the corresponding tree branch by selecting a node with a higher value. Indeed the difference ($V_9 - V_2$) is the maximum cost that this decision maker is willing to pay for further information, and the more information there is, the higher this cost becomes; in other words, the closer the revised probabilities come to extreme values 1 and 0. This information is more or less useful, so it is also known as imperfect information, and what the decision maker is willing to pay is the value of this imperfect information:

$$V_I = (V_9 - V_2)$$

Perfect information is that which transforms revised probabilities into 1 or 0; in other words, that which can ensure the decision maker which of the states will absolutely occur. For example, by taking the case of information i_1, the following result must be obtained:

$$P(s_1/i_1) = \frac{P(s_1) \cdot P(i_1/s_1)}{P(s_1) \cdot P(i_1/s_1) + P(s_2) \cdot P(i_1/s_2)} = 1$$

$$P(s_2/i_1) = \frac{P(s_2) \cdot P(i_1/s_2)}{P(s_1) \cdot P(i_1/s_1) + P(s_2) \cdot P(i_1/s_2)} = 0$$

That is

$$P(s_2) \cdot P(i_1/s_2) = 0 \to P(i_1/s_2) = 0$$

This means that the probability of information i_1 occurring should state s_2 occur is null. This probability will lead to perfect information. The decision maker can calculate what the maximum profit is if perfect information is obtained as follows:

$$V_{\max} = P(s_1) \cdot \max\{Ba_1s_1; Ba_2s_1\} + P(s_2) \cdot \max\{Ba_1s_2; Ba_2s_2\}$$

That is to say, if the decision maker has perfect information and knows the state to take place, he/she will select the decision which provides the highest profit. However, it should be considered that the a priori probabilities remain, so the expected value with perfect information must be calculated.

The decision maker can calculate what the value of this perfect information is as so:

$$V_P = (V_{\max} - V_2)$$

The ratio between the imperfect information value and the perfect information value (PIV) gives us an idea of how close, or how far, the imperfect information is from being perfect:

$$R = \frac{V_I}{V_P}$$

After reading this chapter, readers should be able to understand the nature of low-risk decision problems and their modelling by means of decision trees, to discover the solution method by calculating the EMV, to apply Bayes Theorem to calculate revised probabilities according to new information, and to interpret solutions by calculating the limit values of the costs of further information.

Selected books for further reading can be found in the References section.

7.2 Selecting Suppliers to Manufacture Solar Panels

The firm SUN2 manufactures solar energy panels, and it has received an urgent order for 40 panels from a new German customer which is willing to pay 1,000$ per panel. SUN2 has committed its production capacity to other orders, and the only alternative left to serve the new order is to subcontract the production involved.

SUN2 has located two possible suppliers: Supplier A, based in Romania, which would sell SUN2 each panel for 500$; Supplier B, from Poland, which would sell each panel for 550$. The cost to transport one batch of 40 panels from either of the two suppliers to the German firm is 1,000$.

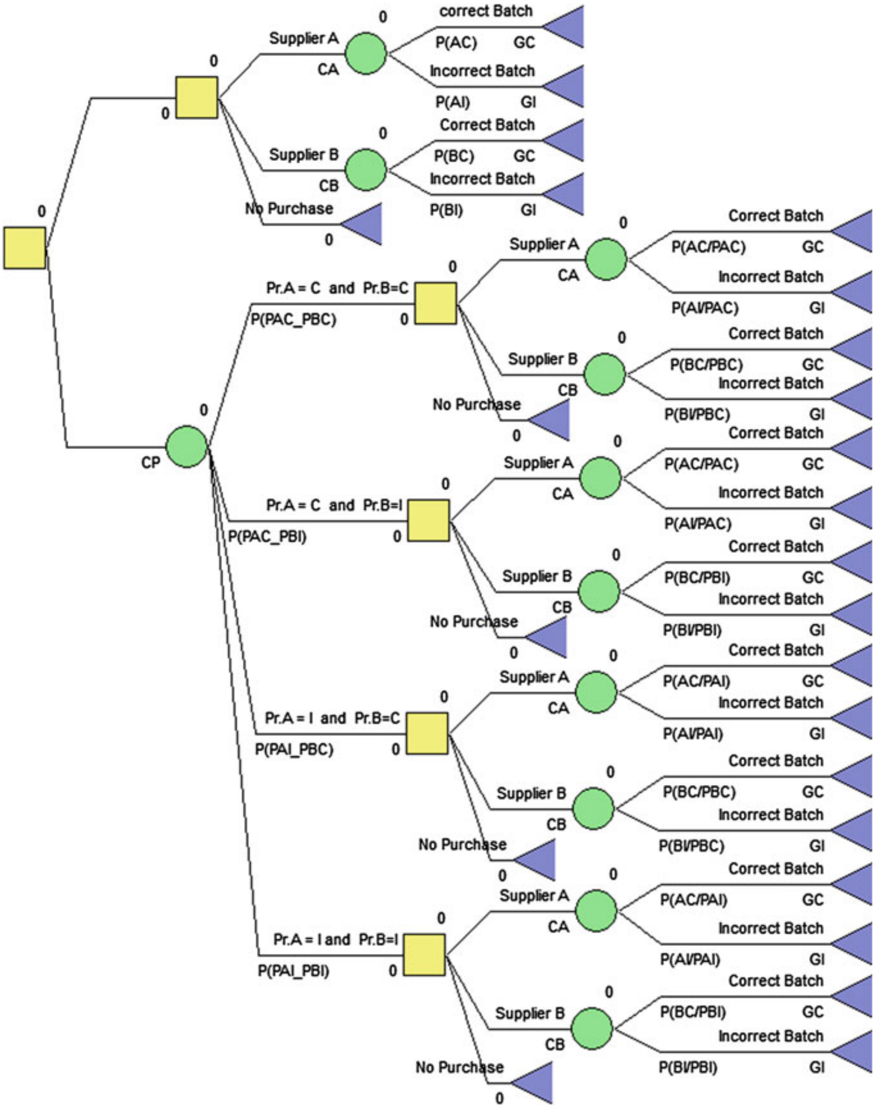
Nonetheless, the customer's technical specifications are very demanding, so SUN2 estimates that Supplier A could supply one correct batch with a probability of 70 %, whereas Supplier B could do it with a probability of 80 %. If the batch were incorrect, SUN2 could sell it to another customer at the price of 200$ per panel.

SUN2 is considering the possibility of asking for a test panel from each supplier to test it before placing an order. SUN2's testing system guarantees that a faulty panel is always detected, while there is a 5 % margin of error with correct panels. The testing system is prepared to assess two test panels, and the overall cost (cost of the two panels + transport + cost of the test) comes to 2,000$.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the decision tree
(c) Up to how much should SUN2 pay for the test?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the decision tree.

Given the following income and expense values:

$$\mathrm{CA} = -20 - 1 = -21$$
$$\mathrm{CB} = -22 - 1 = -23$$
$$\mathrm{GC} = 40$$
$$\mathrm{GI} = 8$$
$$\mathrm{GI} = 8$$

The initial (a priori) probabilities are:

$$P(\mathrm{AC}) = 0.7$$
$$P(\mathrm{AI}) = 0.3$$
$$P(\mathrm{BC}) = 0.8$$
$$P(\mathrm{BI}) = 0.2$$

The conditional probabilties and further information are:

$$P(\mathrm{PAC}/\mathrm{AC}) = 0.95$$
$$P(\mathrm{PAI}/\mathrm{AC}) = 0.05$$
$$P(\mathrm{PAC}/\mathrm{AI}) = 0$$
$$P(\mathrm{PAI}/\mathrm{AI}) = 1$$

$$P(\mathrm{PBC}/\mathrm{BC}) = 0.95$$
$$P(\mathrm{PBI}/\mathrm{BC}) = 0.05$$
$$P(\mathrm{PBC}/\mathrm{BI}) = 0$$
$$P(\mathrm{PBI}/\mathrm{BI}) = 1$$

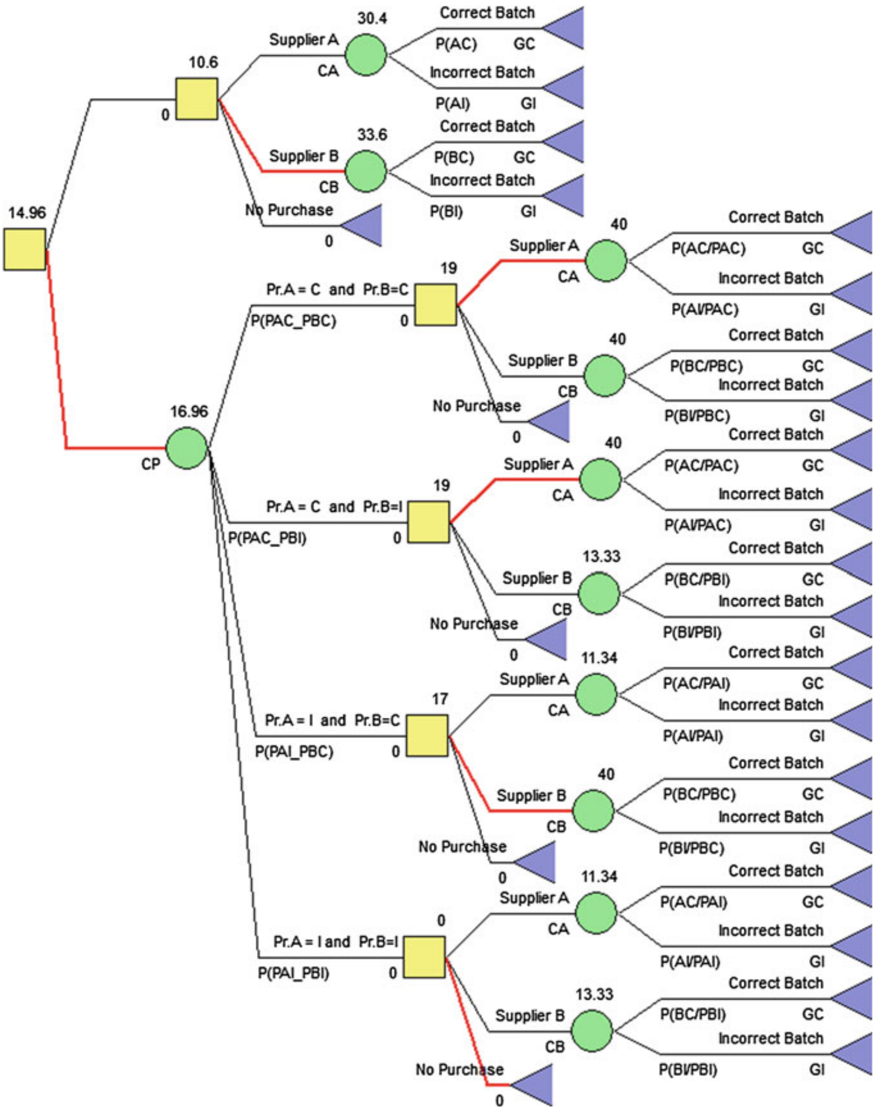
By applying Bayes Theorem, the revised (a posteriori) probabilities are obtained:

$$P(\mathrm{AC/PAC}) = \frac{P(\mathrm{PAC/AC}) \cdot P(\mathrm{AC})}{P(\mathrm{PAC/AC}) \cdot P(\mathrm{AC}) + P(\mathrm{PAC/AI}) \cdot P(\mathrm{AI})} = 1$$
$$P(\mathrm{AI/PAC}) = \frac{P(\mathrm{PAC/AI}) \cdot P(\mathrm{AI})}{P(\mathrm{PAC/AC}) \cdot P(\mathrm{AC}) + P(\mathrm{PAC/AI}) \cdot P(\mathrm{AI})} = 0$$
$$P(\mathrm{AC/PAI}) = \frac{P(\mathrm{PAI/AC}) \cdot P(\mathrm{AC})}{P(\mathrm{PAI/AC}) \cdot P(\mathrm{AC}) + P(\mathrm{PAI/AI}) \cdot P(\mathrm{AI})} = 0.1045$$
$$P(\mathrm{AI/PAI}) = \frac{P(\mathrm{PAI/AI}) \cdot P(\mathrm{AI})}{P(\mathrm{PAI/AC}) \cdot P(\mathrm{AC}) + P(\mathrm{PAI/AI}) \cdot P(\mathrm{AI})} = 0.8955$$
$$P(\mathrm{BC/PBC}) = \frac{P(\mathrm{PBC/BC}) \cdot P(\mathrm{BC})}{P(\mathrm{PBC/BC}) \cdot P(\mathrm{BC}) + P(\mathrm{PBC/BI}) \cdot P(\mathrm{BI})} = 1$$
$$P(\mathrm{BI/PBC}) = \frac{P(\mathrm{PBC/BI}) \cdot P(\mathrm{BI})}{P(\mathrm{PBC/BC}) \cdot P(\mathrm{BC}) + P(\mathrm{PBC/BI}) \cdot P(\mathrm{BI})} = 0$$
$$P(\mathrm{BC/PBI}) = \frac{P(\mathrm{PBI/BC}) \cdot P(\mathrm{BC})}{P(\mathrm{PBI/BC}) \cdot P(\mathrm{BC}) + P(\mathrm{PBI/BI}) \cdot P(\mathrm{BI})} = 0.1667$$
$$P(\mathrm{BI/PBI}) = \frac{P(\mathrm{PBI/BI}) \cdot P(\mathrm{BI})}{P(\mathrm{PBI/BC}) \cdot P(\mathrm{BC}) + P(\mathrm{PBI/BI}) \cdot P(\mathrm{BI})} = 0.8333$$

The occurrence probabilities of further information are:

$$P(\mathrm{PAC}) = P(\mathrm{PAC/AC}) \cdot P(\mathrm{AC}) + P(\mathrm{PAC/AI}) \cdot P(\mathrm{AI}) = 0.665$$
$$P(\mathrm{PAI}) = P(\mathrm{PAI/AC}) \cdot P(\mathrm{AC}) + P(\mathrm{PAI/AI}) \cdot P(\mathrm{AI}) = 1 - P(\mathrm{PAC}) = 0.335$$
$$P(\mathrm{PBC}) = P(\mathrm{PBC/BC}) \cdot P(\mathrm{BC}) + P(\mathrm{PBC/BI}) \cdot P(\mathrm{BI}) = 0.76$$
$$P(\mathrm{PBI}) = P(\mathrm{PBI/BC}) \cdot P(\mathrm{BC}) + P(\mathrm{PBI/BI}) \cdot P(\mathrm{BI}) = 0.24$$
$$P(\mathrm{PAC\_PBC}) = 0.665 \cdot 0.76 = 0.5054$$
$$P(\mathrm{PAC\_PBI}) = 0.665 \cdot 0.24 = 0.1596$$
$$P(\mathrm{PAI\_PBC}) = 0.335 \cdot 0.76 = 0.2546$$
$$P(\mathrm{PAI\_PBI}) = 0.335 \cdot 0.24 = 0.0804$$

By resolving the decision tree, we find:



Thus, test panels should be ordered from both suppliers. Supplier A is chosen if the panels of both suppliers are correct, or if the panel from Supplier A is the only correct panel received. Furthermore, if the panel obtained from Supplier A is faulty, but that from Supplier B is correct, Supplier B should be selected, which leaves the option of not placing any order if both test panels are faulty.

(c) Up to how much should SUN2 pay for the test?

To $16.96 - 10.60 = 6.36\$$ per test.

7.3 Selling a Car

Peter wishes to sell his 8-year-old car to buy a new one. He has placed a for sale ad on the Internet asking for \$1,200 and indicating that the car has 1 month left before its MOT needs renewing. He has received two replies: Ray agrees to pay \$1,200, but after the car has passed its MOT; James is willing to buy it now, but offers \$1,000.

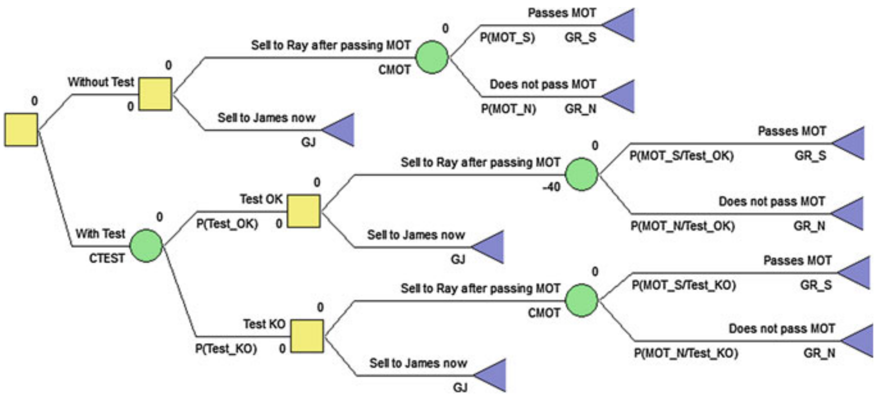
Peter is not absolutely certain that his car will pass its MOT, but he estimates a 50 % probability of it passing it. Lewis, a friend of Peter's, runs a garaje and offers to test his car, and he would only charge \$20. Lewis insists that his test never fails if the car does not pass its MOT, and that his test states that the car would not pass the MOT, but then it does, in only 20 % of cases. Lewis has also explained to Peter that if his car does not pass the MOT, he can mend the car to pass it for \$300.

An MOT costs \$40 every time that a car takes it.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the decision tree
(c) Up to what amount is Peter willing to pay for Lewis' test?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the decision tree

Given the following income and costs values:

$$\begin{aligned}
\text{GR_S} &= 1200 \\
\text{GJ} &= 1000 \\
\text{CITV} &= -40 \\
\text{CREP} &= -300 - 40 = -340 \\
\text{GR_N} &= 1200 - 300 - 40 = -1540 \\
\text{CTEST} &= -20
\end{aligned}$$

The initial (a priori) probabilities:

$$\begin{aligned}
P(\text{MOT_S}) &= 0.5 \\
P(\text{MOT_N}) &= 0.5
\end{aligned}$$

The condition probabilities and further information:

$$\begin{aligned}
P(\text{Test_OK}/\text{MOT_S}) &= 0.8 \\
P(\text{Test_KO}/\text{MOT_S}) &= 0.2 \\
P(\text{Test_OK}/\text{MOT_N}) &= 0 \\
P(\text{Test_KO}/\text{MOT_N}) &= 1
\end{aligned}$$

By applying Bayes Theorem, the revised (a posteriori) probabilties are:

$$P(\text{MOT\_S}/\text{Test\_OK}) = \frac{P(\text{Test\_OK}/\text{MOT\_S}) \cdot P(\text{MOT\_S})}{P(\text{Test\_OK}/\text{MOT\_S}) \cdot P(\text{MOT\_S}) + P(\text{Test\_OK}/\text{MOT\_N}) \cdot P(\text{MOT\_N})} = 1$$

$$P(\text{MOT\_N}/\text{Test\_OK}) = \frac{P(\text{Test\_OK}/\text{MOT\_N}) \cdot P(\text{MOT\_N})}{P(\text{Test\_OK}/\text{MOT\_S}) \cdot P(\text{MOT\_S}) + P(\text{Test\_OK}/\text{MOT\_N}) \cdot P(\text{MOT\_N})} = 0$$

$$P(\text{MOT\_S}/\text{Test\_KO}) = \frac{P(\text{Test\_KO}/\text{MOT\_S}) \cdot P(\text{MOT\_S})}{P(\text{Test\_KO}/\text{MOT\_S}) \cdot P(\text{MOT\_S}) + P(\text{Test\_KO}/\text{MOT\_N}) \cdot P(\text{MOT\_N})} = 0.1667$$

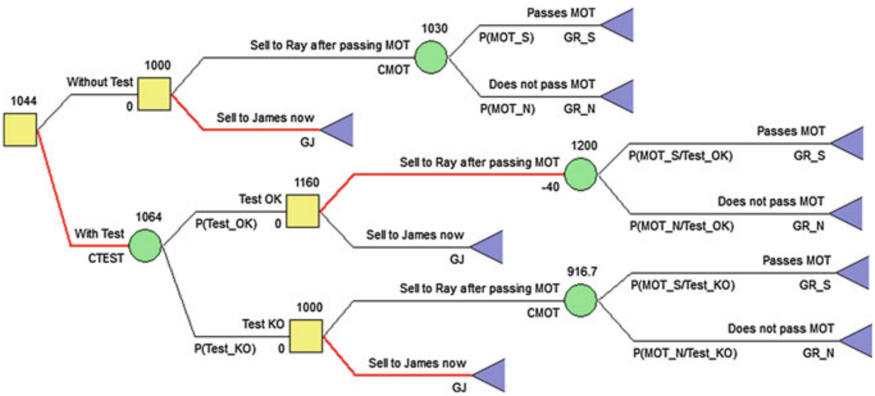
$$P(\text{MOT\_N}/\text{Test\_KO}) = \frac{P(\text{Test\_KO}/\text{MOT\_N}) \cdot P(\text{MOT\_N})}{P(\text{Test\_KO}/\text{MOT\_S}) \cdot P(\text{MOT\_S}) + P(\text{Test\_KO}/\text{MOT\_N}) \cdot P(\text{MOT\_N})} = 0.8333$$

The occurrence probabilities of the further information are:

$$\begin{aligned}
P(\text{Test_OK}) &= P(\text{Test_OK}/\text{MOT_S}) \cdot P(\text{MOT_S}) + P(\text{Test_OK}/\text{MOT_N}) \\
&\quad \cdot P(\text{MOT_N}) \\
&= 0.4
\end{aligned}$$

$$\begin{aligned}
P(\text{Test_KO}) &= P(\text{Test_KO}/\text{MOT_S}) \cdot P(\text{MOT_S}) + P(\text{Test_KO}/\text{MOT_N}) \\
&\quad \cdot P(\text{MOT_N}) \\
&= 1 - P(\text{Test_OK}) = 0.6
\end{aligned}$$

By solving the decision tree:



The test Lewis offers should be done, and if it proves satisfactory, Peter should sell his car to Ray after passing the MOT. However, if the result is not good, the better option is to sell it directly to James.

(c) Up to what amount is Peter willing to pay for Lewis' test?

To $1,064 − 1,000 = $64.

7.4 Football Bets

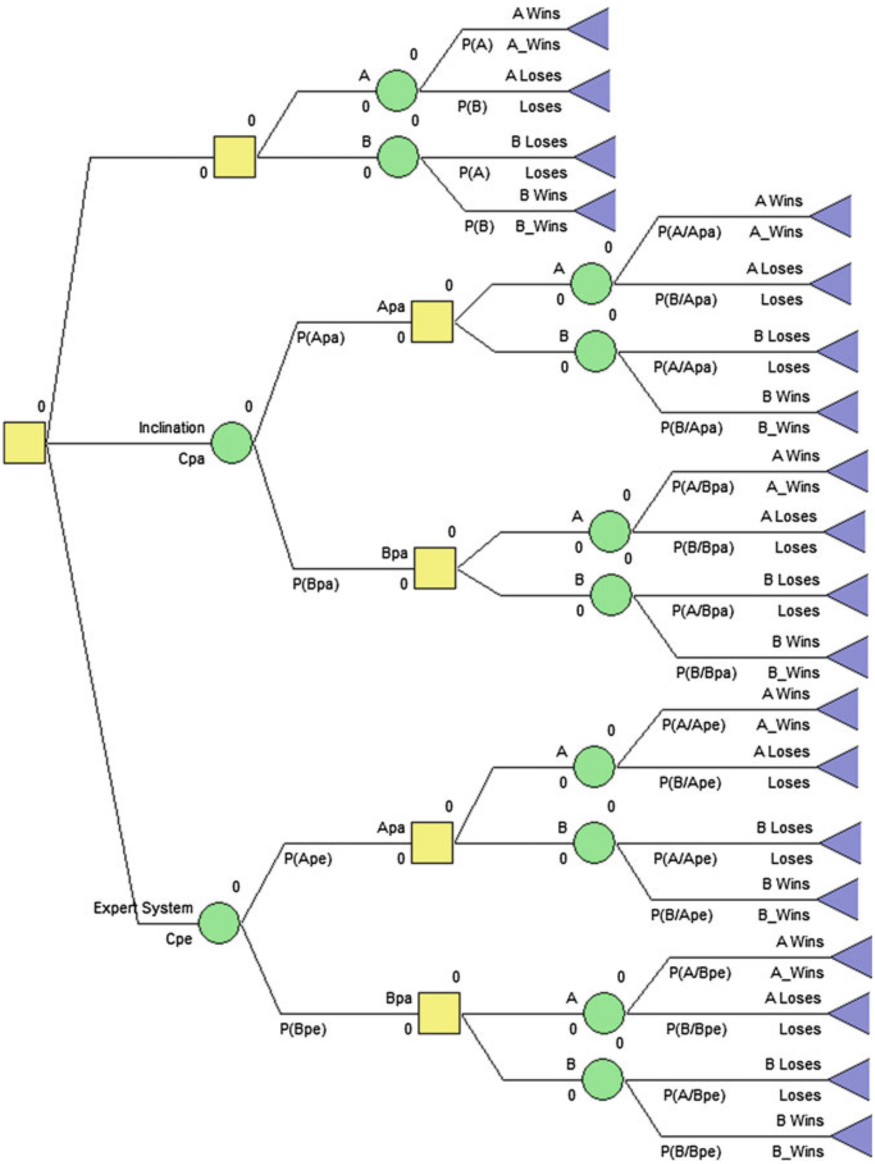
Smith is a big football fan and the 2010 World Cup final is played tomorrow. As his team did not classify, he wishes to bet 100$ on either of the finalists (Holland-Spain). He will place his bet on a well-known Internet portal. Right now, the bets are 2–1 for Spain. This means that if Smith bets for Holland and bets right, he will obtain a profit of 200$, whereas in the case of Spain, he will win a $50 profit.

The Internet portal offers him the option of employing two counselling systems: the first provides him the winner's name according to fans, with a 60 % chance of getting it right, and it costs 5 % of the bet. The second option predicts the winner according to an expert system, and there is a 70 % chance of getting it right and it costs 20 % of the bet. Should Smith decide on contracting either counselling system, Smith will bet 100$ and will also pay the counselling cost.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the decision tree
(c) What is the value of the imperfect information provided by both counselling services?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the decision tree

Given the following income and costs values:

$$\begin{aligned}
\text{A} - \ \text{wins} &= 200 \\
\text{B} - \ \text{wins} &= 50 \\
\text{Loses} &= -100 \\
\text{Cpa} &= -5 \\
\text{Cpe} &= -20
\end{aligned}$$

The initial (a priori) probabilities are:

P(A) = 1/3 where A corresponds to Holland
P(B) = 2/3 where B corresponds to Spain

The conditional probabilities and further information are:

$$\begin{aligned}
P(\text{Apa}/\text{A}) &= 0.6 \\
P(\text{Apa}/\text{B}) &= 0.4 \\
P(\text{Bpa}/\text{A}) &= 0.4 \\
P(\text{Bpa}/\text{B}) &= 0.6
\end{aligned}$$

$$\begin{aligned}
P(\text{Ape}/\text{A}) &= 0.7 \\
P(\text{Ape}/\text{B}) &= 0.3 \\
P(\text{Bpe}/\text{A}) &= 0.3 \\
P(\text{Bpe}/\text{B}) &= 0.7
\end{aligned}$$

By applying Bayes Theorem, the revised (a posteriori) probabilities obtained are:

$$P(\text{A}/\text{Apa}) = \frac{P(\text{Apa}/\text{A}) \cdot P(\text{A})}{P(\text{Apa}/\text{A}) \cdot P(\text{A}) + P(\text{Apa}/\text{B}) \cdot P(\text{B})} = 0.4285$$

$$P(\text{B}/\text{Apa}) = \frac{P(\text{Apa}/\text{B}) \cdot P(\text{B})}{P(\text{Apa}/\text{A}) \cdot P(\text{A}) + P(\text{Apa}/\text{B}) \cdot P(\text{B})} = 0.5714$$

$$P(\text{A}/\text{Bpa}) = \frac{P(\text{Bpa}/\text{A}) \cdot P(\text{A})}{P(\text{Bpa}/\text{A}) \cdot P(\text{A}) + P(\text{Bpa}/\text{B}) \cdot P(\text{B})} = 0.2500$$

$$P(\text{B}/\text{Bpa}) = \frac{P(\text{Bpa}/\text{B}) \cdot P(\text{B})}{P(\text{Bpa}/\text{A}) \cdot P(\text{A}) + P(\text{Bpa}/\text{B}) \cdot P(\text{B})} = 0.7500$$

$$P(\text{A}/\text{Ape}) = \frac{P(\text{Ape}/\text{A}) \cdot P(\text{A})}{P(\text{Ape}/\text{A}) \cdot P(\text{A}) + P(\text{Ape}/\text{B}) \cdot P(\text{B})} = 0.5385$$

$$P(\mathrm{B/Ape}) = \frac{P(\mathrm{Ape/B}) \cdot P(\mathrm{B})}{P(\mathrm{Ape/A}) \cdot P(\mathrm{A}) + P(\mathrm{Ape/B}) \cdot P(\mathrm{B})} = 0.4615$$

$$P(\mathrm{A/Bpe}) = \frac{P(\mathrm{Bpe/A}) \cdot P(\mathrm{A})}{P(\mathrm{Bpe/A}) \cdot P(\mathrm{A}) + P(\mathrm{Bpe/B}) \cdot P(\mathrm{B})} = 0.1764$$

$$P(\mathrm{B/Bpe}) = \frac{P(\mathrm{Bpe/B}) \cdot P(\mathrm{B})}{P(\mathrm{Bpe/A}) \cdot P(\mathrm{A}) + P(\mathrm{Bpe/B}) \cdot P(\mathrm{B})} = 0.8234$$

The occurrence probabilities of the further information are:

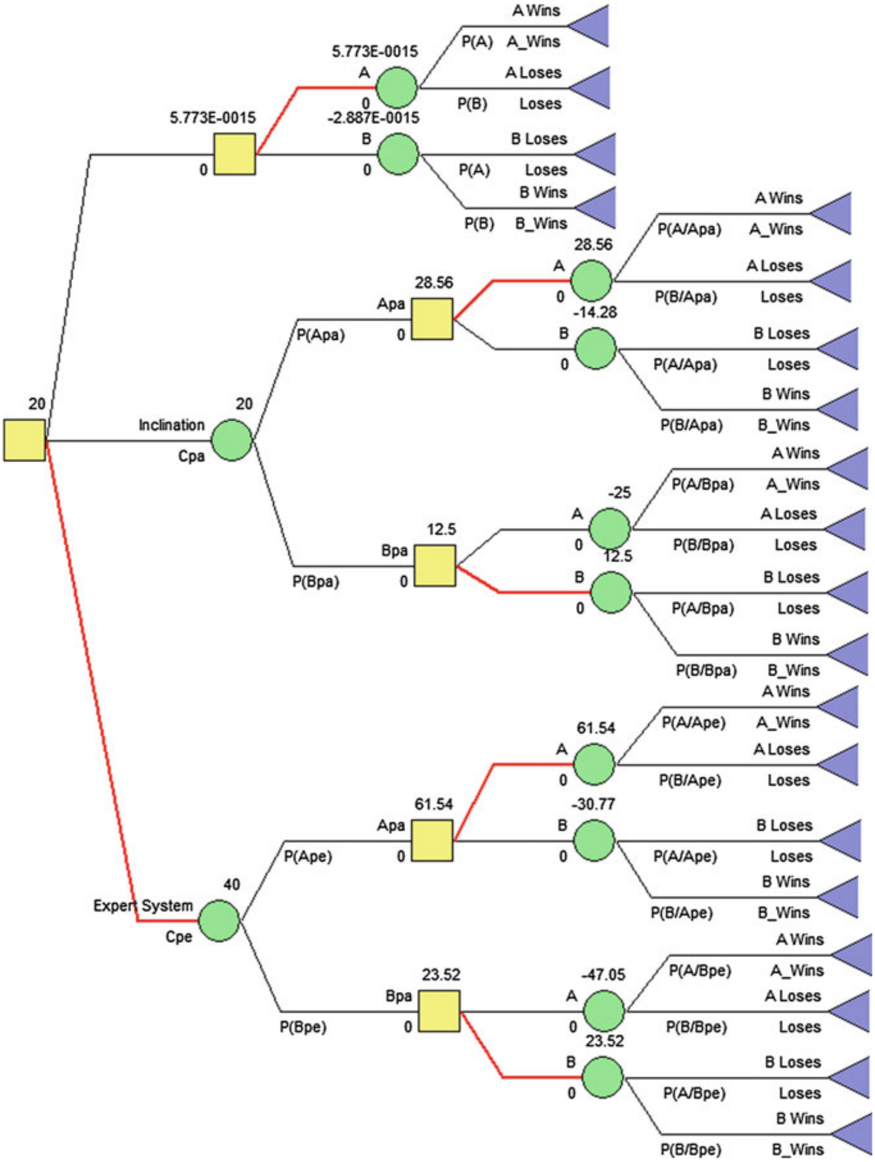
$$P(\mathrm{Apa}) = P(\mathrm{Apa/A}) \cdot P(\mathrm{A}) + P(\mathrm{Apa/B}) \cdot P(\mathrm{B}) = 0.4667$$

$$P(\mathrm{Bpa}) = P(\mathrm{Bpa/A}) \cdot P(\mathrm{A}) + P(\mathrm{Bpa/B}) \cdot P(\mathrm{B}) = 1 - P(\mathrm{Apa}) = 0.5333$$

$$P(\mathrm{Ape}) = P(\mathrm{Ape/A}) \cdot P(\mathrm{A}) + P(\mathrm{Ape/B}) \cdot P(\mathrm{B}) = 0.4333$$

$$P(\mathrm{Bpe}) = P(\mathrm{Bpe/A}) \cdot P(\mathrm{A}) + P(\mathrm{Bpe/B}) \cdot P(\mathrm{B}) = 0.5667$$

By solving the decision tree:



The better option is to choose the expert system and to bet for Holland when the prediction favours Holland, and to place a bet on Spain when the prediction favours Spain.

(c) What is the value of the imperfect information provided by both counselling services?

The imperfect information value is \$20 for fans and \$40 for the expert system.

7.5 Public Tender for Maritime Measuring Instruments

ELECTROSEA is a firm that specialises in maritime measuring apparatus. The U.S. government has just published a public tender to acquire 100 thermosalinographs with some advanced options, which mean that the firm granted the public tender will have to provide a new design and its production. The top price paid per unit is 1,400$.

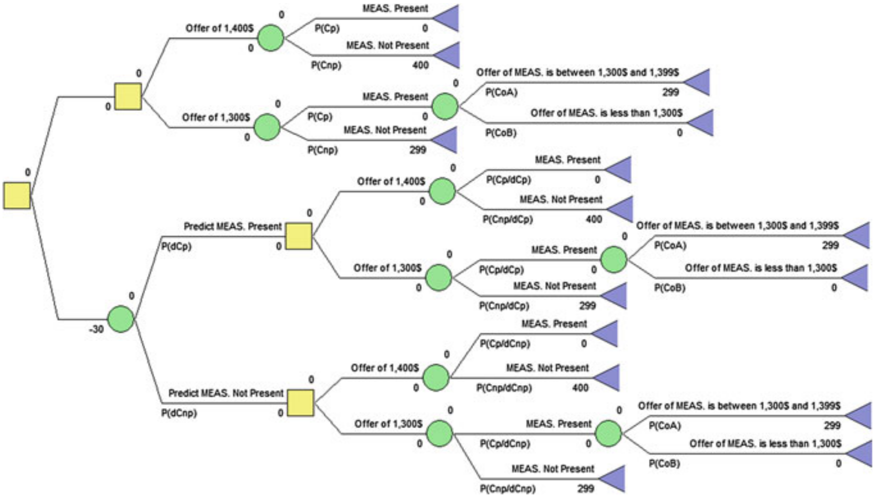
ELECTROSEA has estimated that the final cost of each thermosalinograph is around 1,000$. Nonetheless, the possibility of the firms' main competitor, MEASURESEA, also entering the public tender is considered (they estimate that there is a 50 % probability of this happening). After analysing MEASURESEA's past performance in calls of this kind, it is deduced that it will not offer more than 1,399$ and it is estimated that the probability of its offer being below 1,300$ is 60 %. The remaining competitors do not worry ELECTROSEA as they do not have the technology to design and produce such advanced thermosalinographs.

ELECTROSEA can contract the services of CONCONSULT, a conultancy firm which specialises in counselling firms interested in participating in government public tenders. Specifically, CONCONSULT will conduct a study of MEASURESEA's current public projects and will give its opinion about whether or not this firm will participate in the advanced thermosalinographs call. CONCONSULT guarantees a success rate of 90 % in what it predicts of when a competitor participates, and one of 80 % when it does not. CONCONSULT's fees are 3,000$.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree
(c) Up to what amount can ELECTROSEA pay CONCONSULT? How efficient is the information offered by CONCONSULT?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



All the results are in 100 units.

(b) Solve the tree

The initial (a priori) probabilities are:

$$P(\mathrm{Cp}) = 0.5$$
$$P(\mathrm{Cnp}) = 0.5$$
$$P(\mathrm{CoA}) = 0.4$$
$$P(\mathrm{CoB}) = 0.6$$

The conditional probabilities (further information) are:

$$P(\mathrm{dCp}/\mathrm{Cp}) = 0.9$$
$$P(\mathrm{dCnp}/\mathrm{Cp}) = 0.1$$
$$P(\mathrm{dCp})/\mathrm{Cnp}) = 0.2$$
$$P(\mathrm{dCnp}/\mathrm{Cnp}) = 0.8$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\mathrm{Cp}/\mathrm{dCp}) = \frac{P(\mathrm{dCp}/\mathrm{Cp}) \cdot P(\mathrm{Cp})}{P(\mathrm{dCp}/\mathrm{Cp}) \cdot P(\mathrm{Cp}) + P(\mathrm{dCp})/\mathrm{Cnp}) \cdot P(\mathrm{Cnp})} = 0.8182$$

$$P(\mathrm{Cnp}/\mathrm{dCp}) = \frac{P(\mathrm{dCp}/\mathrm{Cnp}) \cdot P(\mathrm{Cnp})}{P(\mathrm{dCp}/\mathrm{Cp}) \cdot P(\mathrm{Cp}) + P(\mathrm{dCp})/\mathrm{Cnp}) \cdot P(\mathrm{Cnp})} = 0.1818$$

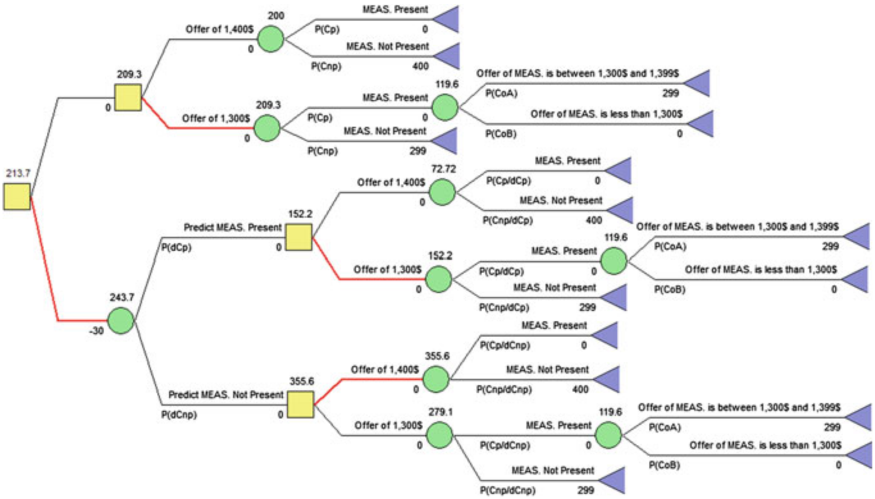
$$P(\mathrm{Cp}/\mathrm{dCnp}) = \frac{P(\mathrm{dCnp}/\mathrm{Cp}) \cdot P(\mathrm{Cp})}{P(\mathrm{dCnp}/\mathrm{Cp}) \cdot P(\mathrm{Cp}) + P(\mathrm{dCnp})/\mathrm{Cnp}) \cdot P(\mathrm{Cnp})} = 0.1111$$

$$P(\mathrm{Cnp}/\mathrm{dCnp}) = \frac{P(\mathrm{dCnp})/\mathrm{Cnp}) \cdot P(\mathrm{Cnp})}{P(\mathrm{dCnp}/\mathrm{Cp}) \cdot P(\mathrm{Cp}) + P(\mathrm{dCnp})/\mathrm{Cnp}) \cdot P(\mathrm{Cnp})} = 0.8889$$

The occurrence probabilities of the further information are:

$$P(\mathrm{dCp}) = P(\mathrm{dCp}/\mathrm{Cp}) \cdot P(\mathrm{Cp}) + P(\mathrm{dCp})/\mathrm{Cnp}) \cdot P(\mathrm{Cnp}) = 0.55$$

$$P(\mathrm{dCnp}) = P(\mathrm{dCnp}/\mathrm{Cp}) \cdot P(\mathrm{Cp}) + P(\mathrm{dCnp})/\mathrm{Cnp}) \cdot P(\mathrm{Cnp}) = 1 - P(\mathrm{dCp}) = 0.45$$



According to the tree solution, the consulting firm should be resorted to. If the consulting firm predicts that MEASURESEA will participate in the tender, ELECTROSEA should make an offer of \$1,300. Yet if the prediction suggests that the rival firm will not take part, ELECTROSEA's offer should be \$1,400.

(c) Up to what amount can ELECTROSEA pay CONCONSULT? How efficient is the information offered by CONCONSULT?

The earnings without resorting to the consultancy firm will be \$20,930, while they will be \$24,370 if it is resorted to. Therefore, the maximum quantity to pay the consultant will be \$3,440. The efficiency of the information offered will be 69.7 %.

7.6 Providing Personal Loans

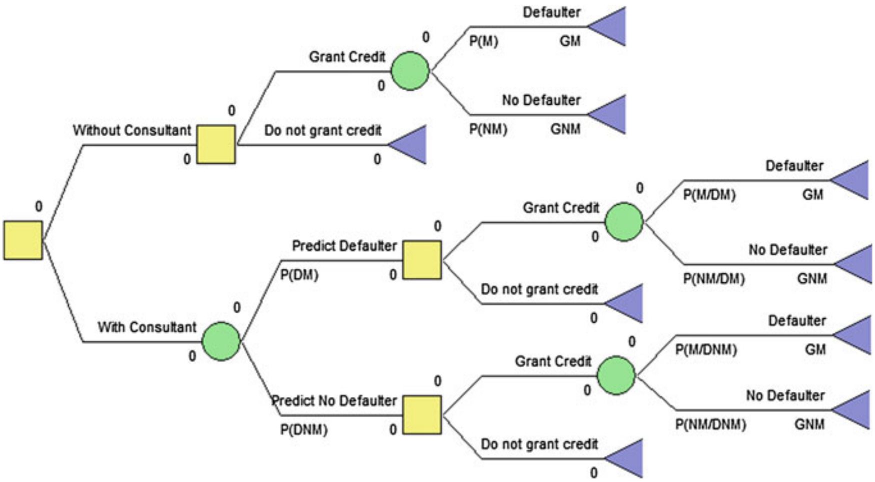
CARCREDIT is a lending agency that specialises in providing personal loans to purchase new cars with. Normally, it automatically provides loans of up to \$30,000 over a maximum period of 4 years. Nevertheless, the recent rise in arrears of payment, which affects 1 in 4 customers, has meant that it has had to reconsider its credit-loaning strategy. This agency obtains, on average, a 10 % profit from the capital loaned if customers return it, but loses 20 % of the capital loaned with defaulters.

CARCREDIT can contract the services of MORCONSULT, a consultancy firm specialised in making rapid evaluations (under 24 h) of potential customers, and it charges a percent of the credit to be loaned. The consultancy firm publishes the success rates obtained to date: it is successful in 70 % of the cases in which a potential customer becomes a defaulting debtor, and in 80 % of the cases where a potential customer completely pays back the loan.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree
(c) What percentage of the credit to be loaned should CARCREDIT pay MORCONSULT?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree



The initial (a priori) probabilities are:

$$P(\mathrm{M}) = 1/4$$
$$P(\mathrm{NM}) = 3/4$$

The conditional probabilties and further information are:

$$\begin{aligned}
&\mathrm{CC} = 100\\
&\mathrm{GNM} = 0.1 \cdot \mathrm{CC} = 0.1 \cdot 100 = 10\\
&\mathrm{GM} = -0.2 \cdot \mathrm{CC} = -0.2 \cdot 100 = -20\\
&P(\mathrm{DM/M}) = 0.7\\
&P(\mathrm{DNM/M}) = 0.3\\
&P(\mathrm{DM/NM}) = 0.2\\
&P(\mathrm{DNM/NM}) = 0.8
\end{aligned}$$

By applying Bayes Theorem, the revised (a posteriori) probabilies are:

$$P(\mathrm{M/DM}) = \frac{P(\mathrm{DM/M}) \cdot P(\mathrm{M})}{P(\mathrm{DM/M}) \cdot P(\mathrm{M}) + P(\mathrm{DM/NM}) \cdot P(\mathrm{NM})} = 0.5385$$

$$P(\mathrm{NM/DM}) = \frac{P(\mathrm{DM/NM}) \cdot P(\mathrm{NM})}{P(\mathrm{DM/M}) \cdot P(\mathrm{M}) + P(\mathrm{DM/NM}) \cdot P(\mathrm{NM})} = 0.4615$$

$$P(\mathrm{M/DNM}) = \frac{P(\mathrm{DNM/M}) \cdot P(\mathrm{M})}{P(\mathrm{DNM/M}) \cdot P(\mathrm{M}) + P(\mathrm{DNM/NM}) \cdot P(\mathrm{NM})} = 0.1111$$

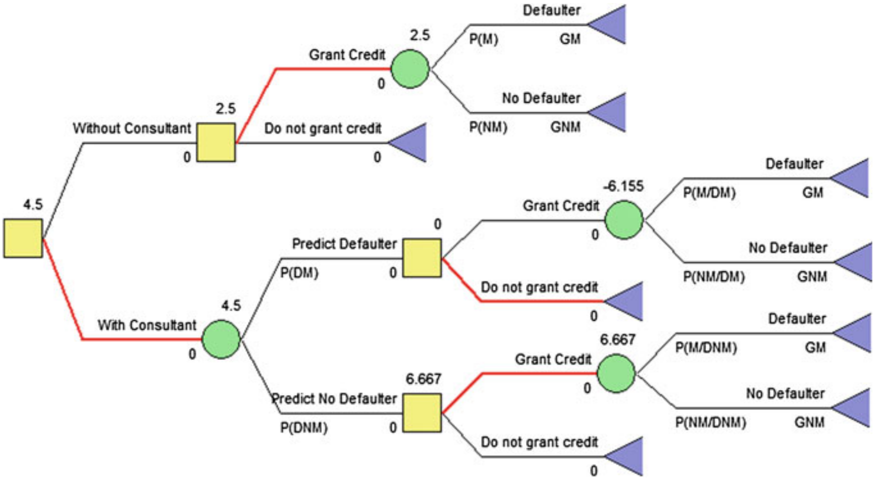
$$P(\mathrm{NM/DNM}) = \frac{P(\mathrm{DNM/NM}) \cdot P(\mathrm{NM})}{P(\mathrm{DNM/M}) \cdot P(\mathrm{M}) + P(\mathrm{DNM/NM}) \cdot P(\mathrm{NM})} = 0.8889$$

The occurrence probabilities of the further information are:

$$P(\mathrm{DM}) = P(\mathrm{DM/M}) \cdot P(\mathrm{M}) + P(\mathrm{DM/NM}) \cdot P(\mathrm{NM}) = 0.325$$

$$P(\mathrm{DNM}) = P(\mathrm{DNM/M}) \cdot P(\mathrm{M}) + P(\mathrm{DNM/NM}) \cdot P(\mathrm{NM}) = 0.675$$

By solving the decision tree



According to the solution obtained, CARCREDIT should resort to MORCONSULT. Should the consultancy agency predict the possibility of arrears of payment, no credit should be loaned. If MORCONSULT's assessment is positive, credit should be loaned.

(c) What percentage of the credit to be loaned should CARCREDIT pay MORCONSULT?

The earnings without resorting to the consultanty firm will be 2.50 %, whereas they will be 4.50 % if the consultancy firm is employed. Therefore, the maximum quantity to pay MORCONSULT should be 2.00 %.

7.7 Spying in Genetic Research

GATACA is a firm which undertakes genetic research. The discoveries made in its R&D department generate patents of high economic value. In the last year however, it has seen how its main competitor, GENETICS, has arrived before it in patenting its most important discoveries, which has led to a loss of 3 million $.

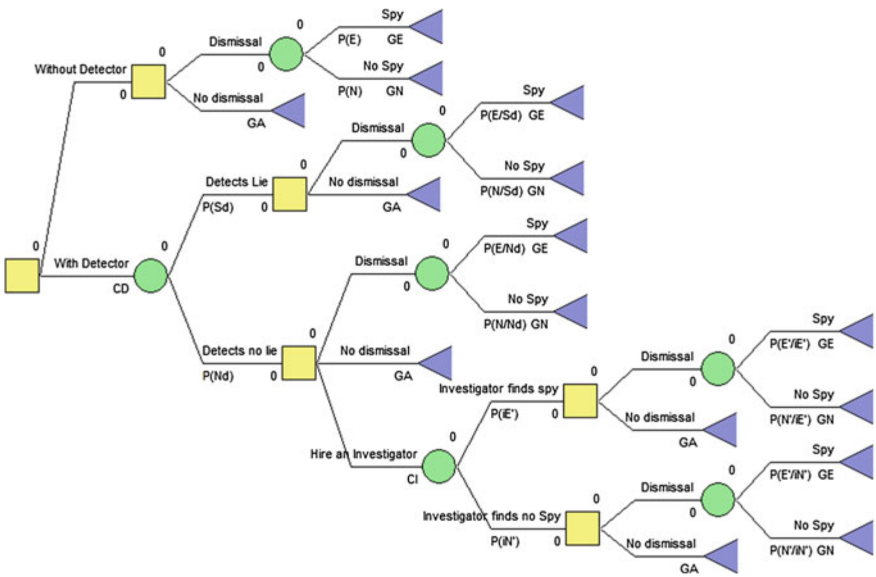
GATACA's Board of Governors hypotheses that Paul Piller, the most highly qualified technician in the R&D department, is a spy and has been informing GENETICS about GATACA's works. Nonetheless, they are not altogether sure because they estimate that the probability of being right is 70 %. If they dismiss Paul Piller and he was not a spy, they will lose their best researcher, who is quite likely to go and work for GENETICS. This can entail a loss of 12 million $ over the next year.

The Board of Governors could make Paul Piller do a lie detector test. The cost of this process is estimated at 50,000$. Besides, the results would not be definitive because it detects only 85 % of liers. Should the lie detector results favour Paul Piller, a private investigator could be contracted, one specialised in industrial spying, which would guarantee them a 95 % probability of success of the conclusions drawn. The fee would come to 100,000$.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree
(c) Up to how much should GATACA pay for a private investigator's services?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the tree

Given the following income and costs values:

$$GE = 0$$

$$GN = -12$$

$$GA = -3$$

$$CD = -0.05$$

$$CI = -0.1$$

The initial (a priori) probabilities:

$$P(\mathrm{E}) = 0.7$$
$$P(\mathrm{N}) = 0.3$$

The conditional probabilities and further information:

$$P(\mathrm{Sd/E}) = 0.85$$
$$P(\mathrm{Nd/E}) = 0.15$$
$$P(\mathrm{Sd/N}) = 0$$
$$P(\mathrm{Nd/N}) = 1$$

$$P(\mathrm{E'}) = 0.2593$$
$$P(\mathrm{N'}) = 0.7407$$
$$P(\mathrm{iE'/E'}) = 0.95$$
$$P(\mathrm{iN'/E'}) = 0.05$$
$$P(\mathrm{iE'/N'}) = 0.05$$
$$P(\mathrm{iN'/N'}) = 0.95$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\mathrm{E/Sd}) = \frac{P(\mathrm{Sd/E}) \cdot P(\mathrm{E})}{P(\mathrm{Sd/E}) \cdot P(\mathrm{E}) + P(\mathrm{Sd/N}) \cdot P(\mathrm{N})} = 1$$

$$P(\mathrm{N/Sd}) = \frac{P(\mathrm{Sd/N}) \cdot P(\mathrm{N})/P(\mathrm{Sd})}{P(\mathrm{Sd/E}) \cdot P(\mathrm{E}) + P(\mathrm{Sd/N}) \cdot P(\mathrm{N})} = 0$$

$$P(\mathrm{E/Nd}) = \frac{P(\mathrm{Nd/E}) \cdot P(\mathrm{E})/P(\mathrm{Nd})}{P(\mathrm{Nd/E}) \cdot P(\mathrm{E}) + P(\mathrm{Nd/N}) \cdot P(\mathrm{N})} = 0.2593$$

$$P(\mathrm{N/Nd}) = \frac{P(\mathrm{Nd/N}) \cdot P(\mathrm{N})/P(\mathrm{Nd})}{P(\mathrm{Nd/E}) \cdot P(\mathrm{E}) + P(\mathrm{Nd/N}) \cdot P(\mathrm{N})} = 0.7407$$

$$P(\mathrm{E'/iE'}) = \frac{P(\mathrm{iE'/E'}) \cdot P(\mathrm{E'})/P(\mathrm{iE'})}{P(\mathrm{iE'/E'}) \cdot P(\mathrm{E'}) + P(\mathrm{iE'/N'}) \cdot P(\mathrm{N'})} = 0.8692$$

$$P(\mathrm{N'/iE'}) = \frac{P(\mathrm{iE'/N'}) \cdot P(\mathrm{N'})/P(\mathrm{iE'})}{P(\mathrm{iE'/E'}) \cdot P(\mathrm{E'}) + P(\mathrm{iE'/N'}) \cdot P(\mathrm{N'})} = 0.1307$$

$$P(\mathrm{E'/iN'}) = \frac{P(\mathrm{iN'/E'}) \cdot P(\mathrm{E'})/P(\mathrm{iN'})}{P(\mathrm{iN'/E'}) \cdot P(\mathrm{E'}) + P(\mathrm{iN'/N'}) \cdot P(\mathrm{N'})} = 0.0181$$

$$P(\mathrm{N'/iN'}) = \frac{P(\mathrm{iN'/N'}) \cdot P(\mathrm{N'})/P(\mathrm{iN'})}{P(\mathrm{iN'/E'}) \cdot P(\mathrm{E'}) + P(\mathrm{iN'/N'}) \cdot P(\mathrm{N'})} = 0.9819$$

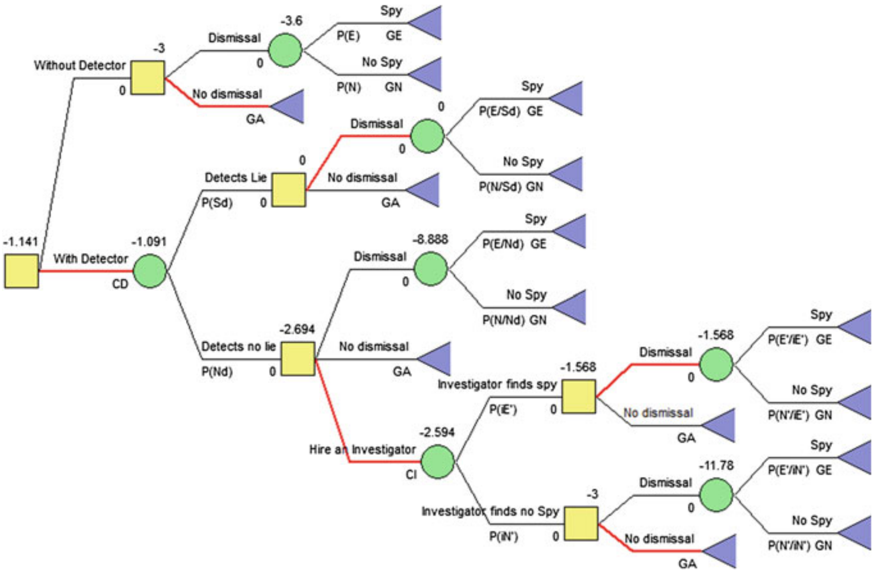
The occurrence probabilties of further information are:

$$P(\mathrm{Sd}) = P(\mathrm{Sd/E}) \cdot P(\mathrm{E}) + P(\mathrm{Sd/N}) \cdot P(\mathrm{N}) = 0.595$$

$$P(\mathrm{Nd}) = P(\mathrm{Nd/E}) \cdot P(\mathrm{E}) + P(\mathrm{Nd/N}) \cdot P(\mathrm{N}) = 0.405$$

$$P(\mathrm{iE'}) = P(\mathrm{iE'/E'}) \cdot P(\mathrm{E'}) + P(\mathrm{iE'/N'}) \cdot P(\mathrm{N'}) = 0.2834$$

$$P(\mathrm{iN'}) = P(\mathrm{iN'/E'}) \cdot P(\mathrm{E'}) + P(\mathrm{iN'/N'}) \cdot P(\mathrm{N'}) = 0.7166$$



The results are expressed in millions of dollars.

Depending on the solution obtained, Paul Piller will do a lie detector test. If Paul Piller is found to have lied, he will be dismissed. However, if the lie detector test proves negative, a private investigator should be contracted. If the private investigator states that Paul Piller has lied, he will be dismissed, but if he states that Paul Piller did not lie, he shall remain in the firm.

(c) Up to how much should GATACA pay for a private investigator's services?

Up to $3 - 2.694 = 0.406$ million \$.

7.8 Choosing an Elliptical Trainer Bike

Peter has decided to buy an elliptical trainer bike to practice sport from home. After looking at several models on the Internet, he cannot make up his mind between a standard model costing \$300 and a professional one priced at \$1,000.

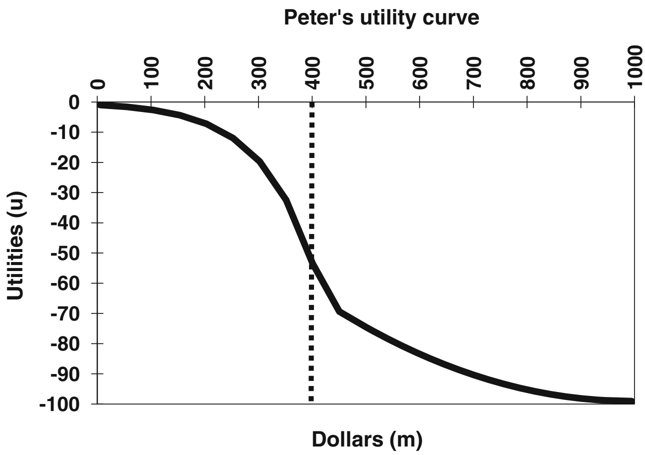


Fig. 7.2 Peter's utility curve

Peter estimates that if he buys the standard model, and this meets his expectations, the utilities from it might amount to 100, whereas if not, they would amount to 10 because, in this case, he would probably tire of it and would use it only occasionally. With the professional bike, these utilities would increase and lower 50 %, respectively. Peter estimates that the probability of a professional model meeting his expectations would be 90 %, whereas it would be 45 % with a standard model.

Peter is considering asking Balti for counselling, who is an Elipdoor instructor and a friend of his. What he is not sure about is asking him for counselling on a standard bike or on a professional one. In exhange, Peter would invite Balti to a meal. In any case, Peter estimates that Balti's recommendation is not altogether reliable and assigns him a 10 % error margin.

Figure 7.2 shows Peter's utility curve, which corresponds to the following equations:

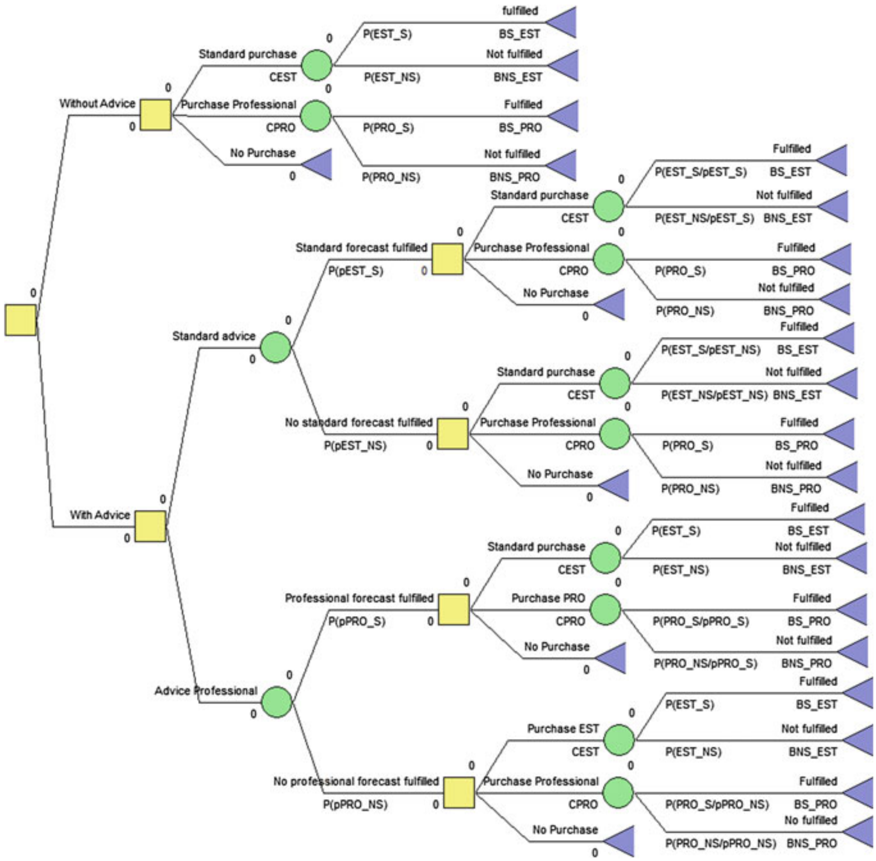
$$u = -e^{\frac{m}{100}} + 1 \quad ; \quad 0 \leq m \leq 400$$

$$u = \frac{m^2}{10000} - \frac{m}{5} \quad ; \quad 400 < m \leq 1000$$

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree
(c) What kind of restaurant must Peter take Balti to? That is, how much money is he interested in paying for the meal at the most?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the tree

Given the following income and costs values:

$$\text{CEST} = -\exp(300/100) + 1 = -19.09$$

$$\text{CPRO} = 100^2/10000 - 1000/5 = -199$$

$$\text{BS\_EST} = 100$$

$$\text{BNS\_EST} = -10$$

$$\text{BS\_PRO} = 150$$

$$\text{BNS\_PRO} = -15$$

The initial (a priori) probabilities are:

$$P(\text{EST\_S}) = 0.45$$
$$P(\text{EST\_NS}) = 1 - 0.45 = 0.55$$
$$P(\text{PRO\_S}) = 0.9$$
$$P(\text{PRO\_NS}) = 1 - 0.9 = 0.1$$

The conditional probabilities and further information are:

$$P(\text{pEST\_S}/\text{EST\_S}) = 0.9$$
$$P(\text{pEST\_S}/\text{EST\_NS}) = 0.1$$
$$P(\text{pEST\_NS}/\text{EST\_S}) = 0.1$$
$$P(\text{pEST\_NS}/\text{EST\_NS}) = 0.9$$
$$P(\text{pPRO\_S}/\text{PRO\_S}) = 0.9$$
$$P(\text{pPRO\_S}/\text{PRO\_NS}) = 0.1$$
$$P(\text{pPRO\_NS}/\text{PRO\_S}) = 0.1$$
$$P(\text{pPRO\_NS}/\text{PRO\_NS}) = 0.9$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\text{EST\_S}/\text{pEST\_S}) = \frac{P(\text{pEST\_S}/\text{EST\_S}) \cdot P(\text{EST\_S})}{P(\text{pEST\_S}/\text{EST\_S}) \cdot P(\text{EST\_S}) + P(\text{pEST\_S}/\text{EST\_NS}) \cdot P(\text{EST\_NS})} = 0.8804$$

$$P(\text{EST\_NS}/\ \text{pEST\_S}) = \frac{P(\text{pEST\_S}/\text{EST\_NS}) \cdot P(\text{EST\_NS})}{P(\text{pEST\_S}/\text{EST\_S}) \times P(\text{EST\_S}) + P(\text{pEST\_S}/\text{EST\_NS}) \times P(\text{EST\_NS})} = 0.1196$$

$$P(\text{EST\_S}/\text{pEST\_NS}) = \frac{P(\text{pEST\_NS}/\text{EST\_S}) \cdot P(\text{EST\_S})}{P(\text{pEST\_NS}/\text{EST\_S}) \cdot P(\text{EST\_S}) + P(\text{pEST\_NS}/\text{EST\_NS}) \cdot P(\text{EST\_NS})} = 0.0833$$

$$P(\text{EST\_NS}/\ \text{pEST\_NS}) = \frac{P(\text{pEST\_NS}/\text{EST\_NS}) \cdot P(\text{EST\_NS})}{P(\text{pEST\_NS}/\text{EST\_S}) \cdot P(\text{EST\_S}) + P(\text{pEST\_NS}/\text{EST\_NS}) \cdot P(\text{EST\_NS})} = 0.9167$$

$$P(\text{PRO\_S}/\text{pPRO\_S}) = \frac{P(\text{pPRO\_S}/\text{PRO\_S}) \cdot P(\text{PRO\_S})}{P(\text{pPRO\_S}/\text{PRO\_S}) \cdot P(\text{PRO\_S}) + P(\text{pPRO\_S}/\text{PRO\_NS}) \cdot P(\text{PRO\_NS})} = 0.9878$$

$$P(\text{PRO\_NS}/\text{pPRO\_S}) = \frac{P(\text{pPRO\_S}/\text{PRO\_NS}) \cdot P(\text{PRO\_NS})}{P(\text{pPRO\_S}/\text{PRO\_S}) \cdot P(\text{PRO\_S}) + P(\text{pPRO\_S}/\text{PRO\_NS}) \cdot P(\text{PRO\_NS})} = 0.0122$$

$$P(\text{PRO\_S}/\text{pPRO\_NS}) = \frac{P(\text{pPRO\_NS}/\text{PRO\_S}) \cdot P(\text{PRO\_S})}{P(\text{pPRO\_NS}/\text{PRO\_S}) \cdot P(\text{PRO\_S}) + P(\text{pPRO\_NS}/\text{PRO\_NS}) \cdot P(\text{PRO\_NS})} = 0.5$$

$$P(\text{PRO\_NS}/\text{pPRO\_NS}) = \frac{P(\text{pPRO\_NS}/\text{PRO\_NS}) \cdot P(\text{PRO\_NS})}{P(\text{pPRO\_NS}/\text{PRO\_S}) \cdot P(\text{pPRO\_NS}/\text{PRO\_NS}) \cdot P(\text{PRO\_NS})} = 0.5$$

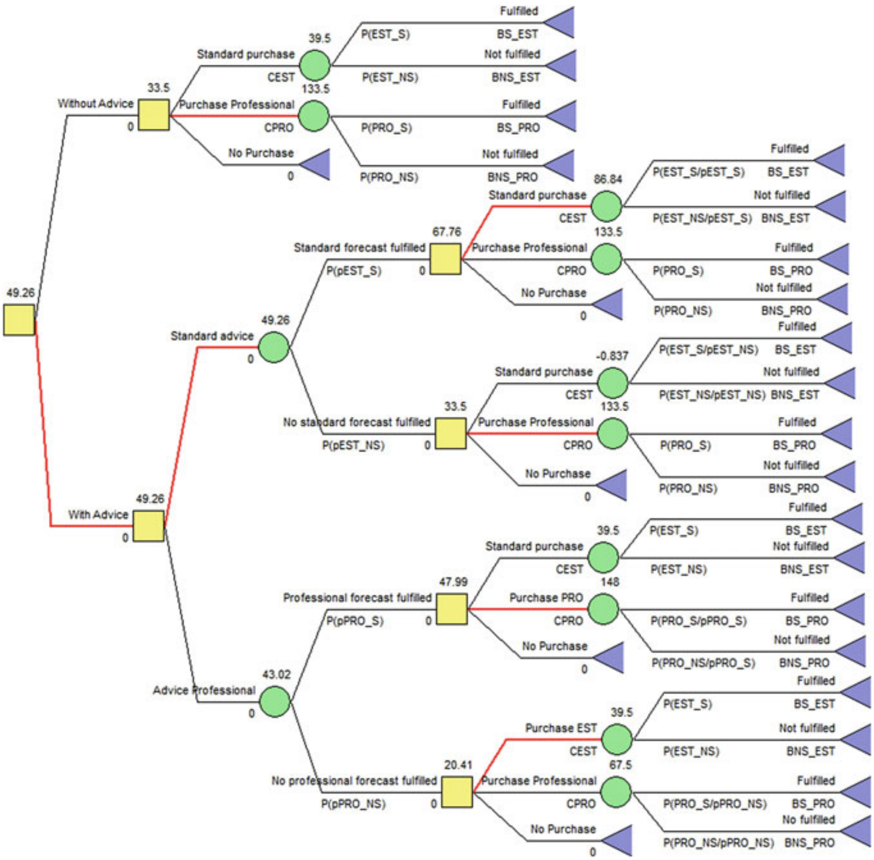
The occurrence probabilities of further information are:

$$\begin{aligned} P(\mathrm{pEST\_S}) &= P(\mathrm{pEST\_S}/\mathrm{EST\_S}) \cdot P(\mathrm{EST\_S}) + P(\mathrm{pEST\_S}/\mathrm{EST\_NS}) \\ &\quad \cdot P(\mathrm{EST\_NS}) \\ &= 0.46 \end{aligned}$$

$$\begin{aligned} P(\mathrm{pEST\_NS}) &= P(\mathrm{pEST\_NS}/\mathrm{EST\_S}) \cdot P(\mathrm{EST\_S}) + P(\mathrm{pEST\_NS}/\mathrm{EST\_NS}) \\ &\quad \cdot P(\mathrm{EST\_NS}) \\ &= 0.54 \end{aligned}$$

$$\begin{aligned} P(\mathrm{pPRO\_S}) &= P(\mathrm{pPRO\_S}/\mathrm{PRO\_S}) \cdot P(\mathrm{PRO\_S}) + P(\mathrm{pPRO\_S}/\mathrm{PRO\_NS}) \\ &\quad \cdot P(\mathrm{PRO\_NS}) \\ &= 0.82 \end{aligned}$$

$$\begin{aligned} P(\mathrm{pPRO\_NS}) &= P(\mathrm{pPRO\_NS}/\mathrm{PRO\_S}) \cdot P(\mathrm{PRO\_S}) + P(\mathrm{pPRO\_NS}/\mathrm{PRO\_NS}) \\ &\quad \cdot P(\mathrm{PRO\_NS}) \\ &= 0.18 \end{aligned}$$



Therefore, Peter should ask Balti for his advice about a standard elliptical trainer bike. If Balti indicates that a standard bike model should meet Peter's expectations, Peter should buy one. Yet if Balti considers that a standard model would not meet his expectations, Peter should buy a professional model.

(c) What kind of restaurant must Peter take Balti to? That is, how much money is he interested in paying for the meal at the most?

If Peter does not ask for advice, he will incur a negative utility as to the possibility of asking for counselling about the standard model: $33.5 - 49.26 = -15.76$

Translated into dollars:

$$\begin{aligned} -\text{EXP}(m/100) + 1 &= -15.76 \\ \text{EXP}(m/100) &= 15.76 + 1 \\ m/100 &= Ln(16.76) \\ m &= 2.82 \cdot 100 = 282 \end{aligned}$$

That is, he could spend up to 282$ to pay for Balti's evening meal.

7.9 Acquiring a New Machine

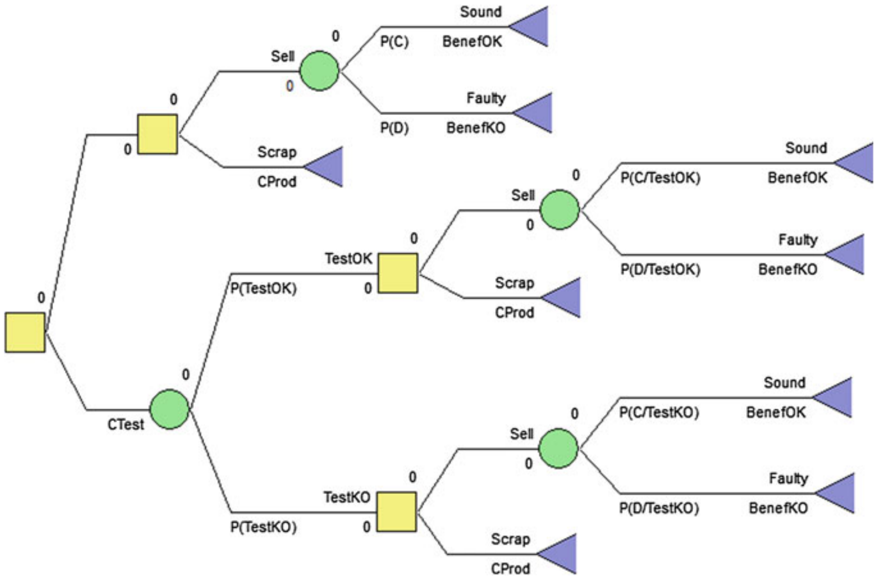
CADTRAM S.A. manufactures driving chains for one customer from the capital goods sector at a selling price of €40 per unit, and it obtains a 100 % profit on the production cost price. Nevertheless, the current production process generates 10 % of chains which do not fulfil tolerance. The contract that CADTRAM holds with its customer states that, if a faulty chain is delivered, the firm has to reimburse it twice the selling price.

The firm owns a machine that can use a test to check whether the chain operates within the specified limits of tolerance. According to a record that it keeps, of the last 1,000 chains which passed this test, 45 were faulty. Moreover, of the 660 that did not pass the test, 33 were sound. The cost per test is €1 for each chain tested.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree
(c) How much money is the firm willing to pay per test at the most?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the tree

Given the following income and costs values:

$$
\begin{aligned}
\text{PVenta} &= 40 \\
\text{CProd} &= -20 \\
\text{Cerror} &= -80 \\
\text{BenefOK} &= \text{PVenta} + \text{CProd} = 40 - 20 = 20 \\
\text{BenefKO} &= \text{PVenta} + \text{CProd} + \text{Cerror} = 40 - 20 - 80 = -60 \\
\text{CTest} &= -1
\end{aligned}
$$

The initial (a priori) probabilities are:

$$
\begin{aligned}
P(\text{C}) &= 0.9 \\
P(\text{D}) &= 0.1
\end{aligned}
$$

The conditional probabilities and further information are:

$$
\begin{aligned}
P(\mathrm{TestOK/C}) &= 1 - 33/660 = 0.95 \\
P(\mathrm{TestKO/C}) &= 33/660 = 0.05 \\
P(\mathrm{TestOK/D}) &= 45/1000 = 0.045 \\
P(\mathrm{TestKO/D}) &= 1 - 45/1000 = 0.955
\end{aligned}
$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are obtained:

$$P(\mathrm{C/TestOK}) = \frac{P(\mathrm{TestOK/C}) \cdot P(\mathrm{C})}{P(\mathrm{TestOK/C}) \cdot P(\mathrm{C}) + P(\mathrm{TestOK/D}) \cdot P(\mathrm{D})} = 0.9948$$

$$P(\mathrm{D/TestOK}) = \frac{P(\mathrm{TestOK/D}) \cdot P(\mathrm{D})}{P(\mathrm{TestOK/C}) \cdot P(\mathrm{C}) + P(\mathrm{TestOK/D}) \cdot P(\mathrm{D})} = 0.0053$$

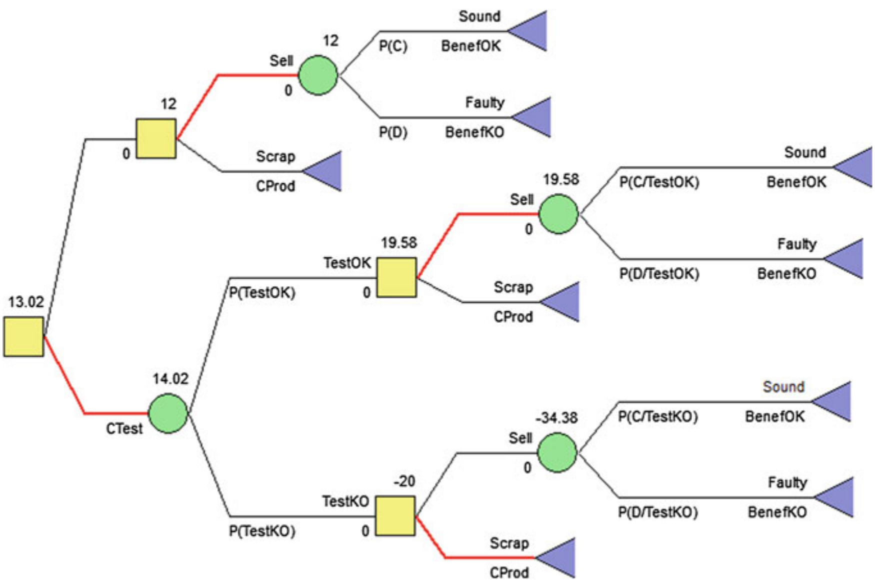
$$P(\mathrm{C/TestKO}) = \frac{P(\mathrm{TestKO/C}) \cdot P(\mathrm{C})}{P(\mathrm{TestKO/C}) \cdot P(\mathrm{C}) + P(\mathrm{TestKO/D}) \cdot P(\mathrm{D})} = 0.3203$$

$$P(\mathrm{D/TestKO}) = \frac{P(\mathrm{TestKO/C}) \cdot P(\mathrm{D})}{P(\mathrm{TestKO/C}) \cdot P(\mathrm{C}) + P(\mathrm{TestKO/D}) \cdot P(\mathrm{D})} = 0.6797$$

The occurrence probabilities of the further information are:

$$P(\mathrm{TestOK}) = P(\mathrm{TestOK/C}) \cdot P(\mathrm{C}) + P(\mathrm{TestOK/D}) \cdot P(\mathrm{D}) = 0.8595$$

$$
\begin{aligned}
P(\mathrm{TestKO}) &= P(\mathrm{TestKO/C}) \cdot P(\mathrm{C}) + P(\mathrm{TestKO/D}) \cdot P(\mathrm{D}) = 1 - P(\mathrm{TestOK}) \\
&= 0.1405
\end{aligned}
$$



Therefore, the test will be run. If it is positive, the product will be sold; if it is negative, the product will be scrapped.

(c) How much money is the firm willing to pay per test at the most?

Up to $14.02 - 12 = 2.02$.

7.10 Wine Tasting

Shane and Peter are two friends who are very keen on oenology. Peter has invited Shane to his house and has decided on a bet: Peter has randomly chosen one bottle from the wall of his wine cellar, where there are around 700 bottles of Rioja wines and some 300 bottles of Ribera del Duero wines, and he has filled a wine glass with the chosen wine.

If Shane correctly identifies Peter's wine, he must pay Shane \$8, but if Shane fails, he must pay Peter \$10.

Shane can try to guess which wine it is without smelling it or tasting it. He also has the option of smelling the wine if he pays \$2 to Peter. Having smelt the wine, he can try guessing the type of wine, or can even taste it by paying Peter \$2.

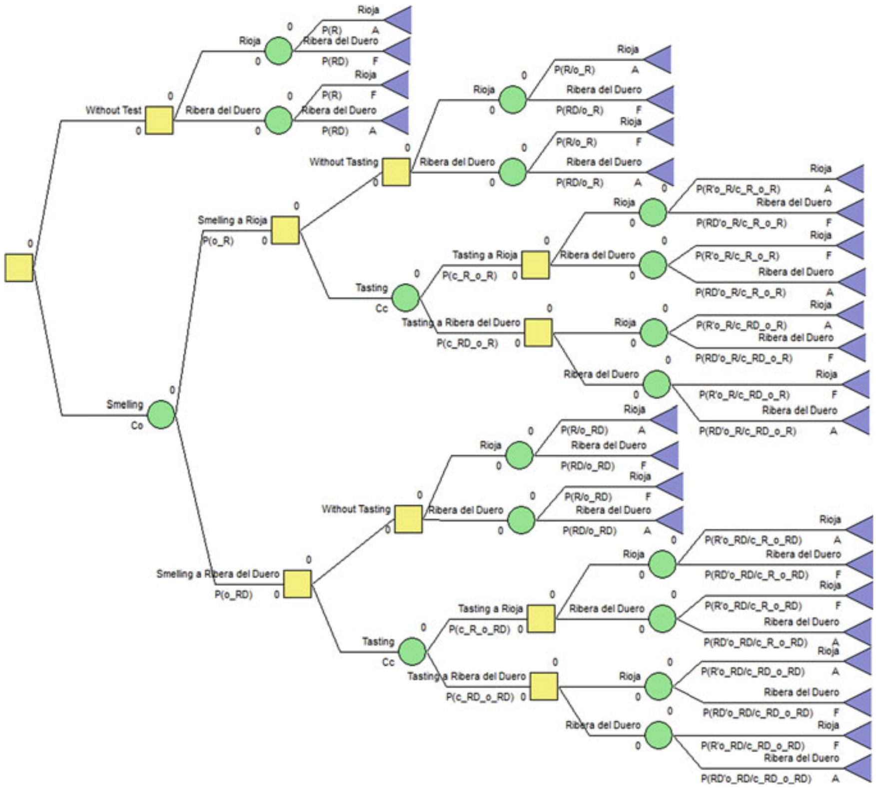
Shane estimates that the probability of succeeding by smell if it is a Rioja wine is 85 %, but this drops to 70 % for a Ribera del Duero wine.

However, he also estimates that the probability of success when tasting a Rioja wine is 75 %, but it is 85 % for a Ribera del Duero wine.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree. What strategy should Shane adopt?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the tree. What strategy should Shane adopt?

Given the following income and costs values:

$$\begin{aligned} \text{A} &= 8 \\ \text{F} &= -10 \\ \text{Co} &= -2 \\ \text{Cc} &= -2 \end{aligned}$$

The initial (a priori) probabilities are:

$$\begin{aligned} P(\text{R}) &= 0.7 \\ P(\text{RD}) &= 0.3 \end{aligned}$$

The conditional probabilities and further information are:

$$\begin{aligned} P(\text{o\_R/R}) &= 0.85 \\ P(\text{o\_RD/R}) &= 0.15 \\ P(\text{o\_R/RD}) &= 0.3 \\ P(\text{o\_RD/RD}) &= 0.7 \end{aligned}$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\text{R/ o\_R}) = \frac{P(\text{o\_R/R}) \cdot P(\text{R})}{P(\text{o\_R/R}) \cdot P(\text{R}) + P(\text{o\_R/RD}) \cdot P(\text{RD})} = 0.8686$$

$$P(\text{RD/o\_R}) = \frac{P(\text{o\_R/RD}) \cdot P(\text{RD})}{(\text{o\_R/R}) \cdot P(\text{o\_R/RD}) \cdot P(\text{RD})} = 0.1314$$

$$P(\text{R/ o\_RD}) = \frac{P(\text{o\_RD/R}) \cdot P(\text{R})}{P(\text{o\_RD/R}) \cdot P(\text{R}) + P(\text{o\_RD/RD}) \cdot P(\text{RD})} = 0.3333$$

$$P(\text{RD/ o\_RD}) = \frac{P(\text{o\_RD/RD}) \cdot P(\text{RD})}{P(\text{o\_RD/R}) \cdot P(\text{R}) + P(\text{o\_RD/RD}) \cdot P(\text{RD})} = 0.6667$$

The occurrence probabilities of the further information are:

$$P(\text{o\_R}) = P(\text{o\_R/R}) \cdot P(\text{R}) + P(\text{o\_R/RD}) \cdot P(\text{RD}) = 0.685$$

$$P(\text{o\_RD}) = P(\text{o\_RD/R}) \cdot P(\text{R}) + P(\text{o\_RD/RD}) \cdot P(\text{RD}) = 0.315$$

The probability for tasting is:

$$\begin{aligned} P(\text{R}'\text{o\_R}) &= P(\text{R/o\_R}) = 0.8686 \\ P(\text{RD}'\text{O\_r}) &= P(\text{RD/o\_R}) = 0.1314 \\ P(\text{R}'\text{o\_RD}) &= P(\text{R/o\_RD}) = 0.3333 \\ P(\text{RD}'\text{o\_RD}) &= P(\text{RD/ o\_RD}) = 0.6667 \end{aligned}$$

Information about the tasting is:

$$\begin{aligned} P(\text{c\_R\_o\_R/R}'\text{o\_R}) &= 0.75 \\ P(\text{c\_RD\_o\_R/R}'\text{o\_R}) &= 0.25 \\ P(\text{c\_R\_o\_R/RD}'\text{o\_R}) &= 0.15 \\ P(\text{c\_RD\_o\_R/RD}'\text{o\_R}) &= 0.85 \\ P(\text{c\_R\_o\_RD/R}'\text{o\_RD}) &= 0.75 \\ P(\text{c\_RD\_o\_RD/R}'\text{o\_RD}) &= 0.25 \\ P(\text{c\_R\_o\_RD/RD}'\text{o\_RD}) &= 0.15 \\ P(\text{c\_RD\_o\_RD/RD}'\text{o\_RD}) &= 0.85 \end{aligned}$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\mathrm{R'o\_R/c\_R\_o\_R})\ =\ \_\mathrm{R})\ = \frac{P(\mathrm{c\_R\_o\_R/R'o\_R})\cdot P(\mathrm{R'o\_R})}{P(\mathrm{c\_R\_o\_R/R'o\_R})\cdot P(\mathrm{R'o\_R}) + P(\mathrm{c\_R\_o\_R/RD'o\_R})\cdot P(\mathrm{RD'O\_r})} = 0.9706$$

$$P(\mathrm{RD'o\_R/c\_R\_o\_R}) = \frac{P(\mathrm{c\_R\_o\_R/RD'o\_R})\cdot P(\mathrm{RD'O\_r})}{P(\mathrm{c\_R\_o\_R/R'o\_R})\cdot P(\mathrm{R'o\_R}) + P(\mathrm{c\_R\_o\_R/RD'o\_R})\cdot P(\mathrm{RD'O\_r})} = 0.0294$$

$$P(\mathrm{R'o\_R/c\_RD\_o\_R}) = \frac{P(\mathrm{c\_RD\_o\_R/R'o\_R})\cdot P(\mathrm{R'o\_R})}{P(\mathrm{c\_RD\_o\_R/R'o\_R})\cdot P(\mathrm{R'o\_R}) + P(\mathrm{c\_RD\_o\_R/RD'o\_R})\cdot P(\mathrm{RD'o\_R})} = 0.6604$$

$$P(\mathrm{RD'o\_R/c\_RD\_o\_R}) = \frac{P(\mathrm{c\_RD\_o\_R/RD'o\_R})\cdot P(\mathrm{RD'o\_R})}{P(\mathrm{c\_RD\_o\_R/R'o\_R})\cdot P(\mathrm{R'o\_R}) + P(\mathrm{c\_RD\_o\_R/RD'o\_R})\cdot P(\mathrm{RD'o\_R})} = 0.3397$$

$$P(\mathrm{R'o\_RD/c\_R\_o\_RD}) = \frac{P(\mathrm{c\_R\_o\_RD/R'o\_RD})\cdot P(\mathrm{R'o\_RD})}{P(\mathrm{c\_R\_o\_RD/R'o\_RD})\cdot P(\mathrm{R'o\_RD}) + P(\mathrm{c\_R\_o\_RD/RD'o\_RD})\cdot P(\mathrm{RD'o\_RD})} = 0.7142$$

$$P(\mathrm{RD'o\_RD/\ c\_R\_o\_RD}) = \frac{P(\mathrm{c\_R\_o\_RD/RD'o\_RD})\cdot P(\mathrm{RD'o\_RD})}{P(\mathrm{c\_R\_o\_RD/R'o\_RD})\cdot P(\mathrm{R'o\_RD}) + P(\mathrm{c\_R\_o\_RD/RD'o\_RD})\cdot P(\mathrm{RD'o\_RD})} = 0.2857$$

$$P(\mathrm{R'o\_RD/c\_RD\_o\_RD}) = \frac{P(\mathrm{c\_RD\_o\_RD/R'o\_RD})\cdot P(\mathrm{R'o\_RD})}{P(\mathrm{c\_RD\_o\_RD/R'o\_RD})\cdot P(\mathrm{R'o\_RD}) + P(\mathrm{c\_RD\_o\_RD/RD'o\_RD})\cdot P(\mathrm{RD'o\_RD})} = 0.1282$$

$$P(\mathrm{RD'o\_RD/c\_RD\_o\_RD}) = \frac{P(\mathrm{c\_RD\_o\_RD/RD'o\_RD})\cdot P(\mathrm{RD'o\_RD})}{P(\mathrm{c\_RD\_o\_RD/R'o\_RD})\cdot P(\mathrm{R'o\_RD}) + P(\mathrm{c\_RD\_o\_RD/RD'o\_RD})\cdot P(\mathrm{RD'o\_RD})} = 0.8718$$

The occurrence probabilities of the further information are:

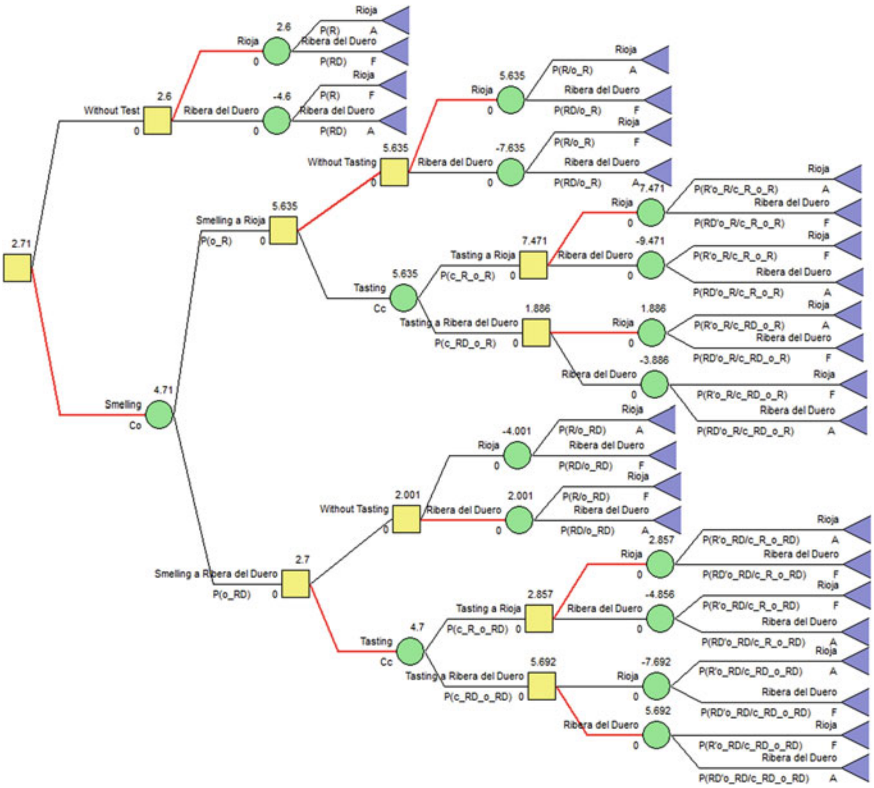
$$\begin{aligned} P(\mathrm{c\_R\_o\_R}) &= P(\mathrm{c\_R\_o\_R/R'o\_R})\ \cdot P(\mathrm{R'o\_R})\ + P(\mathrm{c\_R\_o\_R/RD'o\_R}) \\ &\quad \cdot P(\mathrm{RD'O\_r}) \\ &= 0.6712 \end{aligned}$$

$$\begin{aligned} P(\mathrm{c\_RD\_o\_R}) &= P(\mathrm{c\_RD\_o\_R/R'o\_R})\ \cdot P(\mathrm{R'o\_R})\ + P(\mathrm{c\_RD\_o\_R/RD'o\_R}) \\ &\quad \cdot P(\mathrm{RD'o\_R}) \\ &= 0.3288 \end{aligned}$$

$$\begin{aligned} P(\mathrm{c\_R\_o\_RD}) &= P(\mathrm{c\_R\_o\_RD/R'o\_RD})\ \cdot P(\mathrm{R'o\_RD}) \\ &\quad + P(\mathrm{c\_R\_o\_RD/RD'o\_RD})\ \cdot P(\mathrm{RD'o\_RD}) \\ &= 0.35 \end{aligned}$$

$$\begin{aligned} P(\mathrm{c\_RD\_o\_RD}) &= P(\mathrm{c\_RD\_o\_RD/R'o\_RD})\ \cdot P(\mathrm{R'o\_RD}) \\ &\quad + P(\mathrm{c\_RD\_o\_RD/RD'o\_RD})\ \cdot P(\mathrm{RD'o\_RD}) \\ &= 0.65 \end{aligned}$$

By solving the decision tree:



Therefore, the strategy that Shane should adopt is smelling. If he recognises the Rioja wine aroma, he can bet on a Rioja wine without resorting to tasting. If he detects a Ribera del Duero wine by smelling, he can resort to tasting. If while tasting he detects a Rioja wine, he should bet on a Rioja wine, otherwise he should bet on a Ribera del Duero wine.

7.11 Maintaining Machinery

The firm ADESA is thinking about doing some repairs on one of its production process machines. These repairs can be minor or major. Currently, the percentage of faulty articles that the machine produces appears to be constant, where $p = 0.10$ or $p = 0.25$. Faulty articles are produced randomly and there is no way that ADESA can certainly state if the machine needs minor or major repairs. If minor repairs are done when $p = 0.25$, the probability of obtaining faulty articles lowers to 0.05. If minor repairs are done when $p = 0.10$, or if major repairs are done when

$p = 0.10$ or $p = 0.25$, the proportion of faulty parts comes down to 0. The a priori probability that the percentage of faulty parts in the machine is $p = 0.10$ is 0.70.

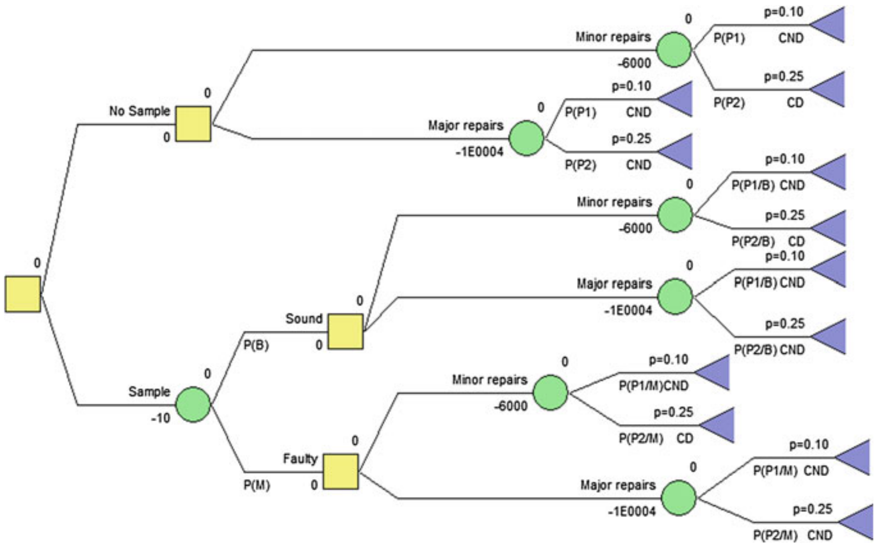
ADESA has just received an order of 1,000 articles. This means a profit of \$50 per unit for ADESA, except in those cases in which the charge of \$200 per faulty article has to be paid. Major machine repairs cost \$10,000 while minor ones are priced at \$6,000. The machine cannot be repaired after commencing the production process for the order.

Before starting the production process for the order, ADESA can sample one article from one test batch at a cost of \$10 per part.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree and obtain the most economic procedures.
(c) Is the cost of the current sampling (\$10/part) suitable? Up to what amount is it willing to pay for perfect information?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the tree and obtain the most economic procedures.

The initial (a priori) probabilities are:

$P(P1) = 0.7$ (probability that the machine has a percentage of faults (B) of $p = 0.10$)

$P(P2) = 0.3$ (probability that the machine has a percentage of faults (B) of $p = 0.25$).

The conditional probabilities (further information) are:

$$P(\mathrm{G}/P1) = 0.90$$
$$P(\mathrm{G}/P2) = 0.75$$
$$P(\mathrm{B}/P1) = 0.10$$
$$P(\mathrm{B}/P2) = 0.25$$

By applying Bayes Theorem, the revised (a posteriori) probabilities (PP) are:

$$P(\mathrm{P1/G}) = \frac{P(P1) \cdot P(\mathrm{G}/P1)}{P(P1) \cdot P(\mathrm{G}/P1) + P(P2) \cdot P(\mathrm{G}/P2)} = 0.7368$$

$$P(\mathrm{P2/G}) = \frac{P(P2) \cdot P(\mathrm{G}/P2)}{P(P2) \cdot P(\mathrm{G}/P2) + P(P1) \cdot P(\mathrm{G}/P2)} = 0.2632$$

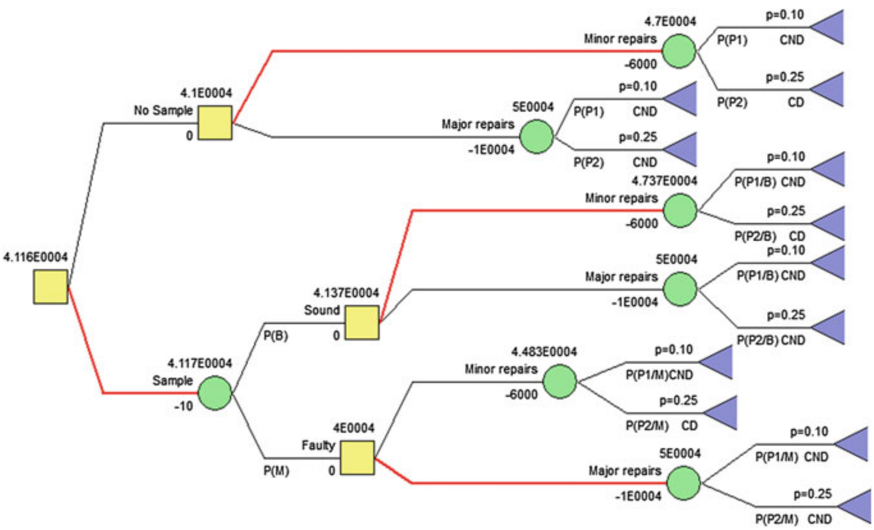
$$P(P1/\mathrm{B}) = \frac{P(P1) \cdot P(\mathrm{B}/P1)}{P(P1) \cdot P(\mathrm{B}/P1) + P(P2) \cdot P(\mathrm{B}/P2)} = 0.4828$$

$$P(P2/\mathrm{B}) = \frac{P(P2) \cdot P(\mathrm{B}/P2)}{P(P2) \cdot P(\mathrm{B}/P2) + P(P1) \cdot P(\mathrm{B}/P1)} = 0.5172$$

The occurrence probabilities of further information are:

$$P(\mathrm{G}) = P(\mathrm{P1}) \cdot P(\mathrm{G/P1}) + P(\mathrm{P2}) \cdot P(\mathrm{G/P2}) = 0.855$$

$$P(\mathrm{B}) = P(\mathrm{P1}) \cdot P(\mathrm{B/P1}) + P(\mathrm{P2}) \cdot P(\mathrm{B/P2}) = 0.145$$



The firm should sample one article of the test batch. If the result is positive, minor repairs can be carried out, but if it is negative, then major repairs should be done. The EMV is \$41,160.

(c) Is the cost of the current sampling (\$10/part) suitable? Up to what amount is it willing to pay for perfect information?

First of all, the imperfect information sampling value (IISP) is calculated:

$$\text{IISV} = \text{EMV} - \text{EMVPP} = 41170 - 41000 = \$170$$

Therefore, the current cost is adequate as it implies the lower cost of \$170/part. The perfect information value (PIV) is:

$$\text{PIV} = \text{MLU} - \text{EMV} = 42800 - 41000 = \$1,800$$

Thus, up to \$1,800 is paid for perfect information.

7.12 Antidoping Tests

The sporting authorities of ADESA University are considering setting up an antidoping test programme for university interschool athletes. Nevertheless, one of the Sports Committee members and a Quantitative Methods teacher have some doubts as to the use of such tests with student athletes given the possible social costs incurred by a false-positive test. The teacher has decided to conduct a mathematical study of the test process by taking into account the costs and reliability of such tests, and the costs relating to negative and positive errors.

If the antidoping test programme is set up, student athletes will be expected to do an antidoping test. The test result may be positive (suggests a potential user of performance-enhancing drugs) or negative (does not suggest a potential user of performance-enhancing drugs). If the test result is negative, no follow-up action will be taken; yet if the test is positive, follow-up action will be taken to determine whether the athlete or student actually uses performance-enhancing drugs.

The difficulty in this decision model lies in assigning costs to several possibilities. The teacher proposes employing the following potential costs:

- Cost of a test with a false-positive result, C1.
- Cost of a user of performance-enhancing drugs who has not been identified, C2.
- Cost of someone innocent who has to do the doping test and it proves negative, C3.
- Costs of properly identifying a user of performance-enhancing drugs, and the cost of an innocent athlete not doing the test is considered 0.

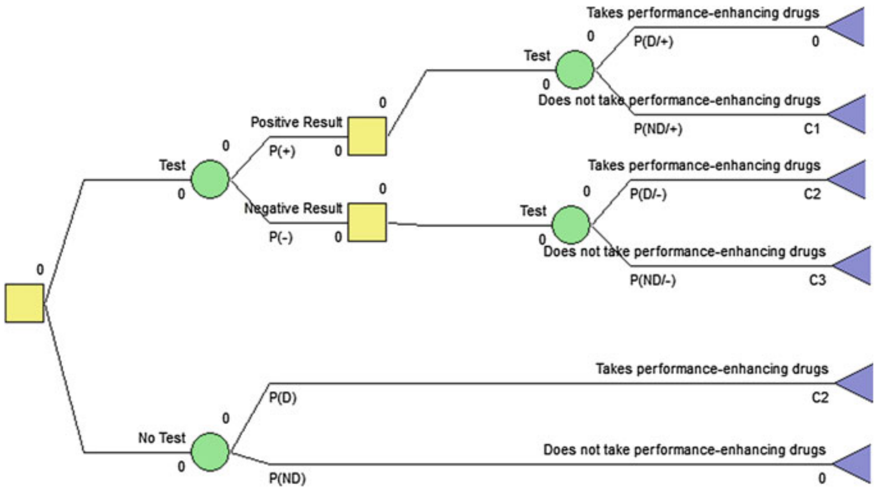
No quantitative values have been assigned to C1, C2 and C3, but a hierarchy of values have been established. In relative terms, C1 is the highest cost and C2 is higher than C3. So: $C1 \geq C2 \geq C3$.

By assuming that test reliability is 95 % and that 5 % of athletes use performance-enhancing drugs, considering the ADESA University Sporting Committee's objective of minimising expected costs helps the teacher solve the decision model.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the tree and obtain the most economic solution.

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the tree and obtain the most economic solution.

Based on the a priori probabilities provided in the problem:

$P(\mathrm{D}) = 0.05$ (probability of an athlete having used performance-enhancing drugs)

$P(\mathrm{ND}) = 0.95$ (probability of an athlete not having used performance-enhancing drugs).

The reliability data of imperfect information are:

$P(+/\mathrm{D}) = 0.95$ (probability of the test being correct and reflecting the true situation)
$P(+/\mathrm{ND}) = 0.05$ (probability of the test being wrong and not reflecting the true situation)
$P(-/\mathrm{D}) = 0.05$ (probability of the test being wrong and not reflecting the true situation)

$P(-/\mathrm{ND}) = 0.95$ (probability of the test being correct and reflecting the true situation).

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\mathrm{D}/+) = \frac{P(\mathrm{D}) \cdot P(\mathrm{TC})}{P(\mathrm{D}) \cdot P(\mathrm{TC}) + P(\mathrm{ND}) \cdot P(\mathrm{TE})} = 0.5$$

$$P(\mathrm{ND}/+) = \frac{P(\mathrm{ND}) \cdot P(\mathrm{TE})}{P(\mathrm{D}) \cdot P(\mathrm{TC}) + P(\mathrm{ND}) \cdot P(\mathrm{TE})} = 0.5$$

$$P(\mathrm{D}/-) = \frac{P(\mathrm{D}) \cdot P(\mathrm{TE})}{P(\mathrm{ND}) \cdot P(\mathrm{TC}) + P(\mathrm{D}) \cdot P(\mathrm{TE})} = 0.9972$$

$$P(\mathrm{ND}/-) = \frac{P(\mathrm{ND}) \cdot P(\mathrm{TC})}{P(\mathrm{ND}) \cdot P(\mathrm{TC}) + P(\mathrm{D}) \cdot P(\mathrm{TE})} = 0.0028$$

The occurrence probabilities of the further information are:

$P(+) = P(\mathrm{D}) \cdot P(\mathrm{TC}) + P(\mathrm{ND}) \cdot P(\mathrm{TE}) = 0.095$ (probability of the test being positive)

$P(-) = P(\mathrm{ND}) \cdot P(\mathrm{TC}) + P(\mathrm{D}) \cdot P(\mathrm{TE}) = 1 - P(+) = 0.905$ (probability of the test being negative)

Cost of not doing test $= 0.05 \cdot \mathrm{C2} + 0.95 \cdot 0 = 0.05 \cdot \mathrm{C2}$

Cost of doing test $= 0.095 \cdot (0.5 \cdot 0 + 0.5 \cdot \mathrm{C1}) + 0.905 \cdot (0.0028 \cdot \mathrm{C2} + 0.9972 \cdot \mathrm{C3})$

$= 0.0475 \cdot \mathrm{C1} + 0.002534 \cdot \mathrm{C2} + 0.902466 \cdot \mathrm{C3}$

Cost of doing test–cost of not doing test

$= 0.0475 \cdot \mathrm{C1} + 0.002534 \cdot \mathrm{C2} + 0.902466 \cdot \mathrm{C3} - 0.05 \cdot \mathrm{C2} >$

$-0.0025 \cdot \mathrm{C2} + 0.002534 \cdot \mathrm{C2} + 0.902466 \cdot \mathrm{C3} = 0.000034 \cdot \mathrm{C2} + 0.902466 \cdot \mathrm{C3} > 0$

The solution is to not set up the doping test programme.

7.13 An Urban Development Purchasing Problem

Jake Farlow, the Chair of Playaview Building, is thinking about presenting a bid to acquire property which will be sold in a sealed bid as part of a foreclosure to pay taxes in Alicante. Initially, the idea is to present an offer of 5 million \$, with a probability of 0.2 of being the highest, for the purpose of obtaining the property for Playaview. Today the date is 1 June. The offers to acquire property must be presented before 15 August. The winning bid will be published on 1 September. If Playaview presents the highest bid and obtains the property, the firm can study constructing and selling a luxury apartments complex. However, a factor that complicates the issue is that the property is currently in an area used only for detached houses. Jake believes that a referendum can be organised in time for the

Table 7.1 Estimations of costs and income

Estimations of income	
Income from selling apartments	$15,000,000
Estimations of costs	
Land	$5,000,000
Building	$8,000,000

Table 7.2 Probabilities of obtaining accurate information

	Changes in area approved	Changes in area not approved
Approved changes in area predicted	0.9	0.2
Approved changes in area not predicted	0.1	0.8

November elections. If the referendum is passed, the property area will be changed and it will be possible to build apartment blocks.

The sealed bid procedure entails Jake having to send a certified cheque covering 10 % of the amount offered. If the offer is rejected, the deposit paid will be returned. If the offer is accepted, the deposit will become the initial payment for the property. However if the offer is accepted, and the bidder does not continue with the purchase, and does not meet the remaining financial obligations within the next 6 months, the deposit will be lost. In this case, the province of Alicante will offer the property to the next highest bid. In order to determine if Playaview must present the offer of 5 million $, Jake carried out a preliminary analysis which gave him an estimation of 0.3 for the probability of passing the referendum to change the area, which gave the estimations of the costs and income indicated in Table 7.1, which would be incurred if the apartments were built.

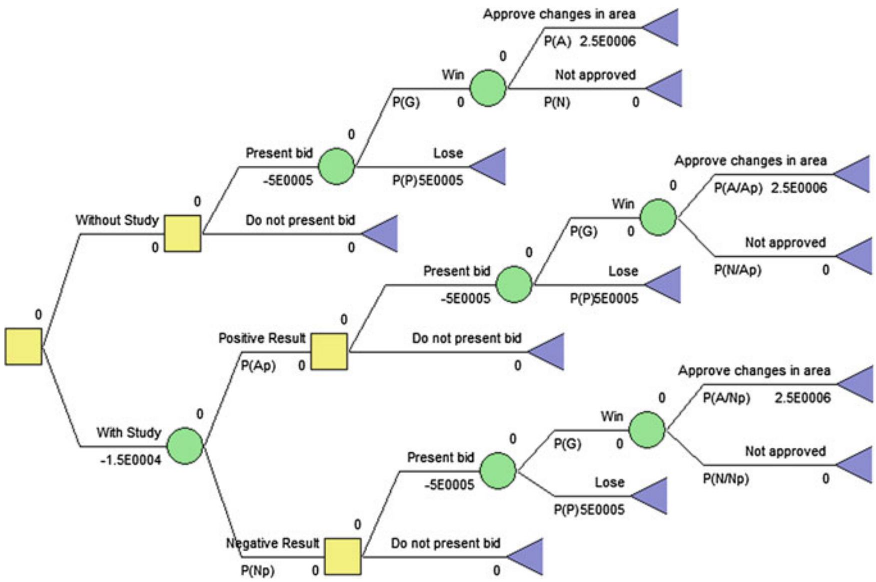
If Playaview acquires the property and the changes in the area are not approved in November, Jake thinks that the best option would be that the firm does not purchase the property.

Jake has suggested that the firm contracts a market research service to conduct a survey among voters, which should provide a better estimation of the possibility of deciding on the changes in the area by referendum. The market research firm is happy to conduct the study, which will be available by 1 August, at a price of 15,000$. Table 7.2 shows the estimations of probability related with the accuracy or preciseness of the market research information:

(a) Draw a decision tree showing the logical sequence of the decision problem.
(b) Indicate the decision strategy that Playaview should adopt if no market research study is done.
(c) Justify if Playaview must resort to a market research firm or not.
(d) Determine the value of the information provided by the market research firm.

Solution

(a) Draw a decision tree showing the logical sequence of the decision problem.



(b) Indicate the decision strategy that Playaview should adopt if no market research study is done.

The initial (a priori) probabilities are:

$$P(\mathrm{G}) = 0.2$$
$$P(P) = 0.8$$
$$P(\mathrm{A}) = 0.3$$
$$P(\mathrm{N}) = 0.7$$

The conditional probabilities (further information) are:

$$P(\mathrm{Ap/A}) = 0.9$$
$$P(\mathrm{Ap/N}) = 0.2$$
$$P(\mathrm{Np/A}) = 0.1$$
$$P(\mathrm{Np/N}) = 0.8$$

The probabilities calculated with Bayes Theorem are as follows:

$$P(\mathrm{A/Ap}) = \frac{P(\mathrm{A}) \cdot P(\mathrm{Ap/A})}{P(\mathrm{A}) \cdot P(\mathrm{Ap/A}) + P(\mathrm{N}) \cdot P(\mathrm{Ap/N})} = 0.6585$$

$$P(\mathrm{N/Ap}) = \frac{P(\mathrm{N}) \cdot P(\mathrm{Ap/N})}{P(\mathrm{A}) \cdot P(\mathrm{Ap/A}) + P(\mathrm{N}) \cdot P(\mathrm{Ap/N})} = 0.3415$$

$$P(\mathrm{A/Np}) = \frac{P(\mathrm{A}) \cdot P(\mathrm{Np/A})}{P(\mathrm{A}) \cdot P(\mathrm{Np/A}) + P(\mathrm{N}) \cdot P(\mathrm{Np/N})} = 0.0508$$

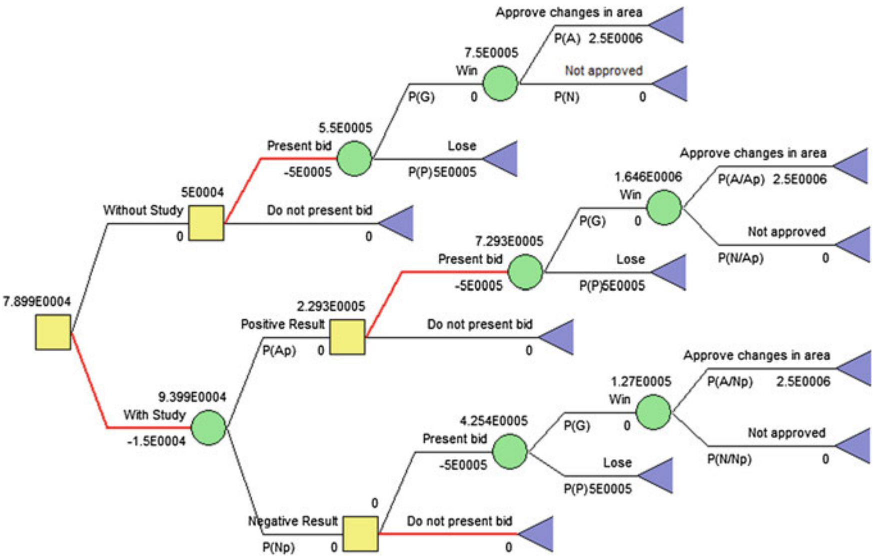
$$P(\mathrm{N/Np}) = \frac{P(\mathrm{N}) \cdot P(\mathrm{Np/N})}{P(\mathrm{A}) \cdot P(\mathrm{Np/A}) + P(\mathrm{N}) \cdot P(\mathrm{Np/N})} = 0.9492.$$

The occurrence probabilities of the further information are:

$$P(\mathrm{Ap}) = P(\mathrm{A}) \cdot P(\mathrm{Ap/A}) + P(\mathrm{N}) \cdot P(\mathrm{Ap/N}) = 0.41$$

$$P(\mathrm{Np}) = P(\mathrm{A}) \cdot P(\mathrm{Np/A}) + P(\mathrm{N}) \cdot P(\mathrm{Np/N}) = 0.59.$$

Therefore if the market research study is not done, the bid which obtains an EMV of 50,000$ should be presented.



(c) Justify if Playaview must resort to a market research firm or not

Jake should resort to the market research firm because he would obtain a higher EMV than 78,990$.

(d) Determine the value of the information provided by the market research firm

IISV = 93990 − 50000 = 43,990$.

7.14 Purchasing a Forklift Truck

The Director of Logistics of a textile firm is in charge of deciding between buying a new electric forklift truck costing \$25,000 or a second-hand 10-year-old costing \$12,500, to handle textile parts weighing up to 2,100 kg and for a height of up to 4 m. The maintenance costs of a new forklift truck for the next 10 years are estimated at \$1,000, whereas the costs of maintaining a second-hand truck are expected to double those of a new one.

Evidently if the second-hand fortklift truck operates properly, it entails a saving of \$11,500, but if it is faulty, it will entail losing the money spent on not only purchasing it, but also on the purchase and maintenance of a new model. Evidently, the Director will be reprimanded by the Manager. Based on the statistical data provided by the seller, the Director of Logistics has established that the proportion of faulty second-hand equipment is 20 % and that the reliability of a new forklift truck is considered 100 %.

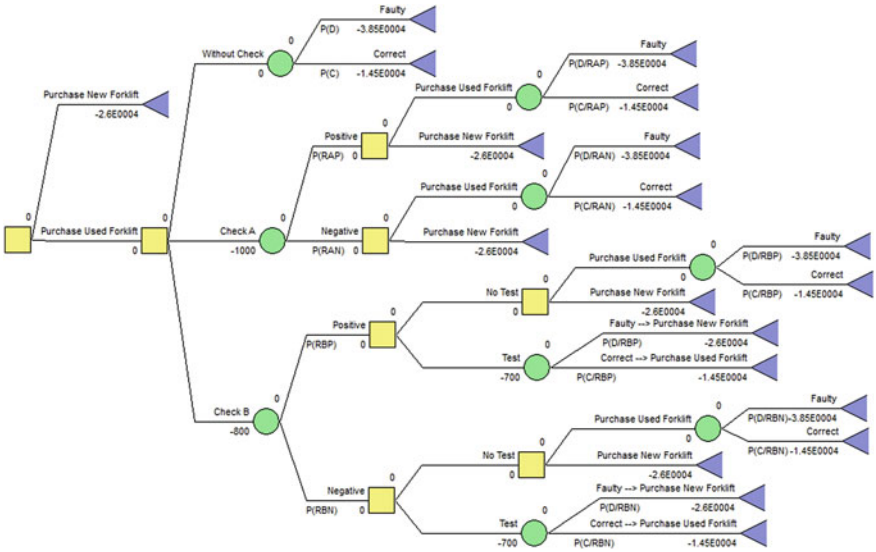
Before purchasing a second-hand forklift truck, two types of mechanical and electric tests can be done:

- <u>TEST A</u>: at a cost of \$1,000, whose diagnosis offers a percentage of failures of 5 % if it is faulty, and of 20 % if it operates properly.
- <u>TEST B</u>: consists in two successive phases. In the first one, whose cost is \$800, a preliminary diagnosis can be made whose probability of error is 15 %. The second phases entails deciding whether to buy or reject the forklift truck, or to complete the test, which costs a further \$700, in order to be absolutely sure about the state the equipment is in.

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)
(b) Solve the decision tree and obtain the most economic decision.
(c) What is the imperfect information value for each test? How much does perfect information cost?

Solution

(a) Draw the corresponding decision tree and indicate with a letter per branch the associated probability or the associated cost (depending on the case)



(b) Solve the decision tree and obtain the most economic decision.

The initial (a priori) probabilities are:

$$P(\text{D}) = 0.2$$
$$P(\text{C}) = 0.8$$

The conditional probabilities (further information) are:

$$P(\text{RAP}/\text{D}) = 0.05$$
$$P(\text{RAN}/\text{D}) = 0.95$$
$$P(\text{RAP}/\text{C}) = 0.80$$
$$P(\text{RAN}/\text{C}) = 0.20$$
$$P(\text{RBP}/\text{D}) = 0.15$$
$$P(\text{RBN}/\text{D}) = 0.85$$
$$P(\text{RBP}/\text{C}) = 0.85$$
$$P(\text{RBN}/\text{C}) = 0.15$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\mathrm{D/RAP}) = \frac{P(\mathrm{D}) \cdot P(\mathrm{RAP/D})}{P(\mathrm{D}) \cdot P(\mathrm{RAP/D}) + P(\mathrm{C}) \cdot P(\mathrm{RAP/C})} = 0.0154$$

$$P(\mathrm{C/RAP}) = \frac{P(\mathrm{C}) \cdot P(\mathrm{RAP/C})}{P(\mathrm{D}) \cdot P(\mathrm{RAP/D}) + P(\mathrm{C}) \cdot P(\mathrm{RAP/C})} = 0.9846$$

$$P(\mathrm{D/RAN}) = \frac{P(\mathrm{D}) \cdot P(\mathrm{RAN/D})}{P(\mathrm{D}) \cdot P(\mathrm{RAN/D}) + P(\mathrm{C}) \cdot P(\mathrm{RAN/C})} = 0.5429$$

$$P(\mathrm{C/RAN}) = \frac{P(\mathrm{C}) \cdot P(\mathrm{RAN/C})}{P(\mathrm{D}) \cdot P(\mathrm{RAN/D}) + P(\mathrm{C}) \cdot P(\mathrm{RAN/C})} = 0.4571$$

$$P(\mathrm{D/RBP}) = \frac{P(\mathrm{D}) \cdot P(\mathrm{RBP/D})}{P(\mathrm{D}) \cdot P(\mathrm{RBP/D}) + P(\mathrm{C}) \cdot P(\mathrm{RBP/C})} = 0.0423$$

$$P(\mathrm{C/RBP}) = \frac{P(\mathrm{C}) \cdot P(\mathrm{RBP/C})}{P(\mathrm{D}) \cdot P(\mathrm{RBP/D}) + P(\mathrm{C}) \cdot P(\mathrm{RBP/C})} = 0.9577$$

$$P(\mathrm{D/RBN}) = \frac{P(\mathrm{D}) \cdot P(\mathrm{RBN/D})}{P(\mathrm{D}) \cdot P(\mathrm{RBN/D}) + P(\mathrm{C}) \cdot P(\mathrm{RBN/C})} = 0.5862$$

$$P(\mathrm{C/RBN}) = \frac{P(\mathrm{C}) \cdot P(\mathrm{RBN/C})}{P(\mathrm{D}) \cdot P(\mathrm{RBN/D}) + P(\mathrm{C}) \cdot P(\mathrm{RBN/C})} = 0.4138$$

The occurrence probabilities of the further information are:

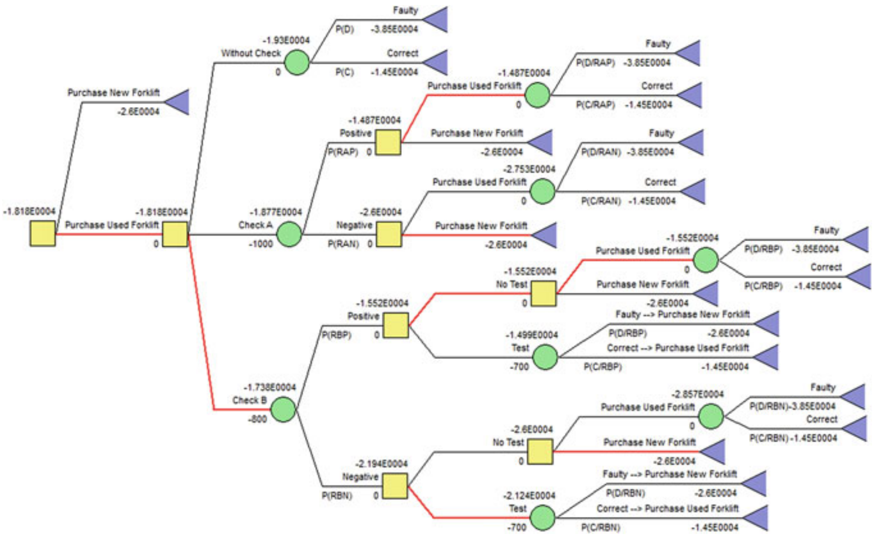
$$P(\mathrm{RAP}) = P(\mathrm{D}) \cdot P(\mathrm{RAP/D}) + P(\mathrm{C}) \cdot P(\mathrm{RAP/C}) = 0.6500$$

$$P(\mathrm{RAN}) = P(\mathrm{D}) \cdot P(\mathrm{RAN/D}) + P(\mathrm{C}) \cdot P(\mathrm{RAN/C}) = 0.3500$$

$$P(\mathrm{RBP}) = P(\mathrm{D}) \cdot P(\mathrm{RBP/D}) + P(\mathrm{C}) \cdot P(\mathrm{RBP/C}) = 0.7100$$

$$P(\mathrm{RBN}) = P(\mathrm{D}) \cdot P(\mathrm{RBN/D}) + P(\mathrm{C}) \cdot P(\mathrm{RBN/C}) = 0.2900$$

The Director of Logistics should opt to do Test B before making a decision. If the first part of the test is positive, then a second-hand forklift truck should be purchased. If the test proves negative, the Director should continue with the test, and if it indicates that the forklift trucks operates properly, the second-hand forklift truck should be purchased, otherwise, a new one should be bought. This procedure has an EMV of $18,180 in costs.



(c) What is the imperfect information value for each test? How much does perfect information cost?

The imperfect information value for test A = −18770 − (−19300) = \$530

The imperfect information value for test B = −17378.8 − (−19300) = \$1,921.2

The perfect information value = MLU − EMV = −16800 − (−19300) = \$2,500

MLU = 0.2 − 26000 + 0.8 − 14500 = −\$16,800.

7.15 Planning to Commercialise a New Irrigation System

Let's assume that you have invented and patented a new irrigation system for house plants. You can opt between commercialising the product yourself or through a firm called PLANTASA, which is known in the market. PLANTASA offers you \$20,000 to sign a contract. If the product is successful, it will sell 200,000 units in the Spanish market. If it is not, it will sell only 10,000 units. PLANTASA pays patent royalties at a rate of \$1 per unit. A market research study done by PLANTASA indicates that there is a 70 % probability that the product will be a success. If, however, you go ahead and commercialise the product yourself, there will be an initial cost of \$90,000 for production and marketing, but each unit will generate two net dollars.

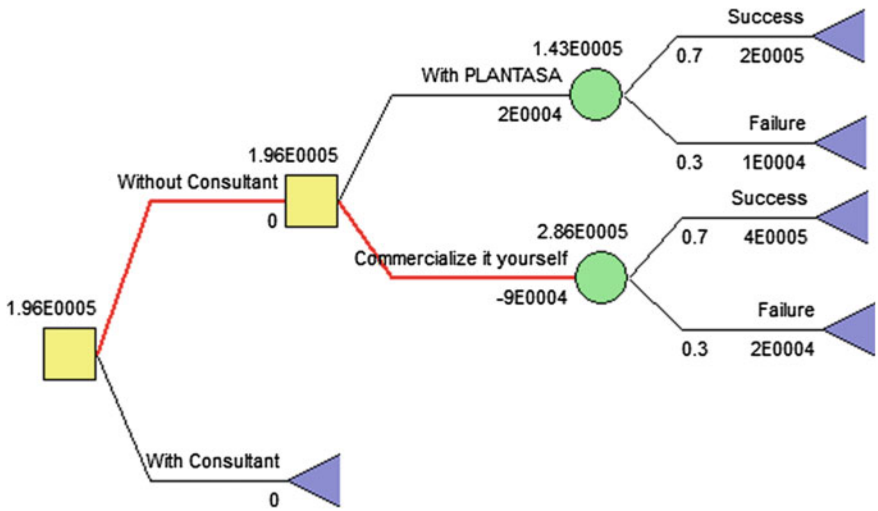
(a) Depending on the information provided, would you accept PLANTASA's offer or would you commercialise the product yourself?

(b) Let's assume that you contract a consultancy firm to conduct some market research into the product's potential success. From past experience, the consultancy firm indicates that when a product is successful, the study fails in predicting the result in 20 % of cases. When the product is not successful, the study predicts correctly 85 % of times. How would the information provided by the consultancy firm affect your former decision?

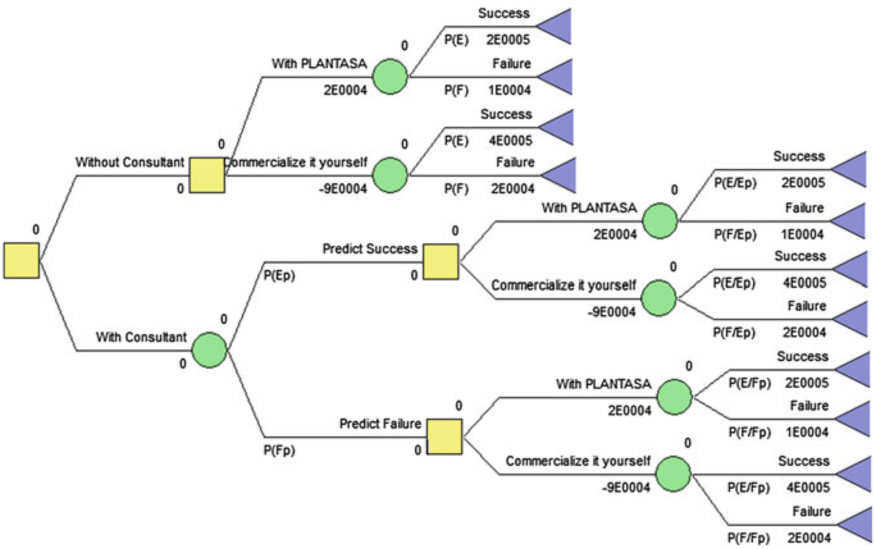
Solution

(a) Depending on the information provided, would you accept PLANTASA's offer or would you commercialise the product yourself?

By considering and solving the corresponding decision tree, it is concluded that you can commercialise the product yourself with an EMV of $196,000.



(b) Let's assume that you contract a consultancy firm to conduct some market research into the product's potential success. From past experience, the consultancy firm indicates when a product is successful, and it offers a prediction error in 20 % of cases. When the product is not successful, the study predicts correctly 85 % of the times. How would the information provided by the consultancy firm affect your former decision?



The initial (a priori) probabilities are:

$$P(\mathrm{E}) = 0.7$$
$$P(\mathrm{F}) = 0.3$$

The conditional probabilities (further information) are:

$$P(\mathrm{Ep/E}) = 0.8$$
$$P(\mathrm{Fp/E}) = 0.2$$
$$P(\mathrm{Ep/F}) = 0.15$$
$$P(\mathrm{Fp/F}) = 0.85$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\mathrm{E/Ep}) = \frac{P(\mathrm{E}) \cdot P(\mathrm{Ep/E})}{P(\mathrm{E}) \cdot P(\mathrm{Ep/E}) + P(\mathrm{F}) \cdot P(\mathrm{Ep/F})} = 0.9256$$

$$P(\mathrm{F/Ep}) = \frac{P(\mathrm{F}) \cdot P(\mathrm{Ep/F})}{P(\mathrm{E}) \cdot P(\mathrm{Ep/E}) + P(\mathrm{F}) \cdot P(\mathrm{Ep/F})} = 0.0744$$

$$P(\mathrm{E/Fp}) = \frac{P(\mathrm{E}) \cdot P(\mathrm{Fp/E})}{P(\mathrm{E}) \cdot P(\mathrm{Fp/E}) + P(\mathrm{F}) \cdot P(\mathrm{Fp/F})} = 0.3544$$

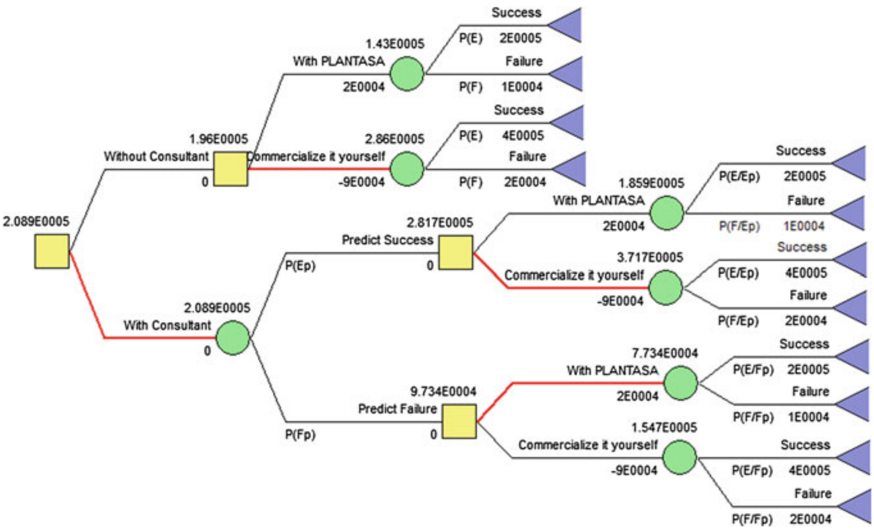
$$P(\mathrm{F/Fp}) = \frac{P(\mathrm{F}) \cdot P(\mathrm{Fp/F})}{P(\mathrm{E}) \cdot P(\mathrm{Fp/E}) + P(\mathrm{F}) \cdot P(\mathrm{Fp/F})} = 0.6456$$

The occurrence probabilities of further information are:

$$P(\text{Ep}) = P(\text{E}) \cdot P(\text{Ep}/\text{E}) + P(\text{F}) \cdot P(\text{Ep}/\text{F}) = 0.6050$$

$$P(\text{Fp}) = P(\text{E}) \cdot P(\text{Fp}/\text{E}) + P(\text{F}) \cdot P(\text{Fp}/\text{F}) = 0.3950$$

It is worth contracting the consultancy firm. If it states that commercialisation will be successful, it is worth commercialising it yourself. However it it predicts failure, then you are recommended to sign a contract with PLANTASA with an EMV of $208,900.



7.16 Maximising Profits in the Film Industry

A film producer is considering filming a picture about the life of a singer who died recently. It is predicted that if the film is not successful, 15 million $ will be lost, but 45 million $ will be earned if it were a success.

The producer has estimated the probability of the film being a success based on these aspects:

- The film script can centre only on the singer's professional life or can include details about the singer's private life. Irrespective of the script, there is a 10 % probability that the film will be successful.
- If the film director is to be well-known Petro Dovar, who will only accept filming the film if it centres on the singer's private life, the probability of the film being a success becomes 50 %. In this case, both the predicted losses and earnings will increase or decrease in 5 million $, respectively, to cover the

additional expenses that this director incurs. If another less well-known director films, the initially estimated profits and probabilities data will not change.

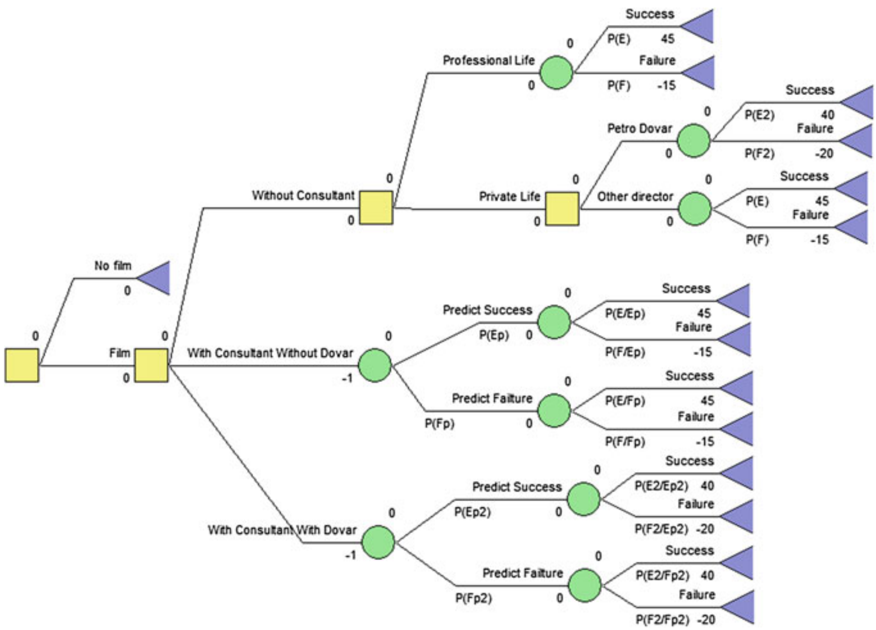
Before filming, the film producer has the option of paying a famous film critic 1 million $ to receive a view about the film's success. In the past, this famous film critic has predicted that films would be a success in 60 % of cases, and that they would flop in 90 % of cases. Evidently, the film critic's predictions will have to consider the film director's characteristics.

The film producer wishes to maximise the expected earnings.

(a) Draw a decision tree to determine the best strategy. Indicate all the strategies to be adopted, the associated earnings and losses, and probabilities.
(b) Establish the probabilities, solve the decision tree and indicate the best strategy.
(c) What is the imperfect information value?

Solution

(a) Draw a decision tree to determine the best strategy. Indicate all the strategies to be adopted, the associated earnings and losses, and probabilities.



(b) Establish the probabilities, solve the decision tree, and indicate the best strategy.

The a priori probabilities without Dovar

$$P(\mathrm{E}) = 0.1$$
$$P(\mathrm{F}) = 0.9$$

The a priori probabilities with Dovar

$$P(\mathrm{E2}) = 0.5$$
$$P(\mathrm{F2}) = 0.5$$

The probabilities related with the film critic.

$$P(\mathrm{Ep}/\mathrm{E}) = 0.6$$
$$P(\mathrm{Fp}/\mathrm{E}) = 0.4$$
$$P(\mathrm{Fp}/\mathrm{F}) = 0.9$$
$$P(\mathrm{Ep}/\mathrm{F}) = 0.1$$

The a posteriori probabilities without Dovar

$$P(\mathrm{E}/\mathrm{Ep}) = \frac{P(\mathrm{E}) \cdot P(\mathrm{Ep}/\mathrm{E})}{P(\mathrm{E}) \cdot P(\mathrm{Ep}/\mathrm{E}) + P(\mathrm{F}) \cdot P(\mathrm{Ep}/\mathrm{F})} = 0.4000$$

$$P(\mathrm{F}/\mathrm{Ep}) = \frac{P(\mathrm{F}) \cdot P(\mathrm{Ep}/\mathrm{F})}{P(\mathrm{E}) \cdot P(\mathrm{Ep}/\mathrm{E}) + P(\mathrm{F}) \cdot P(\mathrm{Ep}/\mathrm{F})} = 0.6000$$

$$P(\mathrm{E}/\mathrm{Fp}) = \frac{P(\mathrm{E}) \cdot P(\mathrm{Fp}/\mathrm{E})}{P(\mathrm{E}) \cdot P(\mathrm{Fp}/\mathrm{E}) + P(\mathrm{F}) \cdot P(\mathrm{Fp}/\mathrm{F})} = 0.0471$$

$$P(\mathrm{F}/\mathrm{Fp}) = \frac{P(\mathrm{F}) \cdot P(\mathrm{Fp}/\mathrm{F})}{P(\mathrm{E}) \cdot P(\mathrm{Fp}/\mathrm{E}) + P(\mathrm{F}) \cdot P(\mathrm{Fp}/\mathrm{F})} = 0.9529$$

The a posteriori probabilities with Dovar

$$P(\mathrm{E2}/\mathrm{Ep2}) = \frac{P(\mathrm{E2}) \cdot P(\mathrm{Ep}/\mathrm{E})}{P(\mathrm{E2}) \cdot P(\mathrm{Ep}/\mathrm{E}) + P(\mathrm{F2}) \cdot P(\mathrm{Ep}/\mathrm{F})} = 0.8571$$

$$P(\mathrm{F2}/\mathrm{Ep2}) = \frac{P(\mathrm{F2}) \cdot P(\mathrm{Ep}/\mathrm{F})}{P(\mathrm{E2}) \cdot P(\mathrm{Ep}/\mathrm{E}) + P(\mathrm{F2}) \cdot P(\mathrm{Ep}/\mathrm{F})} = 0.1429$$

$$P(\mathrm{E2}/\mathrm{Fp2}) = \frac{P(E2) \cdot P(Fp/E)}{P(E2) \cdot P(Fp/E) + P(F2) \cdot P(Fp/F)} = 0.3077$$

$$P(\mathrm{F2}/\mathrm{Fp2}) = \frac{P(\mathrm{F2}) \cdot P(\mathrm{F}p/\mathrm{F})}{P(\mathrm{E2}) \cdot P(\mathrm{Fp}/\mathrm{E}) + P(\mathrm{F2}) \cdot P(\mathrm{Fp}/\mathrm{F})} = 0.6923$$

The occurrence probabilities of further information are:

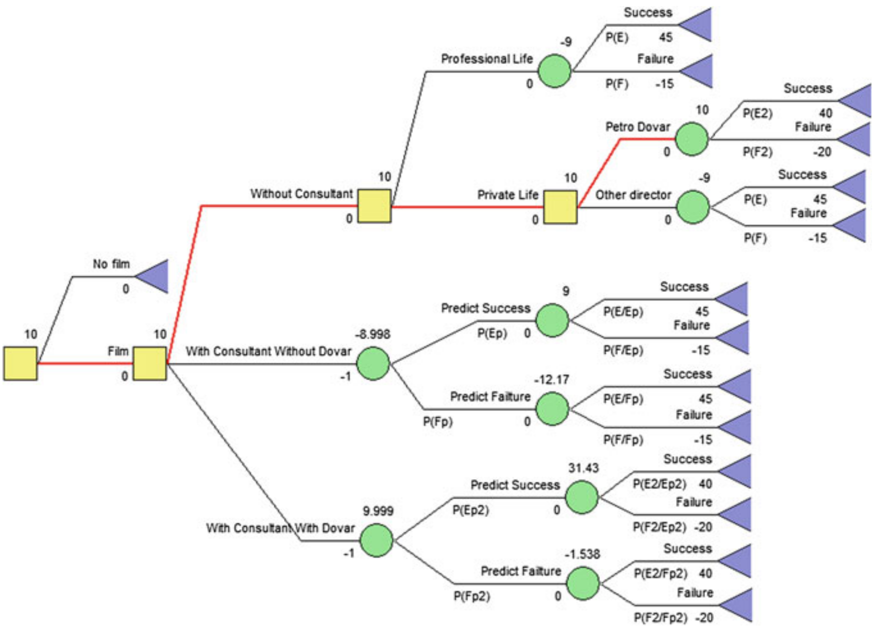
$$P(\mathrm{Ep}) = P(\mathrm{E}) \cdot P(\mathrm{Ep}/\mathrm{E}) + P(\mathrm{F}) \cdot P(\mathrm{Ep}/\mathrm{F}) = 0.1500$$

$$P(\mathrm{Fp}) = P(\mathrm{E}) \cdot P(\mathrm{Fp/E}) + P(\mathrm{F}) \cdot P(\mathrm{Fp/F}) = 0.8500$$

$$P(\mathrm{Ep2}) = P(\mathrm{E2}) \cdot P(\mathrm{Ep/E}) + P(\mathrm{F2}) \cdot P(\mathrm{Ep/F}) = 0.3500$$

$$P(\mathrm{Fp2}) = P(\mathrm{E2}) \cdot P(\mathrm{Fp/E}) + P(\mathrm{F2}) \cdot P(\mathrm{Fp/F}) = 0.6500.$$

The best strategy involves filming the picture, not contracting the film critic, and filming a picture based on the singer's private life with Petro Dovar. The expected profit is estimated at 10 million $.



(c) What is the imperfect information value?

The imperfect information sample value (IISV) is:

IISV = EMVPP − EMV = 9.9999 − 10 = −0.0001, which suggests that imperfect information is not suitable at any price in this case.

7.17 Planning Routes for a Vending Machine Firm

To save petrol, the person in charge of replacing articles and who works for ADECAF, an automatic food distribution firm, is planning a daily route to be followed to attend its customers. For this purpose, the firm is left with two options: the main road, because it is generally quicker, although traffic jams tend to form at

Table 7.3 Estimated travelling times

Route	Situation	
	Non-congested main road (s_1)	Congested main road (s_2)
Main road	25	45
Inland road	30	30

Exit 43, which sometimes means long delays; the inland road, a somewhat slower, but more reliable road. The person in charge of replacing items is clear about taking the inland road if there are traffic jams on the main road. Unfortunately, there is no way of knowing about the situation on the main road in advance. It should be pointed out that the driver will always take the main road for the return route because it is safer at night thanks to its signals.

Table 7.3 provides the estimations of the times taken in minutes to travel by both routes.

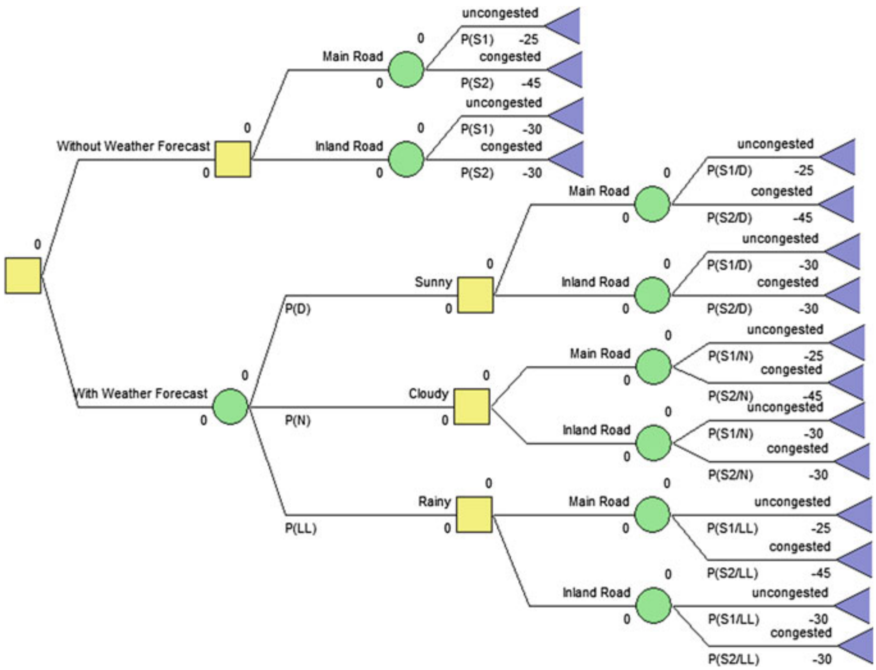
After using the main road for 1 month (20 days), the driver found that it was congested three times on the outgoing route. After a certain time, the driver noticed that the weather seemed to affect traffic on the main road. Indeed, the driver described three climate situations, sunny (D), cloudy (N) and rainy (LL), and he assigned the following conditional probabilities:

$P(D/s_1) = 0.8$	$P(N/s_1) = 0.2$	$P(LL/s_1) = 0$
$P(D/s_2) = 0.1$	$P(N/s_2) = 0.3$	$P(LL/s_2) = 0.6$

(a) Model this problem using a decision tree.
(b) What are the optimum decision strategy and the mean route time?
(c) How efficient is the information in accordance with the weather?

Solution

(a) Model this problem using a decision tree.



(b) What are the optimum decision strategy and the mean route time?

The initial (a priori) probabilities are:

$$P(\mathrm{S1}) = 0.85$$
$$P(\mathrm{S2}) = 0.15$$

The conditional probabilities (further information) are:

$$P(\mathrm{D}/\mathrm{S1}) = 0.8$$
$$P(\mathrm{D}/\mathrm{S2}) = 0.1$$
$$P(\mathrm{N}/\mathrm{S1}) = 0.2$$
$$P(\mathrm{N}/\mathrm{S2}) = 0.3$$
$$P(\mathrm{LL}/\mathrm{S1}) = 0$$
$$P(\mathrm{LL}/\mathrm{S2}) = 0.6$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are obtained:

$$P(\mathrm{S1/D}) = \frac{P(\mathrm{S1}) \cdot P(\mathrm{D/S1})}{P(\mathrm{S1}) \cdot P(\mathrm{D/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{D/S2})} = 0.9784$$

$$P(\mathrm{S2/D}) = \frac{P(\mathrm{S2}) \cdot P(\mathrm{D/S2})}{P(\mathrm{S1}) \cdot P(\mathrm{D/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{D/S2})} = 0.0216$$

$$P(\mathrm{S1/N}) = \frac{P(\mathrm{S1}) \cdot P(\mathrm{N/S1})}{P(\mathrm{S1}) \cdot P(\mathrm{N/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{N/S2})} = 0.7907$$

$$P(\mathrm{S2/N}) = \frac{P(\mathrm{S2}) \cdot P(\mathrm{N/S2})}{P(\mathrm{S1}) \cdot P(\mathrm{N/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{N/S2})} = 0.2093$$

$$P(\mathrm{S1/LL}) = \frac{P(\mathrm{S1}) \cdot P(\mathrm{LL/S1})}{P(\mathrm{S1}) \cdot P(\mathrm{LL/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{LL/S2})} = 0.0000$$

$$P(\mathrm{S2/LL}) = \frac{P(\mathrm{S2}) \cdot P(\mathrm{LL/S2})}{P(\mathrm{S1}) \cdot P(\mathrm{LL/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{LL/S2})} = 1.0000$$

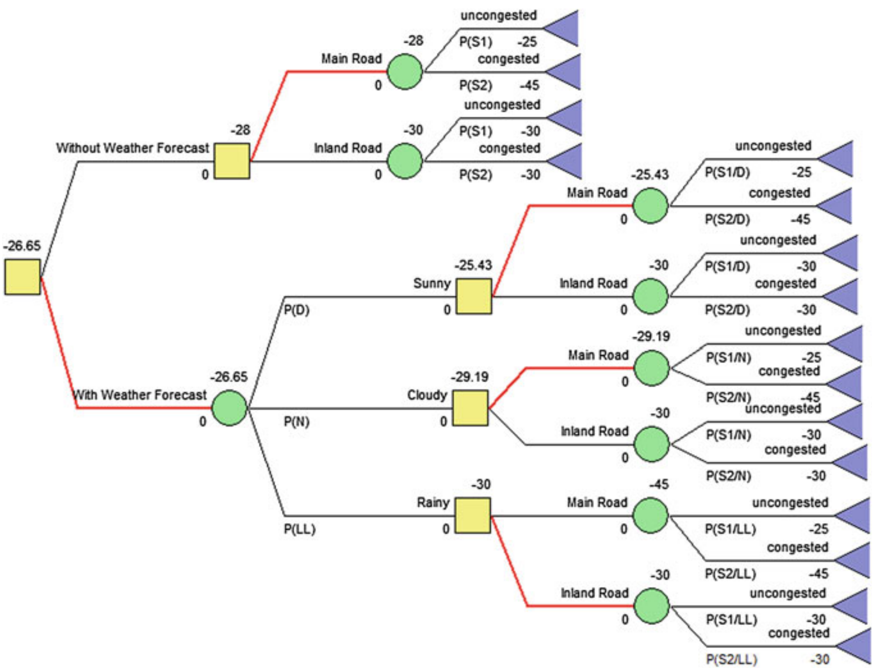
The occurrence probabilities of further information are:

$$P(\mathrm{D}) = P(\mathrm{S1}) \cdot P(\mathrm{D/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{D/S2}) = 0.6950$$

$$P(\mathrm{N}) = P(\mathrm{S1}) \cdot P(\mathrm{N/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{N/S2}) = 0.2150$$

$$P(\mathrm{LL}) = P(\mathrm{S1}) \cdot P(\mathrm{LL/S1}) + P(\mathrm{S2}) \cdot P(\mathrm{LL/S2}) = 0.0900.$$

Solving the tree shows that taking the main road is better if it is sunny or cloudy but if it is rainy, it is better to take the inland road. The mean route time is 26.65 min.



(c) How efficient is the information in accordance with the weather?

$$\text{IISV} = \text{EMV} - \text{EMVPP} = 28 - 26.65 = 1.35$$

$$\text{IVP} = \text{EMV} - \text{MLU} = 28 - 21.25 = 2.25$$

$$\text{Efficiency} = \text{IISV}/\text{PIV} = 0.6 = 60\,\%.$$

7.18 Replacing Petrol Cars with Electric Ones

A firm is contemplating replacing its fleet of petrol cars for one of electric cars. The manufacturer of these electric cars states that if the firm changes, it will make considerable savings while the fleet lasts, but the firm has its doubts. If the manufacturer is right, the firm will save 1 million $. If the new technology fails, as some critics have suggested, the change will cost the firm $450,000. A third possibility is that neither of the two former situations occurs and that the firm continues as it is now with no changes. According to a recent report made by some consultants, the probabilities of these three events are, respectively: 0.25, 0.45 and 0.30.

The firm can use a pilot scheme which, if carried out, will indicate the potential cost or savings involved by changing to electric cars. This scheme includes hiring three electric vehicles for 3 months and using them under normal conditions. This pilot scheme will cost the firm $50,000. The firm's adviser considers that the pilot scheme results will be significant, but not conclusive. To back his view, he provides the table below, which summarises the probabilities based on other firms' experiences (Table 7.4).

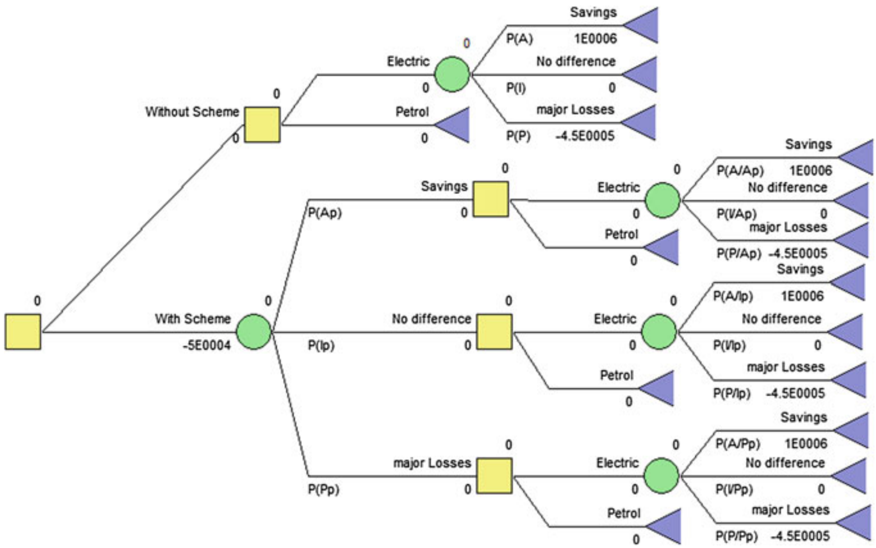
(a) Draw a suitable decision tree which identifies the probability and decision nodes and their respective expected earnings.
(b) What decision should the firm make if it wishes to maximise its expected savings?
(c) What is the largest amount of money that should paid to use the pilot scheme?

Table 7.4 Probabilities in other firms

		A pilot scheme indicates		
		Savings	No change	Losses
With change	Money is saved	0.6	0.3	0.1
	No difference	0.4	0.4	0.2
	Losses	0.1	0.5	0.4

Solution

(a) Draw a suitable decision tree which identifies the probability and decision nodes and their respective expected earnings.



(b) What decision should the firm make if it wishes to maximise its expected savings?

The initial (a priori) probabilities are:

$$P(\mathrm{A}) = 0.25$$
$$P(\mathrm{I}) = 0.3$$
$$P(P) = 0.45$$

The conditional probabilities (further information) are:

$$P(\mathrm{Ap}/\mathrm{A}) = 0.6$$
$$P(\mathrm{Ip}/\mathrm{A}) = 0.3$$
$$P(\mathrm{Pp}/\mathrm{A}) = 0.1$$
$$P(\mathrm{Ap}/\mathrm{I}) = 0.4$$
$$P(\mathrm{Ip}/\mathrm{I}) = 0.4$$
$$P(\mathrm{Pp}/\mathrm{I}) = 0.2$$
$$P(\mathrm{Ap}/P) = 0.1$$
$$P(\mathrm{Ip}/P) = 0.5$$
$$P(\mathrm{Pp}/P) = 0.4$$

By applying Bayes Theorem, the revised (a posteriori) probabilities obtained are:

$$P(\mathrm{A/Ap}) = \frac{P(\mathrm{A}) \cdot P(\mathrm{Ap/A})}{P(\mathrm{A}) \cdot P(\mathrm{Ap/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ap/I}) + P(P) \cdot P(\mathrm{Ap}/P)} = 0.4762$$

$$P(\mathrm{I/Ap}) = \frac{P(\mathrm{I}) \cdot P(\mathrm{Ap/I})}{P(\mathrm{A}) \cdot P(\mathrm{Ap/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ap/I}) + P(P) \cdot P(\mathrm{Ap}/P)} = 0.3810$$

$$P(P/\mathrm{Ap}) = \frac{P(P) \cdot P(\mathrm{Ap}/P)}{P(\mathrm{A}) \cdot P(\mathrm{Ap/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ap/I}) + P(P) \cdot P(\mathrm{Ap}/P)} = 0.1428$$

$$P(\mathrm{A/Ip}) = \frac{P(\mathrm{A}) \cdot P(\mathrm{Ip/A})}{P(\mathrm{A}) \cdot P(\mathrm{Ip/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ip/I}) + P(P) \cdot P(\mathrm{Ip}/P)} = 0.1786$$

$$P(\mathrm{I/Ip}) = \frac{P(\mathrm{I}) \cdot P(\mathrm{Ip/I})}{P(\mathrm{A}) \cdot P(\mathrm{Ip/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ip/I}) + P(P) \cdot P(\mathrm{Ip}/P)} = 0.2857$$

$$P(P/\mathrm{Ip}) = \frac{P(P) \cdot P(\mathrm{Ip}/P)}{P(\mathrm{A}) \cdot P(\mathrm{Ip/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ip/I}) + P(P) \cdot P(\mathrm{Ip}/P)} = 0.5357$$

$$P(\mathrm{A/Pp}) = \frac{P(\mathrm{A}) \cdot P(\mathrm{Pp/A})}{P(\mathrm{A}) \cdot P(\mathrm{Pp/A}) + P(\mathrm{I}) \cdot P(\mathrm{Pp/I}) + P(P) \cdot P(\mathrm{Pp}/P)} = 0.0943$$

$$P(\mathrm{I/Pp}) = \frac{P(\mathrm{I}) \cdot P(\mathrm{Pp/I})}{P(\mathrm{A}) \cdot P(\mathrm{Pp/A}) + P(\mathrm{I}) \cdot P(\mathrm{Pp/I}) + P(P) \cdot P(\mathrm{Pp}/P)} = 0.2264$$

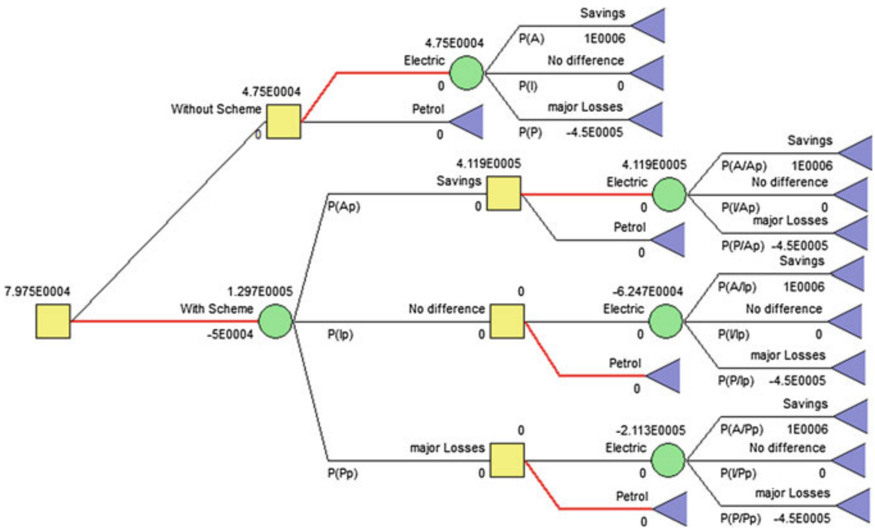
$$P(P/\mathrm{Pp}) = \frac{P(P) \cdot P(\mathrm{Pp}/P)}{P(\mathrm{A}) \cdot P(\mathrm{Pp/A}) + P(\mathrm{I}) \cdot P(\mathrm{Pp/I}) + P(P) \cdot P(\mathrm{Pp}/P)} = 0.6792$$

The occurrence probabilities of further information are:

$$P(\mathrm{Ap}) = P(\mathrm{A}) \cdot P(\mathrm{Ap/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ap/I}) + P(P) \cdot P(\mathrm{Ap}/P) = 0.3150$$

$$P(\mathrm{Ip}) = P(\mathrm{A}) \cdot P(\mathrm{Ip/A}) + P(\mathrm{I}) \cdot P(\mathrm{Ip/I}) + P(P) \cdot P(\mathrm{Ip}/P) = 0.4200$$

$$P(\mathrm{Pp}) = P(\mathrm{A}) \cdot P(\mathrm{Pp/A}) + P(\mathrm{I}) \cdot P(\mathrm{Pp/I}) + P(P) \cdot P(\mathrm{Pp}/P) = 0.2650$$



As the earnings expected from the pilot scheme are comparatively higher than those expected with the a priori probabilities, the firm is recommended to go ahead with the pilot scheme and then change its petrol cars to electric ones.

(c) What is the largest amount of money that should be paid to use the pilot scheme? The maximum amount of money that the firm will have to pay to use the pilot scheme is \$82,250:

$$\text{IISV} = \text{EMVPP} - \text{EMV} = 129750 - 47500 = \$82,250.$$

7.19 A Football Player's Injury

Football player Villade, a forward who plays for Spain, thinks that he has injured his ankle while training today. The Spanish team's doctor believes that the probability of him having an injured ankle is 0.2. Therefore, the question is: should he play in the quarter-final European Cup match against Russia?

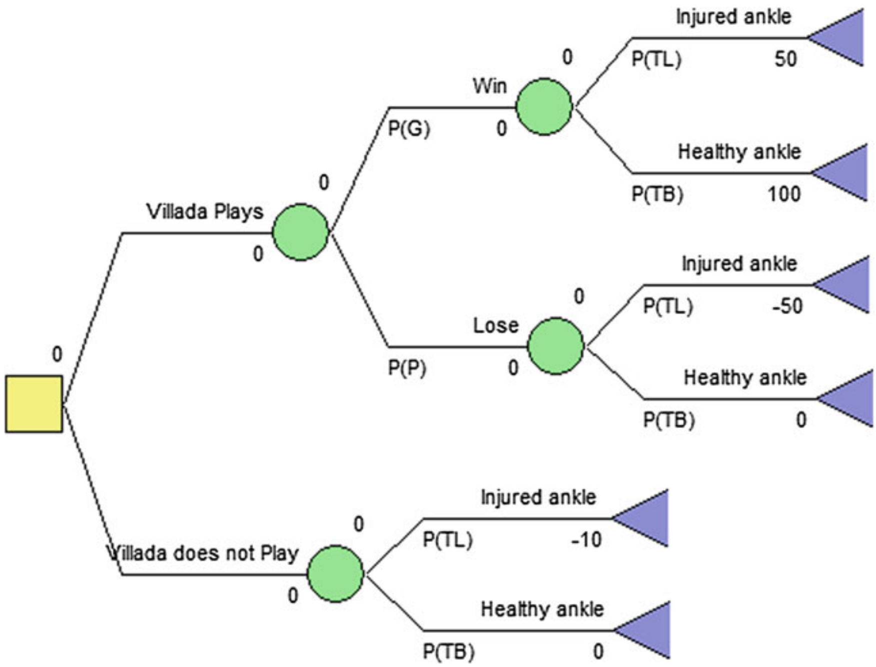
If Villade plays, the coach thinks that there is a probability of 0.1 of Spain winning. However if Villade plays and his ankle is actually injured, then the injury can worsen. Thus, the utilities that the coach employs are: if Villade plays, Spain wins the quarter finals and his ankle is not injured, +100; if Villade plays, Spain wins and Villade's ankle worsens, +50; if Villade plays, Spain loses and Villade's ankle is not injured, 0; if Villade plays, Spain loses and his ankle is injured, −50.

If, conversely, Villade does not play and his ankle is injured, the utility is −10, and it is 0 if it is not injured.

(a) Draw a decision tree for this problem.
(b) What is the largest amount of money that should be paid to use the pilot scheme?
(c) Calculate the perfect information value according to the state of Villade's ankle.
(d) Calculate the perfect information value according to Spain winning the quarter finals.
(e) In the original problem, the probability of Villade's ankle being injured and that of Spain winning the quarter finals of the European Cup 2008 are independent. So, is it possible to employ a decision tree if the probability of winning depends on Villade's ankle being injured? Explain how the structure and data of the decision tree in question (a) would change.

Solution

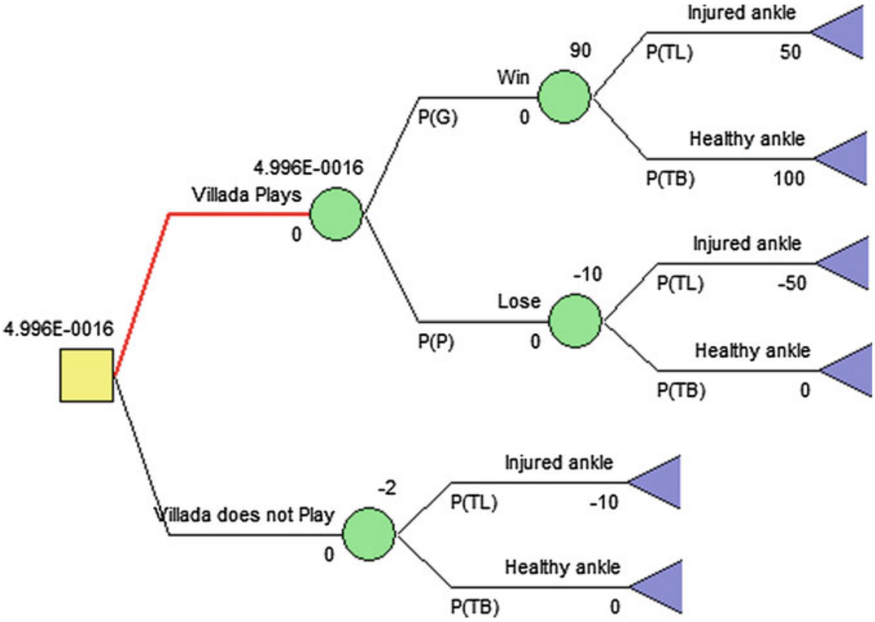
(a) Draw a decision tree for this problem.



(b) Evaluate the tree and indicate the best action to take and the expected utility.

Using the initial (a priori) probabilities:

$$P(\mathrm{G}) = 0.1$$
$$P(P) = 0.9$$
$$P(\mathrm{TL}) = 0.2$$
$$P(\mathrm{TB}) = 0.8$$



The optimum decision is that Villade plays, with a mean utility of 0.

The coach could obtain further information about the state of Villade's ankle through medical tests. He could also acquire more information about the probabilities of winning the quarter finals if he spoke with several expert TV sport commentators.

(c) Calculate the perfect information value according to the state of Villade's ankle.

Based on the perfect information about his ankle, the expected perfect information value (PIV) is:

$$\mathrm{PIV} \; = \; \mathrm{MLU} - \mathrm{EMV} \; = 6 - 0 \; = 6;$$

	Injured ankle $p = 0.2$	Healthy ankle $p = 0.8$	MLU
Plays	-40	**10**	
Does not play	**−10**	0	$= -10 \cdot 0.2 + 10 \cdot 0.8 = 6$

(d) Calculate the perfect information value according to Spain winning the quarter finals.

	Wins $p = 0.1$	Loses $p = 0.9$	MLU
Plays	90	**10**	
Does not play	**−2**	-2	$= 90 \cdot 0.1 + (-2) \cdot 0.9 = 7.2$

$$\text{PIV} = \text{MLU} - \text{EMV} = 7.2 - 0 = 7.2$$

(e) In the original problem, the probability of Villade's ankle being injured and that of Spain winning the quarter finals of the European Cup 2008 are independent. So, is it possible to employ a decision tree if the probability of winning depends on Villade's ankle being injured? Explain how the structure and data of the decision tree in question (a) would change.

Yes, it is possible. It is necessary to place the winning branch after the injured ankle branch, and to use the conditional probabilities in relation to the state of the player's ankle.

7.20 Medical Decisions

Dr. House has a patient who is seriously ill. Without treatment, this patient will live 3 months. The only alternative treatment is a risky operation. The patient will live 1 year after surviving the operation; nevertheless, the probability that the patient does not survive the operation is 0.3.

(a) Assume that U(12) = 1 and U(0) = 0, what minimum value would the utility of living 3 months have to take, U(3) for the patient to prefer the operation? Use a decision tree to obtain your answer.

For the rest of the problem, assume U(3) = 0.8. Dr. House has found out that there is a less risky procedure test which would provide uncertain information to predict whether the patient will survive the operation or not. If the test is positive, the probability that the patient survives the operation increases. This test has the following characteristics:

- The probability that the test results are positive and the patient surviving the operation is 0.9.
- The probability that the test results are positive and the patient not surviving the operation is 0.1.

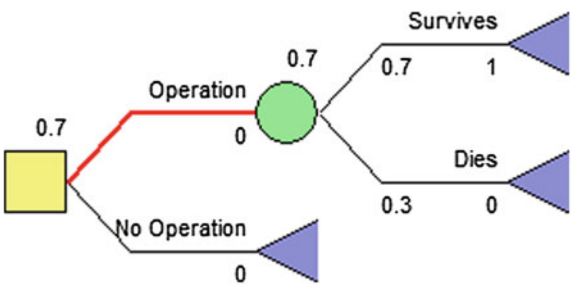
(b) What probability is there of the patient surviving the operation if the test is positive?

Finally, this test may entail fatal complications; that is, the patient might die during the test, with a probability of 0.005.

(c) Draw a decision tree showing all the options and consequences of Dr. House's problem. Should he tell the patient about the possibility of doing the test before deciding on the operation?

Solution

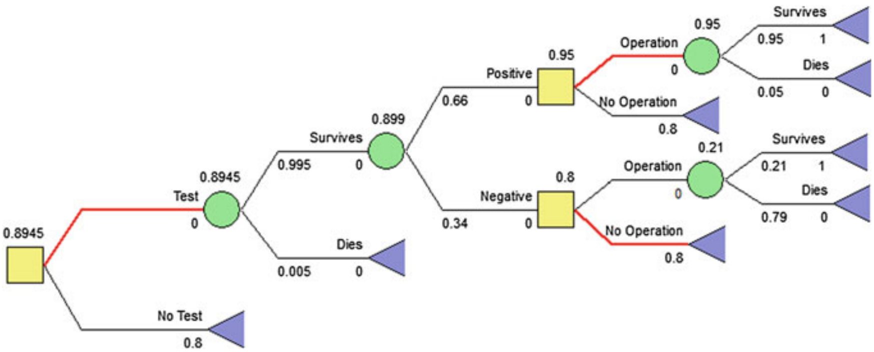
(a) Assume that U(12) = 1 and U(0) = 0, what minimum value would the utility of living 3 months have to take, U(3) for the patient to prefer the operation? Use a decision tree to obtain your answer.



The operation would be preferable for $U(3) < 0.7$.

(b) What probability is there of the patient surviving the operation if the test is positive? P(surviving/positive) = 0.9545.
(c) Draw a decision tree showing all the options and consequences of Dr. House's problem. Should he tell the patient about the possibility of doing the test before deciding on the operation?

Yes, Dr. House should talk to the patient about the test, which should be done. If the result is positive, the operation should be done, otherwise, it should not. The decision tree would be as follows:



7.21 Developing New Textile Products

A textile firm which sells home textile pieces is considering developing a new set of textile products (sheets, bedcovers, towels, etc.) based on a commercial French brand. For this purpose, it must pay the corresponding royalties to use this brand. The Manager of this firm has devised Table 7.5, showing the utilities based on his/her subjective perceptions.

However, the Manager feels reluctant to make this decision. So he/she requests assistance from a market research firm, which implies 500 utilities. By sampling and then analysing the previous results of the consultancy firm, a confidentiality matrix has been drawn up (Table 7.6).

(a) Draw the decision tree to help the Manager make a decision.
(b) Evaluate the decision tree and indicate the decision to be made.

Table 7.5 Utilities according to the managers's perceptions

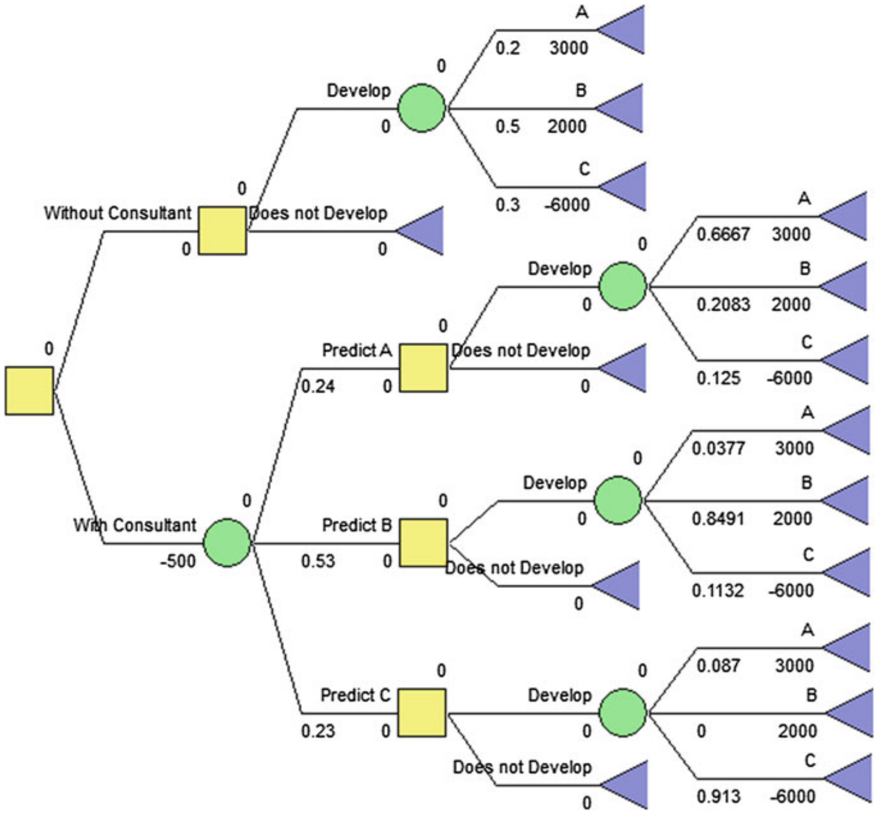
		States of nature		
		Many sales	Average sales	Few sales
		A(0.2)	B(0.5)	C(0.3)
A1	(develop)	3000	2000	−6000
A2	(do not develop)	0	0	0

Table 7.6 Confidentiality matrix

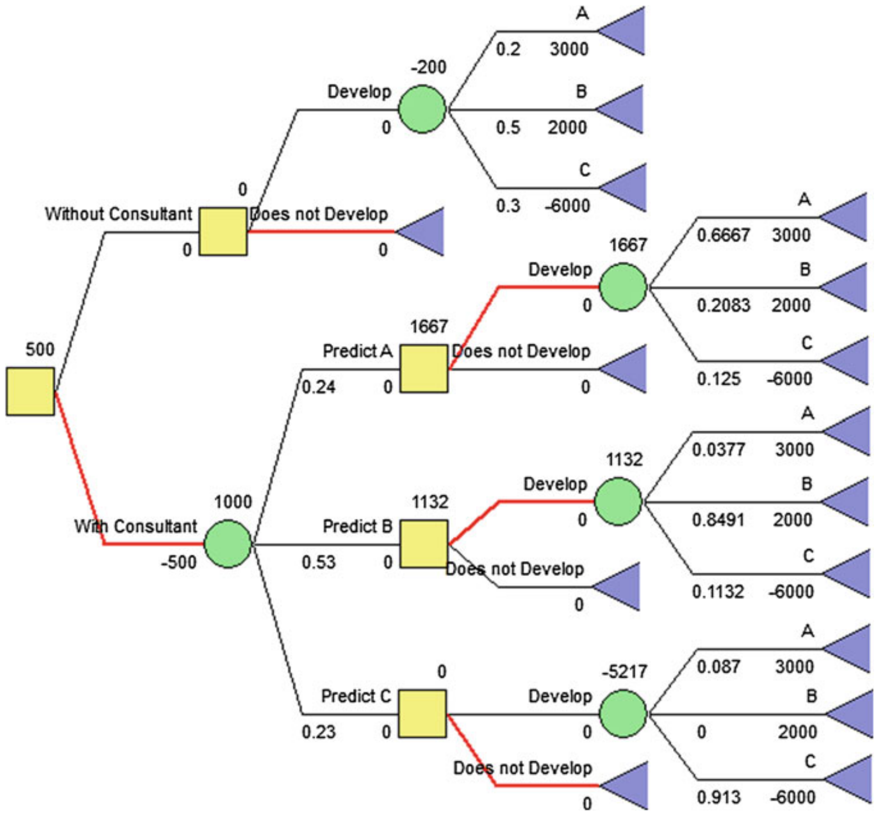
		What actually occurred in the past		
		A	B	C
What the consultancy firm predicted	Ap	0.8	0.1	0.1
	Bp	0.1	0.9	0.2
	Cp	0.1	0.0	0.7

Solution

(a) Draw the decision tree to help the Manager make a decision.



(b) Evaluate the decision tree and indicate the decision to be made.



The decision which emerges from the decision tree is as follows:

Contract the consultancy firm and then wait for its report. If the report predicts many sales or average sales, develop the products, otherwise, do not develop them.

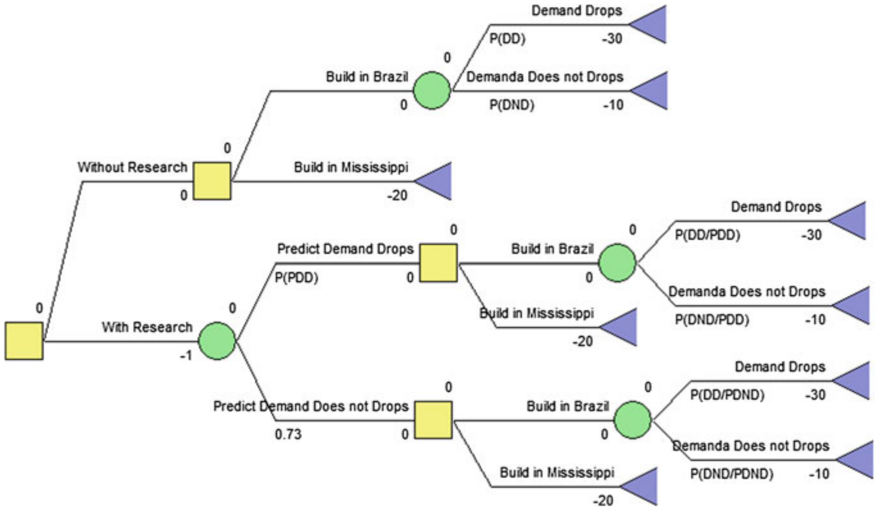
7.22 Locating Production Plants in the Automobile Industry

Tayota, a multinational firm in the automobile sector, is deciding on whether to build, or not, an assembly plant in Brazil or in Mississipi (USA). The cost to build this plant in Brazil is \$10 million, and it is \$20 million in Mississippi. Nonetheless, if the firm builds this plant in Brazil and local demand drops over the following 5 years, the project will be stopped and the firm will lose \$10 million (and it will still have to build a plant in Mississippi). A priori, Tayota believes that the probability of demand for cars dropping in Brazil over the following 5 years is 20 %. For one million $, it can contract a market research firm to analyse demand for cars in Brazil, which will indicate whether demand for cars will drop or not. The market research firm's record indicates that it will predict the occurrence of a drop in demand for 95 % of the time, and of it not occurring for 90 % of the time.

(a) Should Tayota contract the market research firm?
(b) Indicate the optimum strategy that Tayota should adopt and the expected overall cost
(c) Find out the IISV and the PIV.

Solution

(a) Should Tayota contract the market research firm?



The initial (a priori) probabilities are:

$$P(\mathrm{DD}) = 0.2$$
$$P(\mathrm{DND}) = 0.8$$

The conditional probabilities (further information) are:

$$P(\mathrm{PDD}/\mathrm{DD}) = 0.95$$
$$P(\mathrm{PDND}/\mathrm{DD}) = 0.05$$
$$P(\mathrm{PDD}/\mathrm{DND}) = 0.1$$
$$P(\mathrm{PDND}/\mathrm{DND}) = 0.9$$

By applying Bayes Theorem, the revised (a posteriori) probabilities are:

$$P(\mathrm{DD}/\mathrm{PDD}) = \frac{P(\mathrm{DD}) \cdot P(\mathrm{PDD}/\mathrm{DD})}{P(\mathrm{DD}) \cdot P(\mathrm{PDD}/\mathrm{DD}) + P(\mathrm{DND}) \cdot P(\mathrm{PDD}/\mathrm{DND})} = 0.7037$$

$$P(\mathrm{DND}/\mathrm{PDD}) = \frac{P(\mathrm{DND}) \cdot P(\mathrm{PDD}/\mathrm{DND})}{P(\mathrm{DD}) \cdot P(\mathrm{PDD}/\mathrm{DD}) + P(\mathrm{DND}) \cdot P(\mathrm{PDD}/\mathrm{DND})} = 0.2963$$

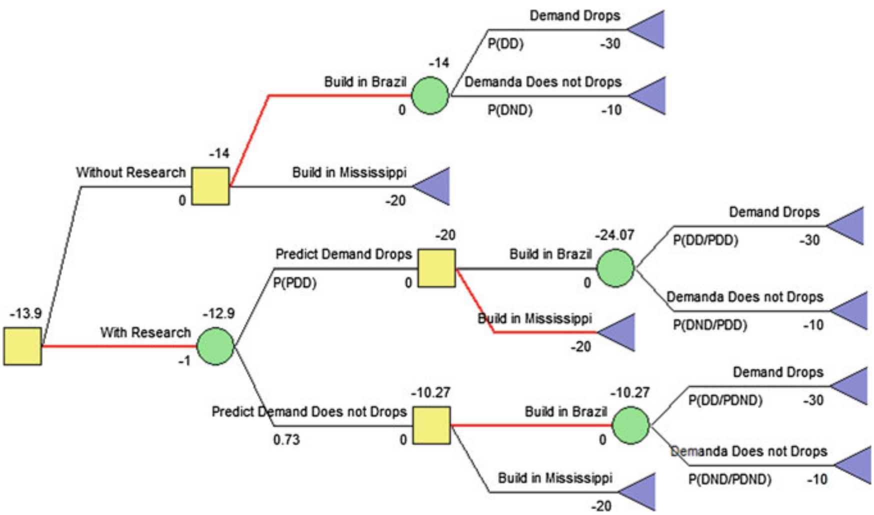
$$P(\mathrm{DD}/\mathrm{PDND}) = \frac{P(\mathrm{DD}) \cdot P(\mathrm{PDND}/\mathrm{DD})}{P(\mathrm{DD}) \cdot P(\mathrm{PDND}/\mathrm{DD}) + P(\mathrm{DND}) \cdot P(\mathrm{PDND}/\mathrm{DND})} = 0.0137$$

$$P(\mathrm{DND}/\mathrm{PDND}) = \frac{P(\mathrm{DND}) \cdot P(\mathrm{PDND}/\mathrm{DND})}{P(\mathrm{DD}) \cdot P(\mathrm{PDND}/\mathrm{DD}) + P(\mathrm{DND}) \cdot P(\mathrm{PDND}/\mathrm{DND})} = 0.9863$$

The occurrence probabilities of further information are:

$$P(\mathrm{PDD}) = P(\mathrm{DD}) \cdot P(\mathrm{PDD}/\mathrm{DD}) + P(\mathrm{DND}) \cdot P(\mathrm{PDD}/\mathrm{DND}) = 0.2700$$

$$P(\mathrm{PDND}) = P(\mathrm{DD}) \cdot P(\mathrm{PDND}/\mathrm{DD}) + P(\mathrm{DND}) \cdot P(\mathrm{PDND}/\mathrm{DND}) = 0.7300$$



(b) Indicate the optimum strategy that Tayota should adopt and the expected overall cost.

According to the solution obtained, if a drop in demand is predicted, the plant must be built in Mississippi; if a drop in demand is not predicted, then it must be built in Brazil. The expected overall cost is 13,900,000$.

(c) Find out the IISV and the PIV.

$$\text{IISV} = \text{EMVPP} - \text{EMV} = -12.9 - (-14) = 1.1 \text{ million \$}$$

$$\text{PIV} = \text{MLU} - \text{EMV} = 0.8 \cdot (-10) + 0.2 \cdot (-20) - (-14) = 2 \text{ million \$} .$$

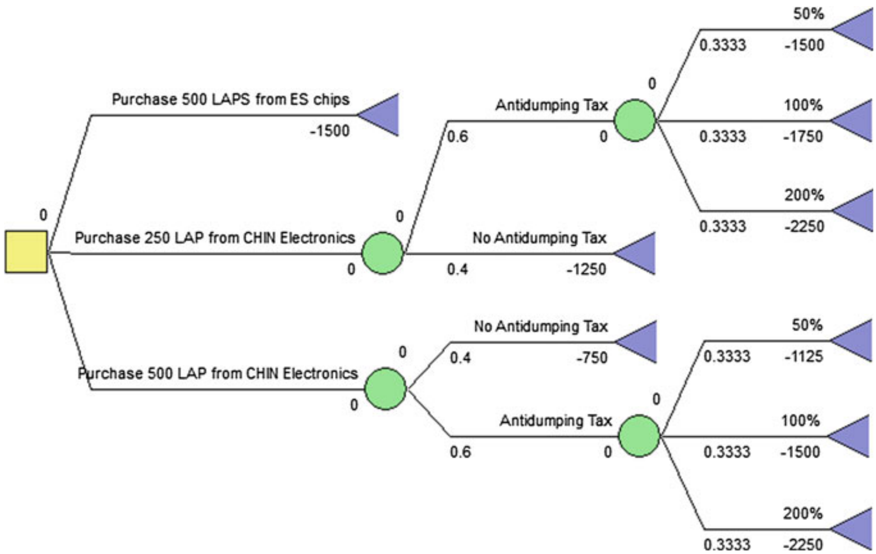
7.23 Selecting Antidumping Purchases and Measures

ADESA Sistemas has agreed to supply 500,000 PC FAX systems to Tarja Warehouses in 90 days at a fixed price. One key component of the FAX systems is an integrated logic circuit chip in a programmable matrix (LAP chip), which is required in each FAX system. In the past, ADESA has purchased these chips from a Spanish chips manufacturer, ES Chips. However, ADESA has received an offer from a Chinese manufacturer, CHIN Electronics, whose price for these chips is lower. This offer is valid for 10 days, and ADESA must decide to buy some or all the LAP chips from CHIN Electronics. Any of the chips ADESA does not purchase from CHIN must be bought from ES. ES Chips will sell the chips to ADESA at \$3.00 per chip for any amount. CHIN accepts orders only in multiples of 250,000 LAP chips, and its offer is \$2.00 per chip for 250,000 chips, and \$1.5 per chip for 500,000 chips, or more. However, this situation is complicated due to a dumping report (an international business practice whereby a firm fixes a lower price for exported goods than for the same goods sold in the country) that has been imposed by ES Chips against CHIN. If this report is admitted by the Spanish government, the CHIN chips will be subject to an antidumping tax. This situation will not be solved until after ADESA has made a decision on its purchase. If ADESA buys the CHIN chips, they will not be delivered until after the antidumping tax comes into effect and the chips will be subject to this tax. According to the terms offered by CHIN, ADESA will have to pay any antidumping tax imposed. ADESA thinks that there is a 60 % probability that the antidumping tax will be imposed. If it is imposed, then it is equally likely that the antidumping tax is 50, 100 or 200 % of the selling price for each LAP chip.p

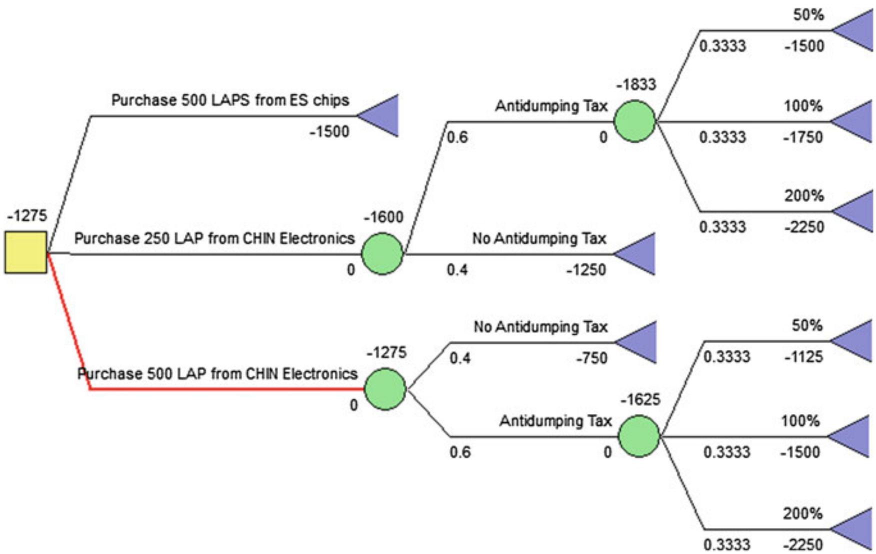
(a) Draw a decision tree for this decision.
(b) Using the EMV decision criterion, determine what the better alternative order is that ADESA can make for LAP chips.

Solution

(a) Draw a decision tree for this decision.



Using the EMV decision criterion, determine what the better alternative order is that ADESA can make for LAP chips.



According to the solution obtained, ADESA should order all 500,000 LAP chips from CHIN Electronics with an expected overall cost of $1,275,000.

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Chapter 8
Games Theory

Abstract This chapter begins with an introduction to Games Theory, describes types of competitive problems, which can be modelled and solved by the Games Theory, and also provides details of solution methods for zero-sum games with two players. Then, it proposes a varied combination of Games Theory problems and provides their corresponding solution. This chapter aims to provide a better understanding of the problems in which more than one decision maker intervenes, who are conflictive. This chapter also proposes problems in which the players, their strategies, and the profits or costs that each would obtain per strategies combination, are identified, and the most suitable strategies should be obtained for the players.

8.1 Introduction

Games theory is the study of conflict and cooperation decision making when more than one decision maker is present. Game theory is used mainly in economics, political science, and psychology where decisions in cooperation and non-cooperation between persons, enterprises or nations are present. The most analysed subject is zero-sum games, in which one person's gains exactly equal the other person's net losses. Moreover, Games Theory has dealt with other more complex situations in which more than two players intervene, and in which the games are not zero-sum.

Games Theory, as a unified discipline, began thanks to Von Neumann's research into the MiniMax Theorem (Von Neumann 1928), especially after his basic work, together with Morgenstern, was published (Von Neumann and Morgenstern 1947), which set the bases of matrix games. Later, and as a result of the contributions of the Optimal Control Theory, it branched off into differential games and into the study of cooperative games. The development of cooperative games is based on the study of the Nucleus Theory and included the Shapley value concept, as well as the works of Aumann and Maschler (1964). There are other

R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5_8, © Springer-Verlag London 2014

authors worth citing: Kreweras (1963), with this collective behaviour study, and Arrow (1974). Then John F. Nash (1950) defined the equilibrium named after him as a way to obtain an optimal strategy for games involving two players or more. Tucker and Luce (1959) collected the problems that were being examined in that decade. If there was a set of strategies in which no player benefits by changing strategy while the others do not change their strategies, then this set of strategies, and the corresponding gains, constitute a Nash equilibrium.

The game's main characteristics are: (i) two decision makers, or more, are involved; (ii) the consequences (results) depend on the actions taken by all the decision makers; (iii) the objectives do not coincide (they can be the absolute opposite). There is also a particularly important characteristic in a game: all the players must make a decision without knowing what the others have decided. Every time players decide, a "move" is made which results in each player's loss or gain. The game can be repeated by generating successive "moves." A game has some similar characteristics to a decision problem, but the "nature" and its aleatory states of a decision problem are transformed into another intelligent decision maker in a game, who will make a decision by thinking about what his or her opponent is thinking.

One game with two players may be represented in a tabular form, and its overall format resembles that provided below (for a 2 × 2 case):

		Player B	
		b_1	b_2
Player A	a_1	(a_{11}, b_{11})	(a_{12}, b_{12})
	a_2	(a_{21}, b_{21})	(a_{22}, b_{22})

Player A can choose between alternatives a_1 and a_2, while Player B can select between b_1 and b_2. Depending on the choice made, the result will be one of the four matrix cells. If Player A opts for strategy a_i and Player B decides on strategy b_j, the result will be a payment of a_{ij} for Player A and a payment of b_{ij} for Player B.

If the game played by two players is a zero-sum game (that is, what one player gains, the other player loses), it is generally represented as shown below:

		Player B	
		b_1	b_2
Player A	a_1	$(a_{11}, -a_{11})$	$(a_{12}, -a_{12})$
	a_2	$(a_{21}, -a_{21})$	$(a_{22}, -a_{22})$

In this case, the table is simplified by assuming that the values contained in the cells imply gains for the player of the rows (Player A) and losses for the player of the columns (Player B):

		Player B	
		b_1	b_2
Player A	a_1	a_{11}	a_{12}
	a_2	a_{21}	a_{22}

If a game has an equilibrium point, it can be solved by applying the MiniMax Method: each player will choose the alternative which provides the best result among the possible results that can be presented:

		Player B		
		b_1	b_2	
Player A	a_1	a_{11}	a_{12}	$min\{a_{11}, a_{12}\}$
	a_2	a_{21}	a_{22}	$min\{a_{21}, a_{22}\}$
		$max\{a_{11}, a_{21}\}$	$max\{a_{12}, a_{22}\}$	

Player A will choose $max\{min\{a_{11},a_{12}\}; min\{a_{21}, a_{22}\}\}$
Player B will choose $min\{max\{a_{11}, a_{21}\}; max\{a_{12}, a_{22}\}\}$.

Should both players choose the same value and it comes from the same matrix cell, we face a pure strategy zero-sum game with two players to be solved; that is, with an equilibrium point for which neither player is interested in leaving. Thus, the player will maintain his/her decision, even when he or she knows the decision that the opponent will make. If several moves are made, the players do not change the decisions they have made, and the game value (the mean payment per move) remains constant and equals the value of the selected cell.

Should the selected values differ, the game has no equilibrium point because the players might be interested in changing their decision when they know their opponent's decision. In this case, players must adopt a mixed strategy, defined by a set of selection probabilities of each alternative. This set of probabilities can be algebraically solved if the game matrix if a 2 × 2 type, as follows:

		Player B		
		b_1	b_2	
Player A	a_1	a_{11}	a_{12}	p
	a_2	a_{21}	a_{22}	$1-p$
		q	$1-q$	

where p is the probability with which Player A should choose alternative a_1 (therefore, $1-p$ is the probability with which alternative a_2 should be chosen), and q is the probability with which Player B should choose alternative b_1 (therefore, $1-p$ is the probability with which alternative b_2 should be chosen).

If both players select their alternatives with these probabilities in successive moves, the game value (the mean payment per move) is the sum of the values of each cell multiplied by the probability that this cell occurs; that is to say:

$$v = a_{11} \cdot p \cdot q + a_{21} \cdot (1-p) \cdot q + a_{12} \cdot p \cdot (1-q) + a_{22} \cdot (1-p) \cdot (1-q)$$

To determine his or her strategy, Player A considers that probability p is such that it ensures that the player obtains a game value that does not change no matter what his or her opponent does. To go about this, an equation is considered which equals the game value should Player B always choose alternative $b_{1.}$ Should alternative b_2 always be selected, the q values are 1 and 0, respectively. Therefore, the equalled game value equations are:

$$v = a_{11} \cdot p + a_{21} \cdot (1-p) = a_{12} \cdot p + a_{22} \cdot (1-p)$$

from where the p value can be found. Player B considers the same and obtains the q value from the following equation:

$$v = a_{11} \cdot q + a_{12} \cdot (1-q) = a_{21} \cdot q + a_{22} \cdot (1-q)$$

If the game matrix dimensions are greater than 2×2, the algebraic solution is not applicable. In this case, a general linear programming-based method solution can be applied. By assuming a 3×3 matrix, we find that:

		Player B			
		b_1	b_2	b_3	
Player A	a_1	a_{11}	a_{12}	a_{13}	p_1
	a_2	a_{21}	a_{22}	a_{21}	p_2
	a_3	a_{31}	a_{32}	a_{33}	p_3
		q_1	q_2	q_3	

Player A can construct a linear programming model by applying the same concept as if it were a 2×2 matrix; that is, Player A wishes to play with a set of probabilities so that the game value obtained is maintained regardless of the set of probabilities which Player B plays with. To do this, the following steps are taken:

Step 1. Eliminate the negative matrix values by adding a k value to all the cells.
Step 2. Write as many inequalities and alternatives as Player B has with which he or she wishes to ensure the game value:

$$a_{11} \cdot p_1 + a_{21} \cdot p_2 + a_{31} \cdot p_3 \geq v$$

$$a_{12} \cdot p_1 + a_{22} \cdot p_2 + a_{32} \cdot p_3 \geq v$$

$$a_{13} \cdot p_1 + a_{23} \cdot p_2 + a_{33} \cdot p_3 \geq v$$

Step 3. Divide the inequalities by v:

$$a_{11} \cdot \frac{p_1}{v} + a_{21} \cdot \frac{p_2}{v} + a_{31} \cdot \frac{p_3}{v} \geq \frac{v}{v}$$

$$a_{12} \cdot \frac{p_1}{v} + a_{22} \cdot \frac{p_2}{v} + a_{32} \cdot \frac{p_3}{v} \geq \frac{v}{v}$$

$$a_{13} \cdot \frac{p_1}{v} + a_{23} \cdot \frac{p_2}{v} + a_{33} \cdot \frac{p_3}{v} \geq \frac{v}{v}$$

Step 4. Change a variable:

$$\frac{p_i}{v} = x_i$$

$$a_{11} \cdot x_1 + a_{21} \cdot x_2 + a_{31} \cdot x_3 \geq 1$$

$$a_{12} \cdot x_1 + a_{22} \cdot x_2 + a_{32} \cdot x_3 \geq 1$$

$$a_{13} \cdot x_1 + a_{23} \cdot x_2 + a_{33} \cdot x_3 \geq 1$$

Step 5. Construct the objective function using the summation probabilities condition:

$$p_1 + p_2 + p_3 = 1$$

$$\frac{p_1}{v} + \frac{p_2}{v} + \frac{p_3}{v} = \frac{1}{v}$$

$$x_1 + x_2 + x_3 = \frac{1}{v}$$

As the player wishes to maximise v, it is the same as minimising its inverse:

$$\text{Min}\, Z = x_1 + x_2 + x_3$$

Step 6. Formulate the linear programming model:

$$\text{Min}\, Z = x_1 + x_2 + x_3$$

$$a_{11} \cdot x_1 + a_{21} \cdot x_2 + a_{31} \cdot x_3 \geq 1$$

$$a_{12} \cdot x_1 + a_{22} \cdot x_2 + a_{32} \cdot x_3 \geq 1$$

$$a_{13} \cdot x_1 + a_{23} \cdot x_2 + a_{33} \cdot x_3 \geq 1$$

Step 7. Solve the linear programming model and obtain values $\{x_1, x_2, x_3\}$

Step 8. Obtain probabilities $\{p_1, p_2, p_3\}$ by the proportionality of the values obtained:

$$p_i = \frac{x_i}{x_1 + x_2 + x_3}$$

Step 9. Obtain the game value (by subtracting the added value from all the cells):

$$v = \frac{1}{x_1 + x_2 + x_3} - k$$

Player B does a similar calculation to obtain his or her probabilities $\{q_1, q_2, q_3\}$ but wishes, in this case, to minimise the game value. Therefore, the linear programming model is a maximising one and the inequations are lower or equal.

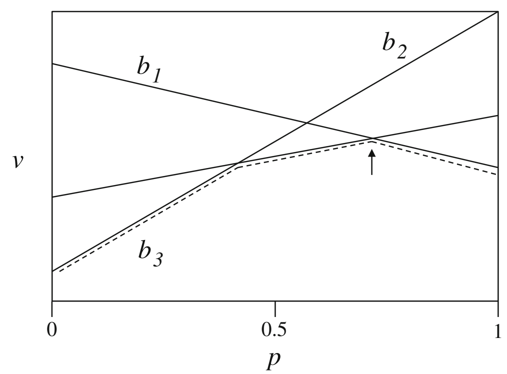
It is obvious that the algebraic solution of the 2 × 2 matrix is much simpler than that of a larger sized matrix. For this reason, one of the exercises prior to applying the general linear programming method involves seeking dominances. A strategy is dominated if there is another better or equal strategy in all cases. By assuming that this is the matrix, we see that:

		Player B		
		b_1	b_2	b_3
Player A	a_1	a_{11}	a_{12}	a_{13}
	a_2	a_{21}	a_{22}	a_{21}
	a_3	a_{31}	a_{32}	a_{33}

What happens is that $a_{31} \cdot \leq a_{11} \wedge a_{32} \cdot \leq a_{12} \wedge a_{33} \cdot \leq a_{13}$. In this case, strategy a_3 is dominated by strategy a_1 (Player A would never select a_3 if he or she had a_1 as an option). So it can be eliminated from the matrix:

		Player B		
		b_1	b_2	b_3
Player A	a_1	a_{11}	a_{12}	a_{13}
	a_2	a_{21}	a_{22}	a_{21}

Fig. 8.1 Solving a 2 × 3 game by the graph method



This exercise involves eliminating dominated strategies and it is done sequentially in both rows and columns until neither the dominated rows nor the columns are left.

If the matrix cannot be reduced to 2×2, but one of its dimensions is 2 (as in the previous example), the Graph Method can be applied to obtain the probabilities that the player with two strategies should play with. The following figure is an example:

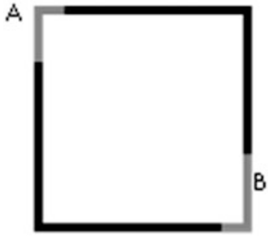
In a graph whose x-axis is probability p and y-axis is game value v, a straight line is represented for each alternative that the opponent has. The quickest way of representation is to locate the points resulting from applying $p = 0$ and $p = 1$. Having represented all the straight lines, the player marks those points which provide him or her with the worst results as he or she selects different p values. In Fig. 8.1, it is assumed that the straight lines represent the player's gains. Thus, the worst values are the lowest ones, and are marked as dashed segments. Of all these points, the player selects the best which, in this case, is the highest (marked by an arrow in Fig. 8.1). This point also indicates game value v.

Game theory has been widely recognised as an important tool in many different fields. Twelve game theorists have won the Nobel Memorial Prize in Economic Sciences: Paul A. Samuelson (1970), Kenneth J. Arrow (1972), John C. Harsanyi (1994), John F. Nash Jr. (1994), Reinhard Selten (1994), Robert E. Lucas Jr. (1995), William Vickrey (1996), Robert J. Aumann (2005), Thomas C. Schelling (2005), Roger B. Myerson (2007), Leonid Hurwicz (2007), Eric S. Maskin (2007).

After reading this chapter, readers should be able to: understand the nature of cooperative and non-cooperative decision problems; identify the players from a games problem and its feasible strategies; calculate the losses and gain associated with each combination of strategies; solve zero-sum games played by two people by applying MiniMax for pure strategies, the algebraic method for mixed strategies with 2×2 matrices, the Graph Method for $2 \times$ M and the general linear programming method for any dimension; obtain optimal solutions that provide the Games Theory for decision making in competitive situations.

8.2 Encounter in a Building

Two people find themselves in opposite corners of a building, A and B. Both are interested in meeting each other, so they start their way to one of the contiguous corners; but, what happens if their selection is wrong?



Evidently, if both move towards the same side, they do not meet, but they meet if they move towards the opposite sides. Therefore, if each player arbitrarily selects a move, the probability of meeting up is ½: on average, meeting up involves two moves, but there can be more.

One possible alternative is to keep still and wait until the other person comes to find us, but, what happens if the two people decide the same tactic?

To simplify the problem, let's take it as a game divided into times. Each time, the two players move simultaneously, and their "move" can be one of three possible moves:

- Move to the right (R)
- Move to the left (L)
- Remain still (O)

Let's consider that the two players meet each other if their movements take them to the same corner or in the two contiguous corners, so they can see each other. What is the best strategy to maximise the probabilities of meeting each other? Use the Games Theory to solve this problem.

(*Note* The linear programming models solution for this game is: $x_1 = x_2 = x_3 = {}^1\!/\!_2$; $z = 3/2yy1 = y_2 = y_3 = {}^1\!/\!_2$; $w = 3/2$.)

Solution

Payoff Matrix:

If both players move towards opposite sides, or if one remains still, the probability of them sighting each other is 100 %. However, if the two players apply the same tactic, the probability of finding each other is 0.

	A/B	q_1 R	q_2 L	q_3 O	MiniMax
p_1	R	0	1	1	0
p_2	L	1	0	1	0
p_3	O	1	1	0	0
	MiniMax	1	1	1	

This is a mixed 3 × 3 strategy game that we can solve by linear programming:

From Player A's viewpoint:	From Player B's viewpoint:
Min $z = X_1 + X_2 + X_3$	Max $w = Y_1 + Y_2 + Y_3$
Subject to:	Subject to:
$X_2 + X_3 \geq 1$	$Y_2 + Y_3 \leq 1$
$X_1 + X_3 \geq 1$	$Y_1 + Y_3 \leq 1$
$X_1 + X_2 \geq 1$	$Y_1 + Y_2 \leq 1$
$X_1, X_2, X_3 \geq 0$	$Y_1, Y_2, Y_3 \geq 0$

The solutions are:

$$X_1 = X_2 = X_3 = 0.5 \text{ and } z = 1.5 \rightarrow p_1 = p_2 = p_3 = 1/3 \text{ and } V = 2/3.$$
$$Y_1 = Y_2 = Y_3 = 0.5 \text{ and } w = 1.5 \rightarrow q_1 = q_2 = q_3 = 1/3 \text{ and } V = 2/3.$$

The best strategy for both players is to choose R, L, O in each move by drawing lots, with a 1/3 probability each. Under these conditions, the probability of finding each other is 2/3. Thus the "mean encounter time" improves to 1.5 moves.

8.3 Products Improvement Strategy Between Competing Firms

Two manufacturers, AdeFab1 and AdeFab2, compete for the sales of two different product lines which are equally profitable. In both cases, AdeFab2's sales volume triples that of AdeFab1. Due to some technological advances, both manufacturers can make substantial improvements to the two products, but they are not sure as to which development and commercialisation strategy they should adopt.

If the improvements to the two products are done at the same time, neither manufacturer can have them ready to sell under 12 months. One alternative is to carry out an "intensive programme" to first develop one of the two products and to try to commercialise it before the competition does. If this were the case, AdeFab2 can have one product ready to sell in 9 months, while AdeFab1 needs 10 months (due to previous commitments in its production installations). Either manufacturer can have the second product ready after another 9-month period.

For either product line, if both manufacturers commercialised the improved models simultaneously, it is estimated that AdeFab1 can increase the percentage of all the future sales of this product by 8 % of the total (from 25 % to 33 %). Likewise, AdeFab1 can increase its total sales in 20, 30 and 40 % of the total if it commercialises the product 2, 6 and 8 months before AdeFab2 does, respectively. Moreover, AdeFab1 can lose 4, 10, 12 and 14 % of the total if AdeFab2 manages to commercialise it 1, 3, 7 and 10 month(s) before it, respectively.

Formulate this problem as if it were a zero-sum game with two players and then establish the strategy the manufacturers must adopt according to the MiniMax criterion.

Solution

Payoff Matrix:

AdeFab2 AdeFab1	Simultaneous improvement Prod 1–12 months Prod 2–12 months	Intensive Programme Prod 1–9 months Prod 2–18 months	Intensive Programme Prod 1–18 months Prod 2–9 months	MiniMax AdeFab1	
Simultaneous improvement Prod 1–12 months Prod 2–12 months	33 + 33 = 66	15 + 55 = 70	55 + 15 = 70	66	←
Intensive Programme Prod 1–10 months Prod 2–19 months	45 + 13 = 58	21 + 21 = 42	65 + 11 =76	42	
Intensive Programme Prod 1–19 months Prod 2–10 months	13 + 45 = 58	11 + 65 = 76	21 + 21 =42	42	
MiniMax AdeFab2	66 ↑	76	76		

This is a Pure Strategy Game. The strategy they must follow is to carry out an improvement plan for the two products simultaneously. The game value is 66; that is, per product line, AdeFab1 will gain 33 % of all sales, which is an 8 % increase as opposed to the initial 25 %. Evidently, AdeFab2 will lose 8 % of all the sales of each product.

8.4 Strategies to Set Oil Prices

In the 1970s, the OPEP countries adopted a price-linking strategy and oil prices went from \$3/barrel to a level above \$30/barrel. Analysts foresaw that prices could rise to over \$100/barrel by the end of the twentieth century. Unexpectedly however, the cartel failed and prices went down again. Therefore, what are the conditions for these cartels to survive? In other words, what determines the equilibrium between cooperation and competition? This problem is similar to the *prisoner's dilemma*.

To simplify matters, it is assumed that only two producers exist, Saudi Arabia and Iran, and that there are only three possible production levels: 2, 4 and

6 million barrels/day. So, depending on the two players' decisions, total world production is 4, 6, 8, 10 and 12 million barrels/day. Accordingly, world prices are 35, 30, 25, 15 or only 10$/barrel. The production costs of one barrel are considered to be $2 for Saudi Arabia and $4 for Iran.

(a) The intention is to maximise the difference between the profit of a producer and that of its competitor. Consider and solve the case as if it were a zero-sum game with two players.
(b) If the idea is to seek OPEP's prime objective by avoiding mutual competition and regulating oil production, sales and prices, what is the best option for Saudi Arabia and for Iran if they cooperate with each other and try to simultaneously obtain the best profit each?

Solution

(a) The intention is to maximise the different between the profit of a producer and that of its competitor. Consider and solve the case as if it were a zero-sum game with two players.

Payoff Matrix:

Saudi Arabia/ Iran	2 million barrels/ day	4 million barrels/ day	6 million barrels/ day	MiniMax Approach
2 million barrels/day	4	−48	−80	−80
4 million barrels/day	60	8	−14	−14
6 million barrels/day	96	34	12	12
MiniMax Approach	96	34	12	

This is a Pure Strategy Game. Thus, even though the dominated strategies are eliminated and the MiniMax approach is used, the production level of 6 million barrels/day will always be more profitable for Saudi Arabia. When both players adopt a dominant strategy, which is a production of 6 million barrels/day in this case, the players gain 48 and 36 million dollars, respectively.

(b) If the idea is to seek OPEP's prime objective by avoiding mutual competition and regulating oil production, sales and prices, what is the best option for Saudi Arabia and for Iran if they cooperate with each other and try to simultaneously obtain the best profit each?

Saudi Arabia/Iran	2 million barrels/day	4 million barrels/day	6 million barrels/day
2 million barrels/day	66/62	56/104	46/126
4 million barrels/day	112/52	92/84	52/66
6 million barrels/day	138/42	78/44	48/36

The best option is that which offers the best profit to them both at the same time; that is, production of 4 million barrels/day, which implies profits of 92 and 84 million dollars for Saudi Arabia and Iran, respectively.

8.5 Playing Spoof

In a variant of the well-known game spoof, two players have two counters each. Each player hides one or two counters in his/her clenched fist at the same time, and guesses the total number of counters that both players will show. The player who rightly guesses the total number of counters shown by both players wins an amount in $ that equals the total number of counters that both players show. Otherwise the game ends in a draw. Tackle the problem as though it were a zero-sum game played between two people and solve it by linear programming, indicating the optimal strategies for each player.

(*Note* The solution of the linear programming models for this game is: $x_1 = 0$, $x_2 = 0.1429, x_3 = 0.1071, x_4 = 0, y, z = 0.25; y_1 = 0, y_2 = 0.1429, y_3 = 0.1071$, $y_4 = 0, y, w = 0.25$).

Solution

Payoff Matrix:

P1/P2	1–2	1–3	2–3	2–4
1–2	0	2	−3	0
1–3	−2	0	0	3
2–3	3	0	0	−4
2–4	0	−3	4	0
+4				
P1/P2	1–1	1–2	2–1	2–2
1–1	4	6	1	4
1–2	2	4	4	7
2–1	7	4	4	0
2–2	4	1	8	4

From Player 1's point of view:

$$
\begin{aligned}
&\text{Minimise}\, z = x_1 + x_2 + x_3 + x_4\\
&4x_1 + 2x_2 + 7x_3 + 4x_4 \geq 1\\
&6x_1 + 4x_2 + 4x_3 + x_4 \geq 1\\
&x_1 + 4x_2 + 4x_3 + 8x_4 \geq 1\\
&4x_1 + 7x_2 + 0x_3 + 4x_4 \geq 1\\
&x_1 = 0\\
&x_2 = 0.429\\
&x_3 = 0.1071\\
&x_4 = 0\\
&z = 0.25
\end{aligned}
$$

Then:

$$
\begin{aligned}
&V = 1/(x_1 + x_2 + x_3 + x_4) - 4 = 0\\
&P_1 = 0\\
&P_2 = 0.572\\
&P_3 = 0.428\\
&P_4 = 0
\end{aligned}
$$

From Player 2's point of view:

$$
\begin{aligned}
&\text{Maximise}\ w = y_1 + y_2 + y_3 + y_4\\
&4y_1 + 6y_2 + 1y_3 + 4y_4 \leq 1\\
&2y1 + 4y_2 + 4y_3 + 7y_4 \leq 1\\
&7y_1 + 4y_2 + 4y_3 + 0y_4 \leq 1\\
&4y_1 + 1y_2 + 8y_3 + 4y_4 \leq 1\\
&y_1 = 0\\
&y_2 = 0.1429\\
&y_3 = 0.1071\\
&y_4 = 0\\
&w = 0.25
\end{aligned}
$$

Then:

$$V = 1/(y_1 + y_2 + y_3 + y_4) - 4 = 0$$
$$q_1 = 0$$
$$q_2 = 0.572$$
$$q_3 = 0.428$$
$$q_4 = 0$$

Both players must play strategies (1, 3) and (2, 3) with probabilities 0.572 and 0.428, respectively.

8.6 Investments Strategy

An investor wishes to invest \$12,000 over a 1-year period, although he/she must decide to invest in market shares and/or gold. The profitability ratios estimated for each investment are based on the possible states of economy (see Table 8.1).

Consider a zero-sum Games Theory model with two players and answer the following:

(a) How should the investor invest the \$12,000 to carry out the optimal strategy?
(b) What mean profitability value is obtained?

Solution

(a) How should the investor invest the \$12,000 to carry out the optimal strategy?

The interpretation of this Games Theory problem is that the investor is playing against the states of nature (the states of economy).

The payoff matrix is the same table as the question posed. It is a 2×4 mixed game strategy. The graphic solution provides a value of $p = 0.33$ (Fig. 8.2).

Therefore, \$4,000 and \$8,000 should be invested in market shares and gold, respectively.

(b) What mean profitability value is obtained?

The optimal game value is 3. Therefore, a mean annual profitability ratio of 3 % is obtained, which is a mean annual value of \$360.

Table 8.1 Profitability of investments to face different states of economy

	Growth	Mean growth	Stabilisation	Poor growth
Market shares	5	4	3	−1
Gold	2	3	4	5

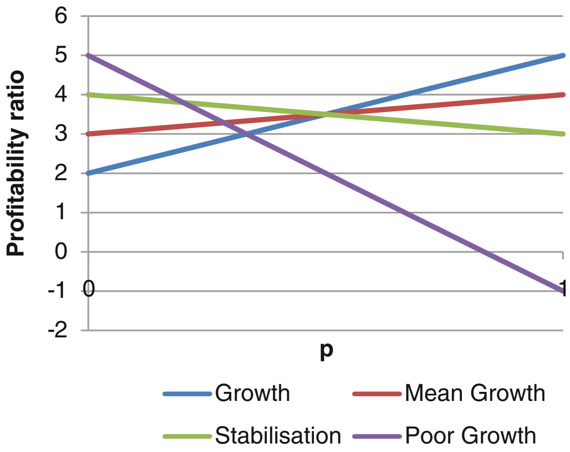


Fig. 8.2 Graphic solution

8.7 Strategy to Select Routes

Elisa is a Business Studies student at EPSA who lives in Bangor, so she has to travel everyday from Bangor to Warrington, and vice versa. Elisa can choose between travelling by one of these two routes: (A) via Chester on the A-55 inland dual carriageway, which passes Mold and Northwich; (B) going though the Wrexham Pass. Route A is a faster 3-lane road, whereas route B is a road with several bends and is exposed to winds. The police force is limited. If it places all its police force on either route, Elisa, who loves driving quickly, with no doubt be fined for speeding, which costs 100$. If the police force is divided 50–50 on both routes, there is a 50 % possibility of Elisa being fined 100$ on route A and 30 % possibility of receiving a similar fine on route B. Besides, route B costs 15$ more in petrol than route A. Devise a strategy for Elisa and for the police.

Solution

Payoff Matrix:

	Elisa/ Police	q1 FA	q2 FB	q3 FA-B
p1	A	−100	0	−50
p2	B	−15	−115	−45

This is a 2 × 3 mixed game strategy that can be solved by linear programming. From Elisa's viewpoint, the linear programming model would be:

$$\begin{aligned}&\text{Min } z = x_1 + x_2\\&15x_1 + 100x_2 \geq 1\\&115x_1 \geq 1\\&65x_1 + 70x_2 \geq 1\end{aligned}$$

The solution of this model is: $x_1 = x_2 = 1/115$ and $z = 2/115$. Thus: $p_1 = x_1/z = 0.5; p_2 = x_2/z = 0.5$. The game value is: $V = u - k = 1/z - k = 115/2 - 115 = -57.5$. Therefore, Elisa should follow the strategy of taking route A or B with a 50 % aleatory probability at a mean cost of 57.5$ as fines and differences in petrol costs.

From the police force's viewpoint:

$$\begin{aligned}&\text{Max } z = y_1 + y_2\\&15y_1 + 115y_2 + 65y_3 \leq 1\\&100y_1 + 70y_3 \leq 1\end{aligned}$$

The solution of this model gives: $y_1 = 1/100$; $y_2 = 17/2300$; $y_3 = 0$ and $z = 2/115$. Therefore, $q_1 = 0.575$, $q_2 = 0.425$ and $q_3 = 0$. Hence, the police strategy is to assign a 57.5 % aleatory probability of all its police force on route A, and a 42.5 % aleatory probability on route B, and the police force should never be divided between the two routes.

8.8 TV Ads Selection Strategy

Two competing mobile phone companies, Telefonita and Vodazone, are sponsors of Formula 1 motor-racing teams and motorcycle teams. For the next season, both firms are thinking of employing more sponsorship by using three 3-minute sponsorship-related ads inserted between the corresponding TV broadcasts of both competitions. Formula 1 is broadcast on TV5, while the motorcycle races are broadcast on Tele 1. Formula 1 has a larger audience, as reflected in its mean audience share of 49 % as opposed to 17 % for motorcycling.

Both firms are considering how to distribute the three ads. However, Telefonita has decided to insert at least one of the ads in each sports broadcast, while Vodazone has not self-imposed this condition. Telefonita and Vodazone have set the objective of maximising the publicity impact of their ads. Telefonita has established that the impact as a percentage of these TV spots can be calculated by the following formula, which agrees with the strategy adopted by both firms:

$$(na\_T\_F1 - na\_V\_F1) * Cpm\_F1 + (na\_T\_M - na\_V\_M) * Cpm\_M$$

where *na_T_F1* is the number of ads inserted by Telefonita on TV5 while the Formula 1 is being broadcast, *na_V_F1* is the number of ads inserted by Vodazone on TV5 during Formula 1, *na_T_M* e is the number of ads inserted by Telefonita on Tele 1 while motorcycling is being broadcast, and *na_V_M* s the number of ads inserted by Vodazone while the motorcycling races are being broadcast. *Cpm_F1* and *Cpm_M* represent the mean audience share for Formula 1 and for motorcycles, respectively. Vodazone also uses the same formulation.

(a) Consider this case as if it were a zero-sum game played by two people
(b) Solve the game. Obtain the game value; what does it mean?

Solution

(a) Consider this case as if it were a zero-sum problem played by two people

Payoff Matrix:

Telefonita/Vodazone	3F1-0M	0F1-3M	2F1-1M	1F1-2M	MiniMax Telefonita
2F1-1M	−0.32	0.64	0	0.32	−0.32
1F1-2M	−0.64	0.32	−0.32	0	−0.64
MiniMax Vodazone	−0.32	0.64	0	0.32	

(b) Solve the game. Obtain the game value; what does it mean?

Telefonita must insert two ads in the Formula 1 and one ad in the motorcycling races. Vodazone must insert its three ads in Formula 1. The game value is −0.32, which indicates that the Vodazone ads insertions have a 32 % publicity impact.

8.9 Advertising Campaign Design

Two firms commercialise two products that they both compete for. Each firm currently has 50 % of the market share. As the two products have been recently improved, both firms are planning to launch an advertising campaign. If neither firm advertises, their market share remains the same. If one of them launches a more powerful advertising campaign, the other loses a proportional percentage of its customers. Market research indicates that it is feasible to reach 50 % of potential customers by TV, 30 % through the press and 20 % on the radio.

(a) Take the problem as if it were a zero-sum Games Theory model played by two people. Indicate the most appropriate advertising means for both firms.
(b) Determine the game value. Can each firm operate with a single pure strategy? What does a pure solution strategy imply?

Solution

(a) Take the problem as if it were a zero-sum Games Theory model played by two people. Indicate the most appropriate advertising means for both firms.

Payoff Matrix:

	No campaign	TV	Press	Radio	MiniMax
No campaign	0.5	0	0.2	0.3	0
TV	1	0.5	1	1	0.5
Press	0.8	0	0.5	0.8	0
Radio	0.7	0	0.2	0.5	0
MiniMax	1	0.5	1	1	

The game solution is based on ensuring the best of the worst for each firm. The optimal game solution is to select strategies TV and TV; that is, both firms must advertise on television.

(b) Determine the game value. Can each firm operate with a single pure strategy? What does a pure solution strategy imply?

The result is the same for both firms, which continue with their 50 % market share. In this case, the game value is 50 % and the two firms adopt the pure strategy. Use of a pure strategy guarantees that neither firm is tempted to select a better strategy.

8.10 Playing Bluff

Two year-5 Business Studies students, Robert and George, are playing the amount foreseen for the end-of-career gala dinner in a game of Bluff. Robert writes a number on a piece of paper. Without showing George what he has written, he tells him what he wrote. Robert might be lying or telling the truth. Then George must guess if Robert is lying or telling the truth, or he can choose to pass and not to continue playing. If George discovers that Robert lied, Robert must pay George the $50 for the gala dinner. If George wrongly accuses Robert of lying, George pays him $25. If Robert told the truth and George guesses this, then Robert pays George $5. If Robert lied and George does not guess that he lied, then George pays Robert $24. Should George decide to pass, then neither player wins or loses anything.

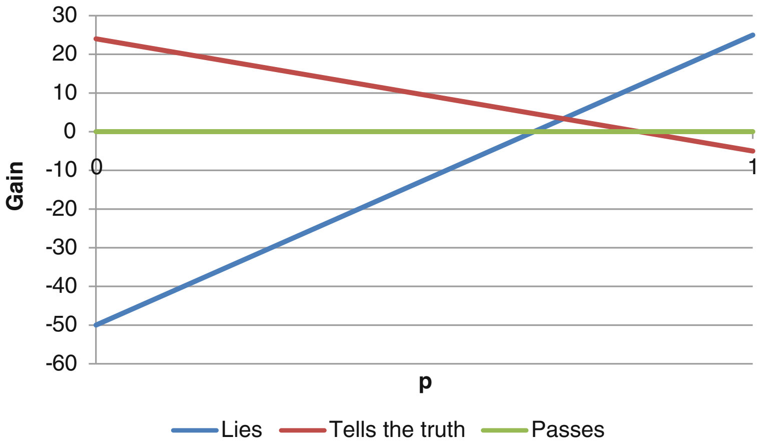


Fig. 8.3 Graphic solution

Determine the game value and the optimal strategies that Robert and George must adopt. (*Note* To determine George's optimal strategy, consider $Y_1 = 0$; $Y_2 = 0$; $Y_3 = 0.020$).

Solution

Payoff Matrix:

Robert/George	Says he lied	Says he told the truth	Pass
Lie	−50	24	0
Tell the truth	25	−5	0

This is a 2 × 3 game with non-dominated strategies. It is solved in a graphic form and a game value of $V = 0$ is obtained. Robert must opt for lying 33 % of the times and for telling the truth 67 % of the times (Fig. 8.3).

In George's case, a linear programming model must be used to determine the optimal strategy:

Add 50 to the original payoff matrix to eliminate the negative values from the initial table:

Robert/George	Lies	Tells the truth	Passes
Lies	0	74	50
Tells the truth	75	45	50

From George's viewpoint, the linear programming model is as follows:

$$\begin{aligned}
&\text{Maximise } z = Y_1 + Y_2 + Y_3\\
&\text{Subject to :}\\
&74Y_2 + 50Y_3 \le 1\\
&75Y_1 + 45Y_2 + 50Y_3 \le 1\\
&Y_1, Y_2, Y_3 \ge 0\\
&\text{By considering that } Y_1 = 0; Y_2 = 0; Y_3 = 0.020 :\\
&Y_1 + Y_2 + Y_3 = 1/u \to u = 50 \to V = 0;\\
&q_1 = Y_1\, u = 0\\
&q_2 = Y_2\, u = 0\\
&q_3 = Y_3\, u = 1
\end{aligned}$$

The optimal strategy for George is to always pass.

8.11 Security Measures to Avoid Snipers

The secret security services protecting the First Lady in the USA have received a warning from a sniper, who is posted in the surroundings of a public school where the First Lady is giving a speech. The security guards can choose from among five cars which drive one behind the other (1, 2, 3, 4 or 5) to transport the First Lady (Fig. 8.4).

The sniper will shoot only once from any of the four spots A, B, C or D. A shot would harm the First Lady if she travelled in a car next to the place where the bullet was shot. For instance, a bullet shot from point A would harm the First Lady if she travelled in car 1 or 2, while a bullet shot at point D would harm whoever is in car 4 or 5. Let us assume that the sniper would be offered award 1 if the First Lady was harmed and 0 if she was unhurt.

(a) By assuming that this is a zero-sum game, consider the payoff matrix from the sniper's viewpoint.
(b) Find and eliminate all the dominated strategies.
(c) We are informed that an optimal strategy for the security guards is that the First Lady travels one third of the times in cars 1, 3 and 5. We are also told that an optimal strategy for the sniper is to shoot one third of the times to point A, one third to D and one third to B or C. Determine the game value for the sniper.

Fig. 8.4 How the cars are arranged to avoid a sniper

Solution

(a) By assuming that this is a zero-sum game, consider the payoff matrix from the sniper's viewpoint.

	First Lady				
Sniper	Car 1	Car 2	Car 3	Car 4	Car 5
Point A	1	1	0	0	0
Point B	0	1	1	0	0
Point C	0	0	1	1	0
Point D	0	0	0	1	1

(b) Find and eliminate all the dominated strategies.
Columns 2 and 4 are dominated. That is, it is always better for the First Lady to travel in car 1 instead of in car 2, and in car 5 instead of in car 4.

(c) We are informed that an optimal strategy for the security guards is that the First Lady travels one third of the times in cars 1, 3 and 5. We are also told that an optimal strategy for the sniper is to shoot one third of the times to point A, one third to D and one third to B or C. Determine the game value for the sniper.

The value expected for the sniper is 1/3.

8.12 A Game Between Friends

Two players independently select a number: 1, 2 or 3. If their numbers are the same, Player 1 pays Player 2 this amount. If they are not the same number, Player 2 pays Player 1 an amount that equals the number Player 1 chose. Determine the game value and the optimal strategies that both players must follow. (*Note* to determine Player 1's optimal strategy, consider $X_1 = 0.1538$; $X_2 = 0.0769$; $X_3 = 0.0513$ and contemplate $Y_1 = 0.0641$; $Y_2 = 0.1026$; $Y_3 = 0.1154$) to determine Player 2's optimal strategy.

Solution

Payoff Matrix:

Player 1	Player 2			Best of the worst
	Select 1	Select 2	Select 3	
Select 1	−1	1	1	−1
Select 2	2	−2	2	−2
Select 3	3	3	−3	−3
Best of the worst	3	3	2	

It is a mixed strategy game and there are no dominances. It is a 3 × 3 game that can be solved by linear programming.

From Player 1's viewpoint, the linear programming model is as follows:

$$\begin{aligned}
&\text{Minimise } z = X_1 + X_2 + X_3\\
&\text{Subject to}\\
&2X_1 + 5X_2 + 6X_3 \geq 1\\
&4X_1 + X_2 + 6X_3 \geq 1\\
&4X_1 + 5X_2 \geq 1\\
&X_1, X_2, X_3 \geq 0
\end{aligned}$$

Bearing in mind that $X_1 = 0.1539;\ X_2 = 0.0769;\ X_3 = 0.0513$:
$X_1 + X_2 + X_3 = 1/u \rightarrow u = 3.5448$ →the game value is: $V = 0.5448$.

The optimal probabilities for Player 1 are:

$$\begin{aligned}
p_1 &= X_1\, u = 0.55\\
p_2 &= X_2\, u = 0.27\\
p_3 &= X_3\, u = 0.18
\end{aligned}$$

From Player 2's viewpoint, the linear programming model is as follows

$$\begin{aligned}
&\text{Maximise } w = Y_1 + Y_2 + Y_3\\
&\text{Subject to}\\
&2Y_1 + 4X_2 + 4X_3 \leq 1\\
&5X_1 + X_2 + 5X_3 \leq 1\\
&6X_1 + 6X_2 \leq 1\\
&Y_1, Y_2, Y_3 \geq 0
\end{aligned}$$

Bearing in mind that $Y_1 = 0.0641;\ Y_2 = 0.1026;\ Y_3 = 0.1154$:
$Y_1 + Y_2 + Y_3 = 1/u \rightarrow u = 3.5448$ →The game value is: $V = 0.5448$.

The optimal probabilities for Player 2 are:

$$p_1 = Y_1\,u = 0.23$$
$$p_2 = Y_2\,u = 0.36$$
$$p_3 = Y_3\,u = 0.41$$

8.13 Strategies to Improve the Market Share of Transport Companies

Two coach firms, A and B, operate the same route between two cities and set on striving for a larger market share. Firms A and B are considering the same three strategies to obtain a larger relative market share as follows:

(1) a1 or b1: Serving refreshments during the journey
(2) a2 or b2: Including DVD on coaches
(3) a3 or b3: Paying daily adverts on the local TV channels in both cities.

It is assumed that, at first, both firms are not making any special effort and that they share the same market share, 50 % each. It is also assumed that each firm cannot adopt more than one of these three strategies at the same time, and that all three strategies cost exactly the same.

It is estimated that if A and B served refreshments during the journey, A would lose 10 % of the market share to favour B, which might indicate that refreshments for firm B are more to the customers' liking. Should firm A serve refreshments and firm B include DVD on its coaches, the loss for A would be 1 point higher. However, if A served refreshments and B advertised on local TV, A would only lose 1 % of its market share.

If A included DVD on its coaches and B served refreshments, A would gain 9 % of the market share. Should both firms include DVD, firm B would gain 8 % of the market share. However, if A included DVD and B advertised, A would gain 6 % of the market share.

Likewise, if A advertised on TV and B served refreshments, it is assumed that A would gain 20 % of the market, which would be detrimental to B; obviously, TV ads seem to be more efficient than serving refreshments. Conversely, A would lose 10 % of the market if it advertised on TV and if B included DVD. Finally, if both firms advertised on TV, A would also lose a market share of 13 %.

Indicate the optimal strategy to be followed by both firms.

Solution

Payoff Matrix:

	b1	b2	b3
a1	−10	−11	−1
a2	9	−8	6
a3	20	−10	−13

It is a pure strategy zero-sum game with two players. Both firms should decide to include DVD on coaches with a game value of −8, and the market share for firm A would be 42 %.

8.14 Negotiation Between the Government and Transport Associations

The government is considering its negotiation strategy with transport associations after petrol prices having increased successively. The government can plan two strategies: negotiation or boycotting negotiation. The transport associations, in turn, can take an aggressive, conciliatory or passive attitude.

(a) Solve this game that is being considered for the government according to the payoff matrix indicated in Table 8.2.
(b) After following your advice in the previous section and having obtained optimal results, the government has decided to consult you about the strategy to adopt to negotiate with fishermen, who have the same rising fuel prices problem. Your colleague has developed a payoff matrix which could simulate the possible alternatives and consequences in accordance with the decision of both parties: the government and the fishermen. These data are provided in Table 8.3. What report must be presented to the government? Describe its strategy, the fishermen's strategy and the game value.

Solution

(a) Solve this game that is being considered for the government according to the payoff matrix indicated in Table 8.2.

It is a mixed strategy game and there are no dominances. By solving it graphically from the government's viewpoint, the government should opt for negotiating with an aleatory probability of 70 % and also for the boycotting negotiation with an aleatory probability of 30 %. The expected game value for the government is 0.1.

Table 8.2 The payoff matrix relating to the negotiations with transport associations

	Transport association			
Government		Aggressive	Conciliatory	Passive
	Negotiation	−2	1	2
	Boycott	5	−2	−3

Table 8.3 The payoff matrix relating to the negotiations with fishermen

	Fishermen				
Government		B1	B2	B3	B4
	A1	50	20	120	−50
	A2	60	20	70	70
	A3	−20	0	−40	75

(b) After following your advice in the previous section and having obtained optimal results, the government has decided to consult you about the strategy to adopt to negotiate with fishermen, who have the same rising fuel prices problem. Your colleague has developed a payoff matrix which could simulate the possible alternatives and consequences in accordance with the decision of both parties: the government and the fishermen. These data are provided in Table 8.3. What report must be presented to the government? Describe its strategy, the fishermen's strategy and the game value.

It is a pure strategy game. The optimal strategy for the government is A2, but it is B2 for the fishermen, with a game value of 20.

8.15 Selecting Locations to Open Fast-Food Restaurants

MacDanald, a fast-food restaurant franchiser, has decided to open two new outlets in Anglesey and Snowdonia (N Wales), with no MacDanald stores, which include three main towns, as Fig. 8.5 illustrates.

If one of the McDanalds is placed nearer to a main town than the other, it will gain 80 % of the market share. If they are located at an equal distance from a main town, they will both gain 50 % of the market; it they are built further away, the market share will be 30 %. The three towns are considered to be equally

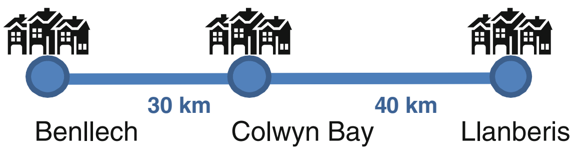


Fig. 8.5 Possible locations where the new restaurants are to be opened

important. MacDanald wishes to know which of the possible locations is the most suitable.

(a) Consider the problem as if it were a zero-sum game played by two people.
(b) Where should each fast-foot restaurant be placed if both restaurants had conflicting objectives; e.g., they belong to different owners?
(c) If both restaurants had the same owner, what locations should be chosen for each restaurant?

Solution

(a) Consider the problem as if it were a zero-sum game played by two people.

Payoff Matrix:

	Benllech	Colwyn Bay	Llanberis	Best of the worst
Benllech	1.5	1.4	1.9	1.4
Colwyn Bay	1.9	1.5	1.9	1.5
Llanberis	1.4	1.4	1.5	1.4
Best of the worst	1.9	1.5	1.9	

(b) Where should each fast-foot restaurant be placed if both restaurants had conflicting objectives; e.g., they belong to different owners?

This is a pure strategy game. The optimal strategy is to locate both fast-food restaurants in Colwyn Bay.

(c) If both restaurants had the same owner, what locations should be chosen for each restaurant?

The locations which offer the best market share for both restaurants are:

– One in Benllech and the other in Llanberis.
– One in Colwyn Bay and the other in Benllech.
– One in Colwyn Bay and the other in Llanberis.

8.16 Strategies for Davis Cup Tennis Heats

Next week, the Spanish team will play the Davis Cup (tennis) against the French team. The Spanish team has two players (Verdesco and Fele), while the French team has three players (Benneteeu, Llodre and Monfels). The following is know about the players' skills: Benneteeu has always beaten Fele; Llodre has always won Verdesco; Verdesco has always played better than Monfels. In any other match, each player has a probability of ½ of winning. The Spanish tennis team trainer has to decide who will play the first individuals match and who will play the second one. After selecting the two players for the individual matches, the

French tennis team trainer must decide who will play the first individual match and who will play the second one. As both trainers wish to maximise the expected number of individual matches won by their team, apply the Games Theory to determine optimal strategies and each team's game value.

Solution

Payoff Matrix:

	Verdesco	Fele	
Benneteeu	0.5	1	1.5
Llodre	1	0.5	1.5
Monfels	0	0.5	0.5

First of all, the French trainer should select Benneteeu and Llodre because they have more probabilities of winning.

Payoff Matrix:

	1st Benneteeu 2nd Llodre	1st Llodre 2nd Benneteeu	MiniMax
1st Verdesco 2nd Fele	0.5 + 0.5 =1	0 + 0 = 0	0
1st Fele 2nd Verdesco	0 + 0=0	0.5 + 0.5=1	0
MaxiMin	1	1	

It is a 2×2 mixed strategy game. Therefore, the optimal solution for the Spanish team is that the game has an aleatory probability of ½ and Verdesco plays first and then Fele afterwards. Then the game has an aleatory probability of ½, so Fele plays first, followed by Verdesco. The French team should opt for Benneteeu-Llodre with a probability of ½ and for Llodre-Benneteeu with a probability of ½. The game value is 0.5.

8.17 Developing a High-Tech Product

BIOSENS and TECHSENS, two competing firms, are developing a new state-of-the-art biomedical sensor, whose total sales will amount to 1,000 million dollars. Both must opt for using the most advanced "e1" or "e2" technology. Should both firms decide on the same technology, BIOSENS would obtain 60 % of sales as it knows how to sell its products better than its competitor, which sells similar products. However, if both opt for different technologies, then TECHSENS would obtain a 60 % market share because it exploits differentiation better.

BIOSENS is contemplating investing 20 million dollars in an advertising campaign which would permit it to extend its market share by five points should it select a different technology to its competitor.

TECHSENS is aware of BIOSENS' intentions, but does not intend to amend its promotion and sales policy, even though BIOSENS does.

(a) Consider this case as if it were a zero-sum game played by two people.
(b) Should BIOSENS launch an advertising campaign? Decide which strategy each firm should adopt
(c) Which firm will earn more? By how much?

Solution

(a) Consider this case as if it were a zero-sum game played by two people

Without an advertising campaign		TECHSENS	
		e1	e2
BIOSENS	e1	60	40
	e2	40	60

With an advertising campaign		TECHSENS	
		e1	e2
BIOSENS	e1	60	45
	e2	45	60

(b) Should BIOSENS launch an advertising campaign? Decide which strategy each firm should adopt.

Without an advertising campaign		TECHSENS		
		e1	e2	
BIOSENS	e1	60	40	40
	e2	40	60	40
		60	60	Not a pure strategy

With an advertising campaign		TECHSENS			
		e1	e2		
BIOSENS	e1	60	40	p	0.5
	e2	40	60	$1-p$	0.5
		q	$1-q$	50	
		0.5	0.5		

Therefore both BIOSENS and TECHSENS will earn 500 million dollars

With an advertising campaign		TECHSENS		
		e1	e2	
BIOSENS	e1	60	45	45
	e2	45	60	45
		60	60	Not a pure strategy

With an advertising campaign		TECHSENS			
		e1	e2		
BIOSENS	e1	60	45	p	0.5
	e2	45	60	$1-p$	0.5
		q	$1-q$	52.5	
		0.5	0.5		

Therefore, BIOSENS will earn 525 million dollars and TECHSENS will earn 475 million dollars. Thus, an advertising campaign must be launched. BIOSENS will earn 25 million more dollars if it invests 20 million dollars in an advertising campaign, which implies an additional profit of 5 million dollars.

(c) Which firm will earn more? By how much?

BIOSENS will earn 50 million more dollars than TECHSENS if it launches an advertising campaign.

8.18 Electoral Strategies in Local Elections

Two regional minority political parties, X and Z, are devising their strategies for the forthcoming local elections. Both have a similar number of votes in a given region. This region is made up of two provinces, A and B, of a similar size and population, which have always been disputing about the rights to the water from a nearby main river which crosses both provinces. Basically, both parties can decide to favour one of the two provinces with its electoral promises about this dispute or it can remain neutral. The campaign directors of both parties have reached the conclusion that if one party favours one the of provinces and the other remains neutral, the former (that which favours one province) will win 70 % of the votes disputed between the two parties. Meanwhile, if one party favours province A and the other favours province B, the former (that which favours province A) will win 60 % of the votes because the inhabitants of A are more aware of the problem.

(a) Contemplate this case as if it were a zero-sum game with two players
(b) Determine which strategy that each political party must follow
(c) Which party will win more votes? By how many?

Solution

(a) Contemplate this case as if it were a zero-sum game with two players

By assuming that the winnings of the player of the rows represent the percentage of votes that he would obtain:

		Party Y		
		FA	FB	N
Party X	FA	50	60	70
	FB	40	50	70
	N	30	30	50

(b) Determine which strategy that each political party must follow

		Party Y			
		FA	FB	N	
Party X	FA	50	60	70	50
	FB	40	50	70	40
	N	30	30	50	30
		50	60	70	

This is a pure strategy game, and both parties should adopt the FA strategy (which favours Province A)

(c) Which party will win more votes? By how many?

There is a draw between both parties, so both will obtain 50 % of the votes.

8.19 Strategies to Develop 3D Television Sets

TONY and TANSUM, two rival firms, are developing new 3D television technology. Both can opt to use a conventional strategy to develop 3D television sets which require special glasses for correct viewing, or they can decide on an innovative strategy to develop a technology that does not require special glasses at all.

The total profits both firms are striving for come to 6,000 million dollars over the next year, when the new TV set models will be available to sell. If both firms opt to develop a technology that requires special glasses, they will share the market share equally. If both decide on the no-glasses technology, TONY will earn 4,800 million dollars as it betters TANSUM's technical specifications. However if TONY opts for the conventional technology while its competitor decides on the innovative strategy, the market will award TANSUM as it will obtain 4.200 million dollars. The strange thing is that if the reverse occurs, TONY will earn only 1,800 million dollars as TANSUM will offer more advanced technology with glasses, so it will offer 3D televisions at today's prices.

(a) Consider this case as if it were a zero-sum game played by two people
(b) Determine which strategy each firm should adopt
(c) Which firm will do better? By how much?

Solution

(a) Consider this case as if it were a zero-sum game played by two people

The percentage of rows for the total earnings of 6,000 million dollars:

		TANSUM	
		ConG	SinG
TONY	ConG	50	30
	SinG	30	80

(b) Determine which strategy each firm should adopt

		TANSUM			
		ConG	SinG		
TONY	ConG	50	30	30	
	SinG	30	80	30	
		50	80	Not a pure strategy	
	ConG	50	30	p	0.714
	SinG	30	80	$1-p$	0.286
		q	$1-q$	44.286	
		0.714	0.286		

Both will gain more weight with the conventional strategy (ConG)

(b) Which firm will do better? By how much?

TONY will obtain 44.3 % of the market = 2,657.14 million dollars
TANSUM will obtain 55.7 % of the market = 3,342.86 million dollars.

8.20 Tennis Playing Strategies

Victor and Anne are playing tennis. Victor only plays two types of service: he serves very close to the central line, or far beyond it and right up to the outer line, but both are sweeping services. When Victor is about to serve, Anne can move to either side (towards the central line or the outer line), or she can remain at an equal distance to both lines to react at the last moment.

Of the six times that Victor has served very close to the central line, and Anne has moved towards the central line, Victor has won three points with an ace. Of the four times Victor has served very close to the central line and Anne has moved to the outer line, Victor has won 1 point with an ace. Of the 10 times that Victor has served close to the central line and Anne has remained in the centre to guess his move, Anne has managed to break three of his services.

Of the 10 times Victor has served very close to the outer line and Anne has moved to the central line, Anne was managed to break only one of his services. Of the 10 times Victor has served very close to the outer line and Anne has moved to the outer line, Anne has managed to break three of his services. Finally of the five times Victor has served very close to the outer line and Anne has remained on the central line to guess his move, Victor has won four points with an ace.

(a) Consider this case as a zero-sum game with two players
(b) Determine the strategies to follow
(c) Given the results, should Victor change his proportion of services?

Solution

(a) Consider this case as a zero-sum game with two players

	Move to central line	Wait	Move to outer line
Central service	0.5	0.7	1
Outer service	0.9	0.8	0.7

(b) Determine the strategies to follow

	Move to central line	Wait	Move to outer line	Min	MaxiMin
Central service	0.5	0.7	1	0.5	0.7
Outer service	0.9	0.8	0.7	0.7	0.7
	0.9	0.8	1		
	0.8	0.8	0.8		

Mixed strategy, which can be solved by the Graph Method (Fig. 8.6)
With the graphic solution, $p = 1/4$
The game value is 0.775.

(c) Given the results, should Victor change his proportion of services?

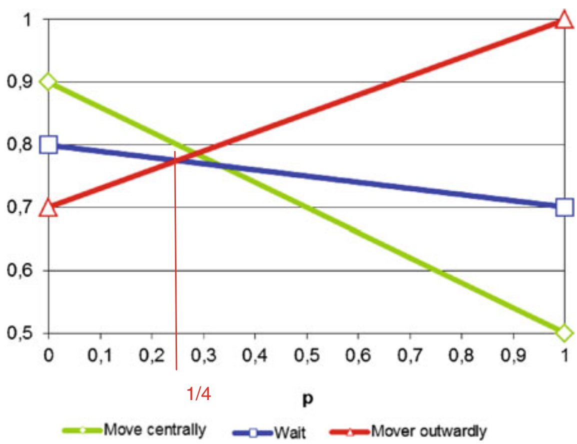
Currently, 44.4 % of Victor's services come close to the central line.
However, Victor must serve this way for 25 % of his services if he wishes to maximise the number of points obtained by aces.

8.21 An Interactive TV Services Market

Two firms TVint and tuTV are the only firms competing in a national market which offers an interactive television service. In the present-day, their market share is 40 and 60 %, respectively. The users can change firm when they wish to by means of an interactive menu.

TVint is considering launching an advertising campaign to increase its market share. If it decides to go ahead with it, the media it may use are: TV, radio or the Internet. TVint has estimated that if launches its advertising campaign on TV, this

Fig. 8.6 Graphic solution



will reach half the tuTV customers and will convince 60 % of them to change company. If it advertises on the radio, it will reach 10 % of tuTV customers and will convince 30 % of them. If it decides to advertise on the Internet, it will reach 20 % of tuTV customers and will convince 70 %.

(a) What conditions must be set up so that TVint can make its decision by considering the problem as if it were a zero-sum game played by two people? Assume that these conditions are viable and consider the problem.
(b) What strategy should TVint adopt?
(c) What additional proportion of the market share will TVint obtain?

Solution

(a) What conditions must be set up so that TVint can make its decision by considering the problem as if it were a zero-sum game played by two people? Assume that these conditions are viable and consider the problem.

In order to consider this a zero-sum problem with two people playing, Tvint must assume that tuTV is considering making the same decision. If it does not acquire further information, it must assume that the audience and the impact that tuTV will achieve will be similar to its own.

		tuTV			
		No	TV	Radio	Internet
Tvint	No	0.4	0.28	0.388	0.344
	TV	0.58	0.46	0.568	0.524
	Radio	0.418	0.298	0.406	0.362
	Internet	0.484	0.364	0.472	0.428

(a) What strategy should TVint adopt?

		tuTV				
		No	TV	Radio	Internet	
Tvint	No	0.4	0.28	0.388	0.344	0.28
	TV	0.58	0.46	0.568	0.524	0.46
	Radio	0.418	0.298	0.406	0.362	0.298
	Internet	0.484	0.364	0.472	0.428	0.364
		0.58	0.46	0.568	0.524	

It is a pure strategy zero-sum game for two players.

TVint will decide to launch an advertising campaign on TV, and tuTV will do the same.

(b) What additional proportion of the market share will TVint obtain?

TVint's market share will increase by 6 %.

8.22 A Security Firm's Service Strategy to Avoid Holdups

The security firm SEGBANK is dedicated to transporting large sums of money between banks. For the purpose of maximising the probabilities of saving the money, it transports if hypothetically faced with a holdup, its transport service must always use two armoured vans because bank robbers tend to centre on one van. Thus, one of the vans transports the money, while the other acts as bait. Of the two armoured vehicles, that which drives ahead is always better armed as the probabilities of the first van resisting holdups by bank robbers are estimated to be 90 %, and 70 % for the van driving behind. One of SEGBANK's employees has filtered this information to a band of bank robbers, which is planning to attack the next assignment.

(a) Consider this problem as a zero-sum problem for two players
(b) What strategy should SEGBANK take? And the robbers?
(c) What probability does SEGBANK have of saving the money from the holdup?

Solution

(a) Consider this problem as a zero-sum problem for two players

	1st van	2nd van
Money-Bait	0.9	1
Bait-Money	1	0.7

(b) What strategy should SEGBANK take? And the robbers?

	q 1st van	$1-q$ 2nd van	
Money-Bait	0.9	1	0.9
Bait-Money	1	0.7	0.7
	1	1	

Hence

$$\begin{aligned} 0.9 \cdot p + 1.0 \cdot (1-p) &= 1.0 \cdot p + 0.7 \cdot (1-p) \quad p = 0.75 \\ 0.9 \cdot q + 1.0 \cdot (1-q) &= 1.0 \cdot q + 0.7 \cdot (1-q) \quad q = 0.75 \\ & \qquad\qquad\qquad\qquad\qquad\quad\; v = 0.925 \end{aligned}$$

SEGBANK will position the van with the money in front with a probability of 0.75.

The bank robbers will attack the first van with a probability of 0.75.

(c) What probability does SEGBANK have of saving the money from the holdup?

The probability of SEGBANK saving the money from the holdup is 0.925.

8.23 Strategies to Improve the Market Share of Airlines

Two low-cost airlines, A and B, operate three flights per day between Madrid and Paris. Both airlines have an equal market share and they strive to improve it. The possible strategies include:

- e1: Serving refreshments free of charge
- e2: Offering a premiere film free of charge
- e3: Increasing their advertising efforts.

It is estimated that if A offers free refreshments and B follows suit, A will lose 10 % of the market share to favour B. If B offers a premiere film, the loss for A will be one point greater. However, if B makes more advertising efforts, A will only lose 1 % of its market share.

If A offers a premier film and B serves free refreshments, A will gain 9 % of the market share. If both airlines opt to offer a free premiere film, B will gain 8 % of the market share. However, if B decides to make more publicity efforts, A will gain 6 %.

Similarly, if A advertises on TV and B serves free refreshments, A is assumed to gain 20 % of the market to the detriment of B. Conversely, A will lose 10 % if B decides to offer the premiere film. Finally, if both airlines advertise on TV, A will also lose 13 % of the market share.

(a) What hypotheses must be fulfilled to solve this problem as a zero-sum game for two players? Consider this problem as a zero-sum problem for two players.
(b) Solve the problem.

Solution

(a) What hypotheses must be fulfilled to solve this problem as a zero-sum game for two players? Consider this problem as a zero-sum problem for two players.

The player's earning must equal the opponent's loss. Both players must have the same information to construct the same payoff matrix.

By assuming the increased market share percentage as payments, over the current 50 %, for the player of the rows we obtain:

		B		
		e1	e2	e3
A	e1	−10	−11	−1
	e2	9	−8	6
	e3	20	−10	−13

(b) Solve the problem

		B			
		e1	e2	e3	
A	e1	−10	−11	−1	−11
	e2	9	−8	6	−8
	e3	20	−10	−13	−13
		20	−8	6	

It is a pure strategy game, and both airlines must select strategy e2. Airline A will lose 8 % of the market share, which airline B will acquire.

8.24 Paying and Avoiding Tax

A firm has two companies, A and B, which must pay Inland Revenue an average of 4 and 12 million dollars a year, respectively. For the two companies, the firm can declare its real income and pay the corresponding tax, or it can manipulate its accountancy and avoid paying tax. Inland Revenue's Inspection Service only has the means to inspect one company every year. If it inspects one company with false income, it will discover fraud, and if the firm has to pay the corresponding tax, plus a fine, which will double the amount evaded.

(a) Consider this case as if it were a zero-sum game for two players
(b) Determine which strategy Inland Revenue should adopt and that which the firm should follow

Solution

(a) Consider this case as if it were a zero-sum game for two players

	rA = declare A's real accountancy fA = manipulate A's accountancy rArB	rAfB	fArB	fAfB
Investigate A	16	4	24	12
Investigate B	16	40	12	36

(b) Determine which strategy Inland Revenue should adopt and that which the firm should follow (Fig. 8.7)

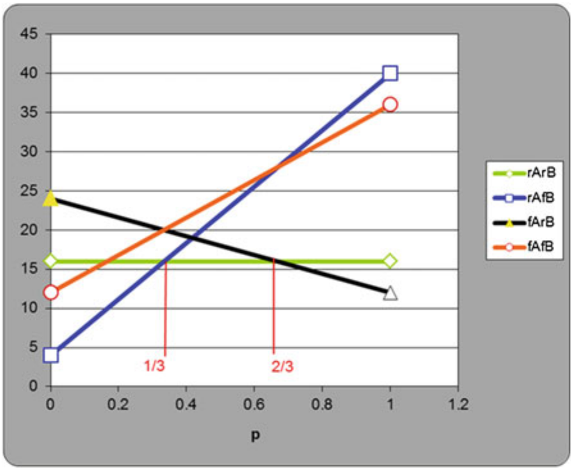
	rArB	rAfB	fArB	fAfB	Min
Investigate A	16	4	24	12	4
Investigate B	16	40	12	36	12
Max	16	40	24	36	

For the graphic solution p = [1/3, 2/3]

Besides strategy fAfB, it is also dominated by the other three.

By eliminating this strategy, this situation is as so:

Fig. 8.7 Graphic solution



	rArB	rAfB	fArB
Investigate A	16	4	24
Investigate B	16	40	12

The game value is 16, therefore it is always convenient for the firm to select strategy rArB.

8.25 Electoral Campaign Strategies

The two main political parties of a given country are considering what their strategies should be for the next European elections. Both parties are debating if they should centre their campaigns to include economic or social matters. Table 8.4 provides the zero-sum game matrix, which establishes the percentage points that favour party A per combination of strategies.

A campaign advisor of party B has proposed two other strategies for this party: an overseas policy and security. With the first policy, party B can obtain 2 % points in its favour and can lose 3 % points depending on the strategy selected by A. However, the security campaign strategy can likewise lose 2 % points or can end in a draw. When considering the new game matrix, the advisor realises that there are no dominances.

(a) Consider this case as if it were a zero-sum game played by two people. What hypothesis must be fulfilled for it to operate like a zero-sum problem played by two people?
(b) Determine the strategy that party A should adopt and the difference encountered with that obtained with the original matrix
(c) Should player B rule out any of his/her strategies?

Solution

(a) Consider this case as if it were a zero-sum game played by two people. What hypothesis must be fulfilled for it to operate like a zero-sum problem played by two people?

Table 8.4 Combination of strategies

A	B	
	Economic	Social
Economic	2	−1
Social	−1	4

Party A must have the same information as party B (including the advisor's recommendations)

B	Economic	Social	Overseas policy	Security
A	r1	r2	r3	r4
Economic	2	−1	3	0
Social	−1	4	−2	2

(b) Determine the strategy that party A should adopt and the difference encountered with that obtained with the original matrix

It is not a pure strategy. When applying the Graph Method, we obtain: (Fig. 8.8)

Party A chooses Econ with a probability of $p = 0.6$

Party A chooses Social with a probability of $(1-p) = 0.4$

Game value = 0.8

The original matrix is:

	B	
A	Economic	Social
Economic	2	−1
Social	−1	4

It is a mixed strategy. By applying the formulation, we obtain:

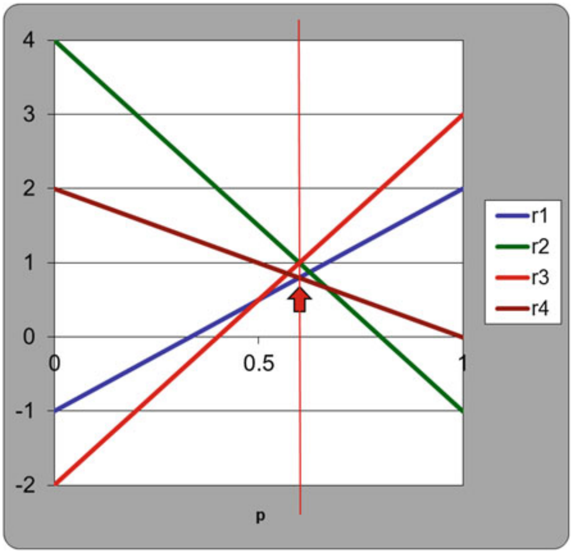


Fig. 8.8 Graphic solution

$$V = 2p + (1 - p) \cdot (-1)$$
$$V = -1p + (1 - p) \cdot 4$$
$$V = 2q + (1 - q) \cdot (-1)$$
$$V = -1q + (1 - q) \cdot 4$$

Party A chooses 2 with a probability of $p = 0.625$
Party A chooses -1 with a probability of $(1-p) = 0.375$
Party B chooses -1 with a probability of $q = 0.625$
Party B chooses 4 with a probability of $(1-q) = 0.375$
The game value is = 0.875
The difference is 0.075.

(c) Should player B rule out any of his/her strategies?

No because they all intervene in the configuration, as the Graph Method shows.

8.26 World Robots Championship

Peter and James are the finalists of the *Robot Wars Championship*, the most famous world robots championships. Peter has two robots: TORNADO and PANZER, while James is the owner of RAER and MANT.

Both must decide which robots they should enter the final with. These robots are characterised mainly by their powerful attack or their defence, and there are three qualities: A = Overturn; B = Collision; C = Destruction.

Based on various competitions that the robots have competed in, there is a classification which indicates the power of the attack or the quality of the defence according to these three characteristics, with a maximum of five points. The classification of these four robots is detailed in Table 8.5 where Aa is power of attack in Turnover, and Ad is quality of defence to face Turnover, etc.

When two robots face each other, the formula which determines *a priori* and approximately the probability of one robot (1) beating the other (2), is:

$$\frac{(Aa_1/Ad_2 + Ba_1/Bd_2 + Ca_1/Cd_2)}{(Aa_2/Ad_1 + Ba_2/Bd_1 + Ca_2/Cd_1)}/2$$

(a) Consider this case as a zero-sum game with two players to maximise the probability of winning

Table 8.5 Classification of the robots according to their characteristics

	Aa	Ad	Ba	Bd	Ca	Cd
TORNADO	4.2	3.7	3.9	4.1	4.5	3.5
PATable NZER	3.4	3.6	4.1	4.7	3.9	4.2
RAER	3.9	4.2	4	3.7	4.6	4.1
MANT	4.2	3.9	4.1	4.1	3.8	4.6

(b) Determine the strategies that each player must follow
(c) Which player has a greater probability of winning?

Solution

(a) Consider this case as a zero-sum game with two players to maximise the probability of winning

By applying the formula to calculate the probability of the robot of the player of the rows winning, we obtain:

	RB1	RB2
RA1	0.471	0.467
RA2	0.473	0.462

(b) Determine the strategies that each player must follow

By applying MiniMax, we obtain:

	RB1	RB2	
RA1	0.471	0.467	0.467
RA2	0.473	0.462	0.462
	0.473	0.467	

It is a pure strategy game: the player of the rows (Peter) will select RA1 (TORNADO) and the player of the columns (James) will opt for RB2 (MANT)

(c) Which player has a greater probability of winning?

The player of the columns (James) with a probability of $1-0.467 = 0.533$.

8.27 Strategies to Increase the Market Share of Travelling Vendors

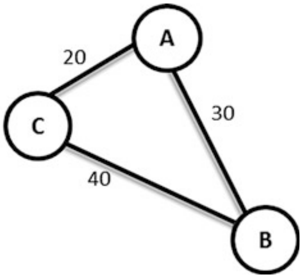
Lewis and Raymond sell pictures in street markets at weekends. Every Saturday, Lewis must select between towns A and B, while Raymond can choose A, B and also C.

Figure 8.9 depicts the locations of the three towns and the distance from each other in kilometres.

Towns A, B and C have 36,000, 38,000 and 40,000 inhabitants, respectively.

Of the inhabitants in the town where Lewis or Raymond (or both) travelled to one Saturday, 1 % purchased pictures. If on one Saturday neither vendor has been in one

Fig. 8.9 Location of towns and distances



of the towns, 0.5 % of the inhabitants travel to the town where Lewis or Raymond is selling. If Lewis and Raymond have chosen the same town, all the interested inhabitants travel there. However, if they decide to sell in different towns, the inhabitants of this town will divide inversely proportional to the distance from their town to the other two. When those visiting the market find only one travelling vendor, this vendor takes all the sales, whereas if the vendors coincide, 60 % of the customers prefer Lewis because they are more familiar with him.

Lewis and Raymond want to increase their market share because the greater it is, the more possibilities there are of increasing it.

(a) Consider this case as if it were a zero-sum game played by two people
(b) Solve the game.

Solution

(a) Consider this case as if it were a zero-sum game played by two people

		Raymond		
	All the customers	A	B	C
Lewis	A	750	940	950
	B	940	760	960
		Raymond		
	Customers for Lewis	A	B	C
Lewis	A	450	493.333	468.571
	B	446.667	456	452
		Raymond		
	Customers for Raymond	A	B	C
Lewis	A	300	446.667	481.429
	B	493.333	304	508
		Raymond		
	Percentage of market for Lewis	A (%)	B (%)	C (%)
Lewis	A	60.0	52.5	49.3
	B	47.5	60.0	47.1

This last market percentage matrix for Lewis will be the game matrix.

(b) Solve the game

		Raymond		
	Percentage of market for Lewis	A (%)	B (%)	C (%)
Lewis	A	60.0	52.5	49.3
	B	47.5	60.0	47.1

Dominances

Solution: Lewis A, Raymond C. The Game value is 49.3 % of the market share for Lewis

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Chapter 9
Dynamic Programming

Abstract This chapter begins with an introduction to dynamic programming, it describes the typology of the problems, which can be divided into subproblems, to be solved by dynamic programming and it explains the formulation to employ for modelling, which focuses on determining the recursive function. Then it proposes a varied set of dynamic programming problems and provides their corresponding solutions. The object of this chapter is to provide a better understanding of modelling multiphase complex problems by means of dynamic programming. Problems are put forward in which the phase, stage, decision, recursive function and the transition function should be defined to then go on to solve the problem to obtain the optimal solution.

9.1 Introduction

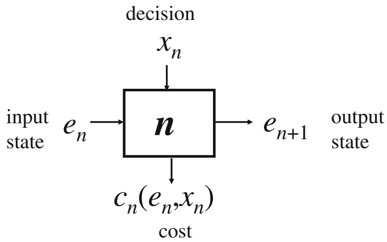
Dynamic programming is a method that can be applied to solve complex problems (e.g. combinatorics problems) by breaking them down into several simpler problems. The problems it is applied to must have the overlapping subproblems characteristic and an optimal substructure. The solving strategy is based first on solving the simplest subproblem in such a way that the solution to this subproblem helps solve the next subproblem, and so on until all the subproblems have been solved and the solution to the complete problem has been obtained.

The term dynamic programming was originally coined by Richard Bellman in the 1940s (Bellman 1954, 1957) to describe the problem solving process in which each problem required the solution of another problem. The Bellman equation provides modelling according to a recursive relation.

The problems that can be solved by dynamic programming have the following characteristics: (i) they can be divided into decision stages; (ii) each stage has a number of states associated with it; (iii) the decision made in a stage transforms an input state into an output state; (iv) given the current stage, the optimal solution for the following stages does not depend on the decisions made in former stages;

R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5_9,

Fig. 9.1 Structure and nomenclature in a dynamic programming problem stage



(v) there is a recursive relation which can identify the optimal decision for a stage once the following stage has been solved; (vi) the last stage must be solved without having to depend on former stages.

A dynamic programming problem can be broken down into a problem with N decision stages. Figure 9.1 depicts the structure of any n stage in which, given an e_n input state, if the decision made is x_n, then the output state is e_{n+1} that generates a cost (or profit) c_n which depends on both the decision and the input state. Figure 9.2 illustrates the N stages sequence of a dynamic programming problem.

The recursive function of a dynamic programming problem allows us to calculate the accumulated cost for an n from stage n to N in association with making decision x_n when we are in state e_n, which is the sum of the lowest accumulated cost from stage $n + 1$ to stage N, plus the cost associated with making this decision x_n in state e_n:

$$f_n(e_n,\ x_n) = f^*_{n+1}(e_{n+1}) + c_n(e_n,\ x_n)$$

According to this recursive relation, if $f^*_{n+1}(e_{n+1})$ were a datum, the subproblem of stage n is reduced to calculate the costs associated with the various x_n decisions that can be made for each possible input state e_n; that is $c_n(e_n,\ x_n)$ which, when added to the optimal costs from the next state to the final $f^*_{n+1}(e_{n+1})$, will provide us with the accumulated costs from stage n to the end, assuming that we know the optimal decisions from stage $n + 1$ to N. Thus, the subproblem from stage n is solved by calculating the minimum (maximum if we are dealing with profits) cost:

$$f^*_n(e_n) = \text{Min}\,\{f_n(e_n,\ x_n)\}$$

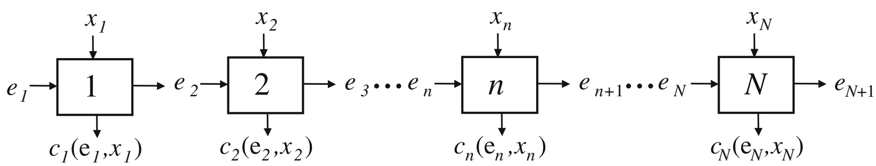


Fig. 9.2 Stages sequence of a dynamic programming problem

Therefore the optimal cost of stage n serves to solve the stage $n-1$ subproblem. The solution to all the stages depends on it being possible to solve the last stage, whose solution is simpler as there are no following stages to consider. In this way, the steps to take to obtain the optimal cost of stage 1 $f_1^*(e_1)$ are:

$$
\begin{aligned}
&f_N^*(e_N) = \mathrm{Min}\{c_N(e_N,\ x_N)\}\\
&f_{N-1}^*(e_{N-1}) = \mathrm{Min}\{f_N^*(e_N) + c_{N-1}(e_{N-1},\ x_{N-1})\}\\
&\ldots\\
&f_n^*(e_n) = \mathrm{Min}\{f_{n+1}^*(e_{n+1}) + c_n(e_n,\ x_n)\}\\
&\ldots\\
&f_2^*(e_2) = \mathrm{Min}\{f_3^*(e_3) + c_2(e_2,\ x_2)\}\\
&f_1^*(e_1) = \mathrm{Min}\{f_2^*(e_2) + c_1(e_1,\ x_1)\}
\end{aligned}
$$

The transformation of an input state into an output state in accordance with the decision made is calculated by a transition function which depends on each problem type:

$$e_{n+1} = e_n \otimes x_n$$

On occasion, this function is a simple addition or subtraction, while it becomes more complex at other times. By applying this transition function, the calculation of the optimal cost of stage n is as follows:

$$f_n^*(e_n) = \mathrm{Min}\ \{f_{n+1}^*(e_n \otimes x_n) + c_n(e_n,\ x_n)\}$$

Hence the optimisation of the stage n subproblem depends exclusively on the input state and the decision to be made because $f_{n+1}^*(e_n \otimes x_n)$ is a datum obtained after having calculated the transition function. This optimisation subproblem can be as simple as a complete enumeration or as complicated as a linear programming problem.

The input states of each e_n stage can take a finite set of values. Nonetheless, the input state of stage 1, e_1, must take a single value as it is the starting point of the problem. Therefore, the optimal solution of the stage 1 subproblem is single and it determines the optional solutions for the following stages.

After obtaining the accumulated cost of stage 1 $f_1^*(e_1)$, this value provides one part of the solution: it is the lowest cost (or highest profit) that can be accomplished with an optimal decision-making policy $\{x_1^*, x_2^*, \ldots x_N^*\}$. The other part of the solution is, precisely, this optimal decisions vector. To obtain it, we can start by finding out the optimal decision in stage 1, x_1^*, for the single value of input state e_1. Next we calculate the input state of stage 2 using the transition function:

$$e_2 = e_1 \otimes x_1^*$$

For this stage 2 input value, e_2, there is an optimal decision, x_2^*, which leads to the optimal accumulated cost of stage 2 $f_2^*(e_2)$. Successively, the optimal decisions are obtained for each stage until final stage N. Note that in each stage, optimal decision x_n^* could be a set of values (all of which lead to the same final cost). In this case, there would be several optimal solution vectors $\{x_n^*\}$ which are differentiated in each bifurcation in the stages in which x_n^* has more than one value, which provide the same final cost. In this case, the problem has multiple solutions.

The recursive function concept does not mean that the solution to a dynamic programming problem has to be recursive from the solution algorithm programming point of view as this would imply the calculations to exponentially grow, which would eliminate the interest of dividing the overall problem into subproblems since no calculations would be saved. Conversely, the dynamic programming approach seeks to solve each subproblem only once: when the solution to a given subproblem has been calculated, it is stored to be simply looked up the next time the same solution is needed. This procedure reduces the complexity of the solution linearly the more stages there are instead of growing exponentially.

Despite there being no standard procedure to model all the problems that can be solved by dynamic programming, there are some well-studied problems; for example:

- The knapsack problem: in this problem, the knapsack (which can be a container, a truck, a plane, etc.) has a certain capacity, and each item that can go into the knapsack has a size and a benefit. The problem is to determine what should go into the knapsack to maximise the total benefit.
- The travelling salesperson problem: in this problem a salesperson must visit a number of cities with a minimum distance or cost. The alternatives are which of the next cities he/she travels to when located in a given city. The cities are separated by distances, time or cost, which must be minimised.
- The equipment replacement problem: in this problem, a given piece of equipment (a machine, a car, etc.) is to be used during successive time periods (months, years, etc.). This equipment has its corresponding running and maintenance costs, which may increase with time. As the equipment becomes older, and its running costs go up, it must be replaced after a given time. For this replacement, a new piece of equipment is purchased (one that is at least not as old as the current one) and the current one is sold, resulting in a net cost for the purchasing cost of a new piece of equipment minus the sale price of the current equipment. The problem lies in determining when it is best to replace equipment.

There are numerous algorithms which employ dynamic programming to solve problems in different science and business areas: Backward induction (Aumann 1995), attice models for protein-DNA binding (Lengauer 1993), the Cocke–Younger–Kasami algorithm (Cocke and Schwartz 1970), (Kasami 1965), (Younger 1967), the Viterbi algorithm (Viterbi 1967), the Earley algorithm (Earley 1970), the Needleman–Wunsch algorithm (Needleman and Wunsch 1970), etc.

The model that the Introduction describes is a deterministic dynamic programming model in which, given a state and a decision, both the immediate payoff and next state are known. If we know either of these only as a probability function, then we have a stochastic dynamic programme. The basic ideas of determining stages, states, decisions and recursive formulae still hold: they simply take into account uncertainties (uncertain payoff or uncertain states).

After reading this chapter, readers should be able to: comprehend the nature of multiphase decision problems that can be modelled by dynamic programming; define the stages of the problem, its input stages and the decisions that can be made; define the transition function between the input state and the output stage according to the decision made in each stage; construct the recursive function of a dynamic programming model. Calculate the optimal costs in each stage, as well as the optimal decisions. Obtain the optimal solutions that provide dynamic programming for decision making.

9.2 Commercialisation of Products Sold Under Licence

The director of a chain of sports stores, AdeMark, is studying the possibility of acquiring a maximum of five licences to sell five sport brand names (Nike, Adidas, Reebok, SportBall and Avia) so they can be commercialised among three stores. The expected profits depend on the brand marks offered in each store, and are provided in Table 9.1.

(a) Determine how to distribute the five sales licences of the sport brand names among the three stores in order to maximise the profits expected to be made by the AdeMark chain.
(b) Assume that the director of the Ademark chain can buy only four of the brand name sales licences for financial reasons. How could the four licences be distributed among the three stores?

Solution

(a) Determine how to distribute the five sales licences of the sport brand names among the three stores in order to maximise the profits expected to be made by the AdeMark chain.

Table 9.1 Profits expected per store depending on the brand names sold

Brand names/shop	1	2	3
0	0	0	0
1	5	6	4
2	9	11	9
3	14	15	13
4	17	19	18
5	21	22	20

n = stores ($N = 3$)
State = e_n = No. of licences available for n;
Decision = x_n = No. of licences to assign to n;
Recursive function: $f_n\ (e_n,\ x_n) = f^*_{n+1}\ (e_n - x_n) + B_n\ (e_n,\ x_n)$, where B_n = expected profits;

The results are offered in the tables below:
$n = 3$ (store 3)

e_3	$x_3^*\ (e_3)$	$f_3^*\ (e_3)$
0	0	0
1	1	4
2	2	9
3	3	13
4	4	18
5	5	20

$n = 2$ (store 2)

e_2	0	1	2	3	4	5	$x_2^*\ (e_2)$	$f_2^*\ (e_2)$
0	0	–	–	–	–	–	0	0
1	4	6	–	–	–	–	6	1
2	9	10	11	–	–	–	11	2
3	13	15	15	15	–	–	15	1, 2, 3
4	18	19	20	19	19	–	20	2
5	20	24	24	24	23	22	24	1, 2, 3

$n = 1$ (store 1)

e_1	0	1	2	3	4	5	$x_1^*\ (e_1)$	$f_1^*\ (e_1)$
5	24	25	24	25	23	21	25	1, 3

From the tables, we can see that the maximum expected profit is 25 with the five licences distributed among the three stores as so:

Store 1	Store 2	Store 3
1	2	2
3	2	0

(b) Assume that the director of the Ademark chain can buy only four of the brand name sales licences for financial reasons. How could the four licences be distributed among the three stores?

We can see that if the maximum number of licences to obtain is four rather than five, this affects only the table of stage 1, which remains as follows:

e_1	0	1	2	3	4	x_1^* (e_1)	f_1^* (e_1)
4	20	20	20	20	17	20	0, 1, 2, 3

The expected profits will be 20 and the optimal distribution of the four licences among the three stores is:

Store 1	Store 2	Store 3
0	2	2
1	1	2
1	2	1
1	3	0
2	2	0
3	1	0

9.3 Investing in an Advertising Campaign

The firm ADESA is planning an advertising campaign for the following season for its three most important products. As all three products are quite different, each investment in advertising is made in only one product. The firm has 6 million dollars for its advertising campaign and it is assumed that the investment in each product should be an integer number over or equal to 1. The Director of the Marketing Department has set the following objective: determine how much to invest in each product to maximise total sales. Table 9.2 shows an estimated increase in the sales of each product in accordance with the investment made in advertising:

Use dynamic programming to solve this problem.

Solution

n = products ($N = 3$)
State = e_n = millions of euros available to invest in product n;
Decision = x_n = millions of euros to invest in product n;
$f_n(e_n, x_n) = f^*_{n+1}(e_n - x_n) + B_n(e_n, x_n)$, where B_n = Expected increase in sales;

Table 9.2 Increase in sales forecast according to the investment made in advertising

Investment in advertising (millions of dollars)	Product		
	1	2	3
1	7	4	6
2	10	8	9
3	14	11	13
4	17	14	15

The results are shown in the tables that follow:

$n = 3$ (product 3)

e_3	$x_3{}^* (e_3)$	$f_3{}^* (e_3)$
1	1	6
2	2	9
3	3	13
4	4	15

$n = 2$ (product 2)

e_2	1	2	3	4	$x_2{}^* (e_2)$	$f_2{}^* (e_2)$
2	10	–	–	–	10	1
3	13	14	–	–	14	2
4	17	17	17	–	17	1, 2, 3
5	19	21	20	20	21	2

$n = 1$ (product 1)

e_1	1	2	3	4	$x_1{}^* (e_1)$	$f_1{}^* (e_1)$
6	28	27	28	27	28	1, 3

From the tables, we can see that the maximum expected increase in sales will be 28 and that the investment made in advertising is distributed among the three products as shown below:

Product 1	Increase	Product 2	Increase	Product 3	Increase	Total
1	7	2	8	3	13	28
3	14	2	8	1	6	28

9.4 Planning Routes to Refill Products in Vending Machines

A firm based in Warrington is the supplier of vending machines and it supplies coffee, snacks and refreshments. This firm has machines set up in firms in Warrington, Runcorn, Widness and Frodsham. The technician in charge of maintaining and refilling these machines has to visit these towns every day; that is, he starts in Warrington, then goes to Runcorn, next on to Widnes and then arrives in Frodsham before returning to Warrington. How can he/she minimise the distance travelled? The distances (in km) between the towns visited are provided in Fig. 9.3.

Consider and solve this problem with a dynamic programming model.

(a) Define the phases, states, decisions and recursive function
(b) Solve the problem.

Solution

(a) Define the phases, states, decisions and recursive function

N = no. of routes to travel ($n = 1\ldots\ 4$)
State = e_n = town where I am and towns visited C (1: Warrington; 2: = Runcorn; 3: = Widness; 4: = Frodsham) in n;
Decision = x_n = town to go to in stage n;
Recursive function: $f_n\ (e_n, x_n) = f^*(x_n, C) + c\ (e_n, x_n)$;

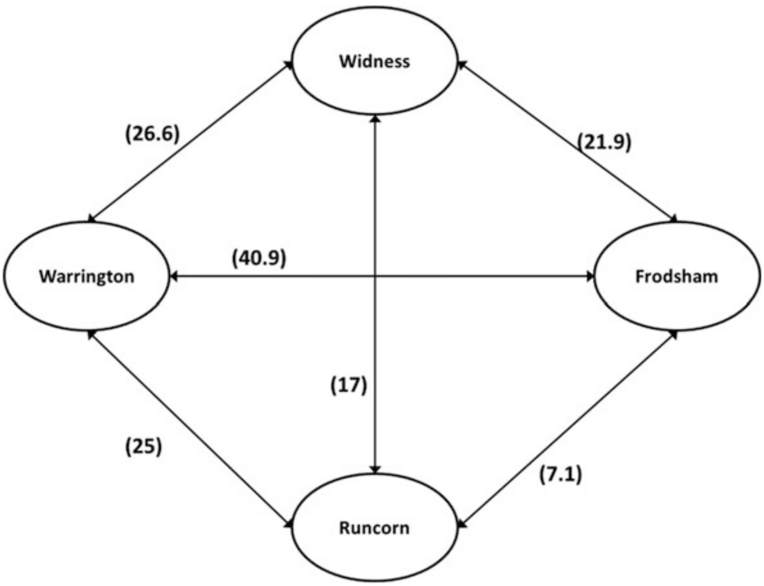


Fig. 9.3 Distances between towns with vending machines

(b) Solve the problem.

$n = 4$

e_4/x_4	f_4	x_4
2 (2, 3, 4)	25	1
3 (2, 3, 4)	40.9	1
4 (2, 3, 4)	26.6	1

$n = 3$

e_3/x_3	2	3	4	f_3	x_3
2 (3, 2)	–	–	43.6	43.6	4
2 (4, 2)	–	48	–	48	3
3 (2, 3)	–	–	48.5	48.5	4
3 (4, 3)	32.1	–	–	32.1	2
4 (4, 2)	–	62.8	–	62.8	3
4 (4, 3)	42	–	–	42	2

$n = 2$

e_2/x_2	2	3	4	f_2	x_2
2 (2)	–	55.6	79.8	55.6	3
3 (3)	50.7	–	63.9	50.7	2
4 (4)	65	54	–	54	3

$n = 1$

e_1/x_1	2	3	4	f_1	x_1
1 (1)	80.6	91.6	80.6	80.6	2, 4

There are two possible solutions:

x_0	1	1
x_1	2	4
x_2	3	3
x_3	4	2
x_4	1	1

Leave Warrington, visit Runcorn and Frodsham and Widness and return to Warrington. Leave Warrington, visit Widness, Runcorn and Fordsham, and return to Warrington.

9.5 Planning an Electoral Campaign

Candidate Arnold Schwarzenegger has been nominated to be elected in California (USA). The funds available there come to around 10,000 (in thousands of dollars). Although the Committee in charge wishes to start the electoral campaign in the five districts in the state of California, limited funds indicate otherwise. Table 9.3 lists Schwarzenegger's estimated voting population to win the elections in each district and the quantity of funds required to start an effective campaign in each district.

The Committee in charge of the campaign has decided that every district receives all the assigned funds or none. How can the available funds be assigned to win the election in the largest number of districts and, therefore, the largest number of total votes? Solve this problem with a dynamic programming model.

Solution

$N = 5$ districts.
State $= e_n =$ funds available to be assigned to each district;
Decision $= x_n =$ funds to be assigned;
$V_n =$ Votes obtained in stage n;
Recursive function: $f_n\ (e_n, x_n) = f^*_{n+1}\ (e_n - x_n) + V_n$

$n = 5$

Table 9.3 Electoral population and the funds needed to start campaigns

District	Voting population (thousands of people)	Funds required (thousands of $)
1	3,100	3,500
2	2,600	2,500
3	3,500	4,000
4	2,800	3,000
5	2,400	2,000

e_5	f_5*	x_5*
0	0	0

$n = 4$

e_4/x_4	0	f_4*	x_4*
0	0	0	0

$n = 3$

e_3/x_3	4000	f_3*	x_3*
4,000	3,500	3,500	4,000

$n = 2$

e_2/x_2	2500	f_2*	x_2*
6,500	7,100	7,100	2,500

$n = 1$

e_1/x_1	3500	f_1*	x_1*
10,000	3,100	9,200	3,500

The optimal solution is to assign funds to districts 1, 2 and 3 to obtain 9,200 votes.

9.6 Seeking a Study Strategy

A year 3 Business Studies student at EPSA has 7 days to prepare the final exams of four subject matters. The student wishes to determine a study strategy which maximises the total marks obtained in all the subject matters. To do this, the student has checked the content of these subject matters and has estimated the

Table 9.4 Marks expected per subject matter according to the days studied

No. of days	Estimated marks			
	Subject matter 1	Subject matter 2	Subject matter 3	Subject matter 4
1	3	5	2	6
2	5	5	4	7
3	6	6	7	9
4	7	9	8	9

mark that could be obtained in each subject matter if he or she studied for 1, 2, 3 or 4 days. The student wishes to study all the subject matters. This information is summarised in Table 9.4.

Consider the dynamic programming model which provides this student with an optimal study strategy.

Solution

$N = 4$, number of subject matters.
Decision $= x_n =$ number of study days assigned to subject matter n;
State $= e_n =$ number of study days still available before starting to study subject matter n;
$marks_n\ (x_n) =$ estimated marks for subject matter n if x_n study days are assigned;
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(e_n - x_n) + \text{notas}_n(x_n)$

$n = 4$

e_4	f_4*	x_4*
1	6	1
2	7	2
3	9	3
4	9	4

$n = 3$

e_3	x_3				f_3*	x_3*
	1	2	3	4		
2	$2 + 6 = 8$	–	–	–	8	1
3	$2 + 7 = 9$	$4 + 6 = 10$	–	–	10	2
4	$2 + 9 = 11$	$4 + 7 = 11$	$7 + 6 = 13$	–	13	3
5	$2 + 9 = 11$	$4 + 9 = 13$	$7 + 7 = 14$	$8 + 6 = 14$	14	3, 4

$n = 2$

e_2	x_2				f_2*	x_2*
	1	2	3	4		
3	5 + 8 = 13	–	–	–	13	1
4	5 + 10 = 15	5 + 8 = 13	–	–	15	1
5	5 + 13 = 18	5 + 10 = 15	6 + 8 = 14	–	18	1
6	5 + 14 = 19	5 + 13 = 18	6 + 10 = 16	9 + 8 = 17	19	1

$n = 1$

e_1	x_1				f_1*	x_1*
	1	2	3	4		
7	3 + 19 = 22	5 + 18 = 23	6 + 15 = 21	7 + 13 = 20	23	2

Using the tables, the optimal study strategy obtained is:
Spend 2 study days on subject matter 1 to obtain an estimated mark of 5.
Spend 1 study day on subject matter 2 to obtain an estimated mark of 5.
Spend 3 study days on subject matter 3 to obtain an estimated mark of 7.
Spend 1 study day on subject matter 4 to obtain an estimated mark of 6.

9.7 Production Planning in a Textile Firm

A textile firm holds a contract with a Portuguese firm to deliver the following number of metres of material over the next 3 months: month 1,200 m; month 2,300 m; month 3,300 m. For each metre produced in months 1 and 2, a variable cost of \$10 is incurred; for each metre produced in month 3, a variable cost of \$12 is incurred. The storage cost is \$1.5 for each metre in the inventory at the end of 1 month. The cost to prepare production over 1 month is \$250. The metres produced in 1 month can cover the demand of this month or that of a future month. Let's assume that the production generated in each month must be a multiple of 100. Given that the initial inventory level is zero, use dynamic programming to determine the optimal production schedule.

Solution

$N = 3$ months
Decision $= x_n =$ Quantity to be produced in n;
State $= e_{n-1} =$ Inventory at the end of $n-1$;
Recursive function:

$$\{f_n(e_{n-1}, x_n) = f^*_{n+1}(e_{n-1} + x_n - D_n) + (CV \cdot x_n + Ca \cdot e_n + Cp(if\ x_n > 0)\}$$

$n = 3$

e_2	x_3	f_3
300	0	0
200	100	1,450
100	200	2,650
0	300	3,850

$n = 2$

e_1/x_2	0	100	200	300	400	500	600	f_2	x_2
600	450	–	–	–	–	–	–	0	0
500	1,750	1,700	–	–	–	–	–	1,700	0
400	2,800	3,000	2,700	–	–	–	–	2,700	200
300	3,850	4,050	4,000	3,700	–	–	–	3,700	300
200	–	5,100	5,050	5,000	4,700	–	–	4,700	400
100	–	–	6,100	6,050	6,000	5,700	–	5,700	500
0	–	–	–	7,100	7,050	7,000	6,700	6,700	600

$n = 1$

e_0/x_1	200	300	400	500	600	700	800	f_1	x_1
0	8,950	9,100	9,250	9,400	9,650	9,700	9,600	8,950	200

The optimal production schedule with a total cost of 8,950$ would be: to produce 200 m in month 1, 600 m in month 2 and 0 m in month 3.

9.8 Frozen Cakes Production Planning

The firm Elis's Cheesecake produces the favourite dessert of Chicago. The firm's Chair, Marc Schulman, wants to establish the production of frozen cakes for the summer season. The maximum monthly production capacity is 40,000 frozen cakes. The demand forecasts for the month of June are 20,000 cakes, 30,000 for July and 30,000 for August.

The required production processes are mixing, baking, decorating, packing and freezing. The containers which process the mixture are large enough to optimally produce batches of 10,000 cakes. Each stored frozen cake has a monthly inventory cost of $10. The production costs per cake depend on the number of production batches with 10,000 produced units per month, as shown in Table 9.5. Fixed costs are estimated to be $30,000.

The cakes have a 1-month expiry date, so the cakes that are not sold in the month they are produced will serve only for the next month. The firm starts with an initial inventory of zero units and does not wish to have any inventory left at the end of August.

(a) Consider this problem with dynamic programming by describing the stage, the decision variable, the state and the recursive function. Specify the possible states in each phase and the decisions that can be made
(b) Solve the problem.

Solution

(a) Consider this problem with dynamic programming by describing the stage, the decision variable, the state and the recursive function. Specify the possible states in each phase and the decisions that can be made

$N = 3$ months
Decision = x_n = cakes to produce in n;
State = e_{n-1} = cakes available in $n-1$;
Recursive function: $\{f_{n-1}(e_{n-1}, x_n) = f^*_n(e_{n-1} + x_n - d_n) + Cf_n + Cp \cdot x_n + Ca \cdot e_n\}$

where
Cf_n = The fixed costs in n, if $x_n = 0 \rightarrow Cf_n = 0$;
Cp = Production costs;
Ca = Storage costs.

Table 9.5 Unit production costs per cake

Batches	Production cost per cake ($)
0	0
1	10
2	10
3	5
4	5

(b) Solve the problem

e_3	x_3	f_3*
0	30	180,000
10	20	230,000
20	10	130,000
30	0	0

e_2/x_2	0	10	20	30	40	f_2	x_2
0	–	–	–	360,000	560,000	360,000	30
10	–	–	410,000	510,000	560,000	410,000	20
20	–	310,000	560,000	510,000	530,000	310,000	10

e_1/x_1	0	10	20	30	40	f_2	x_2
0	–	–	590,000	690,000	740,000	590,000	20

The optimal solution is to produce exactly the demand units forecast for each month; that is, 20,000 cakes in June, 30,000 in July and 30,000 in August, with a total cost of $590,000.

9.9 Managing Hire Cars in Travel Agencies

A travel agency, TURALSA, organises 1-week journey around southern Egypt. The travel agency has a contract to provide groups of tourists with seven, four, seven and eight rented four-wheel-drive vehicles, respectively, for the next 4 weeks. The travel agency subcontracts an Egyptian car hire firm, HEZ EGYPT, which covers its car hire requirements. HEZ EGYPT charges a weekly hiring rate of $220 for each four-wheel-drive vehicles, plus a set rate of $500 for any weekly hire transaction. However, TURALSA can opt not to return hire vehicles at the end of the week, in which case the agency will be responsible only for the weekly hire ($220). What is the optimal way for TURALSA to manage the car hire situation in Egypt?

Solution

$N = 4$ weeks;
Decision $= x_n =$ number of vehicles rented in week n ($n = 1,\ldots 4$);
State $= e_n =$ Vehicles rented in week $n - 1$ ($n = 1,\ldots, 4$);
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(x_n) + \text{CostToRent} \cdot x_n + \text{Fixed Cost}\ (if\ x_n > e_{n-1})$

$n = 4$

e_3	x_4	f_4
7	8	2,260
8	8	1,760

$n = 3$

e_2/x_3	7	8	f_3*	x_3*
4	4,300	4,020	4,020	8
7	3,800	4,020	3,800	7
8	4,300	3,520	3,520	8

$n = 2$

e_1/x_2	4	7	8	f_2*	x_2*
7	4,900	5,340	5,780	4,900	4
8	5,400	5,840	5,280	5,280	8

$n = 1$

e_0/x_1	7	8	f_1	x_1
0	6,940	7,040	6,940	7

The four-wheel-drive vehicles rented for weeks 1–4 are 7, 4, 8, 8 with a cost of $6,940.

9.10 Fruit and Vegetable Production Planning

An agricultural firm situated in Bussot grows fruit and vegetables in a greenhouse dedicated to R&D + I measuring 10 × 20 m. For the next season, it is planning three types of products to do different experiments with: tomatoes, artichokes and red peppers. The experimental greenhouse is arranged in 20m rows. The rows of tomatoes and peppers are 2 m wide, while that of artichokes is 3 m wide. The experiments that the firm is most interested in are done with tomato plants, and those that it is least interested in are done with artichoke plants. On a scale of preference from 1 to 10, it assigns 10 to the tomato rows, 7 to the pepper rows and 3 to the artichoke rows. Moreover, due to technical restrictions, it is necessary to plant at least one row of artichokes and no more than two rows of tomatoes. Formulate and solve the dynamic programming model that helps determine how many rows of each product must be planted by considering that it wishes to maximise the preference score given to each product.

Solution

$N = 3$ types of products (tomatoes, artichokes, red peppers)
Decision $= x_n =$ number of rows to plant of each n;
State $= e_n =$ metres wide available in n;
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(e_n - x_n a_n) + p_n x_n$

where $a_n =$ width of row n; and $p_n =$ preference of row n;
$n = 3$ (red peppers) ($a_3 = 2$; $p_3 = 7$)

e_3	x_3^*	f_3^*
0	0	0
1	0	0
2	1	7
3	1	7
4	2	14
5	2	14
7	3	21

$n = 2$ (artichokes) ($a_2 = 3$; $p_2 = 3$)

e_2/x_2	1	2	3	f_2^*	x_2^*
6	10	6	–	10	1
8	17	13	–	17	1
10	24	20	9	24	1

$n = 1$ (tomatoes) ($a_1 = 2$; $p_1 = 10$)

e_1/x_1	0	1	2	f_1^*	x_1^*
10	24	27	30	30	2

The optimal solution is to plant two rows of tomato plants, one row of artichokes and one row of red peppers with a maximum scored obtained of 30.

9.11 Production Planning in a Recycled Glass Firm

A supplier of recycled glass parts, MEDITERRASA, forecasts the demand of an article for the following 4 months of 100, 140, 210 and 180 units, respectively. The firm must hold only the sufficient stocks to meet the demand of each month, or it may hold some surplus stock to meet the demand of two or more successive months with a storage cost of \$1.2 per month for each surplus stock unit. MEDITERRASA calculates that the production costs for the following months, which varies depending on the units produced, is \$15, \$12, \$10 and \$14 , respectively. A preparation cost of \$200 is incurred each time a manufacturing order is done. The firm wishes to develop a production plan which minimises the total costs incurred by production orders, production and maintaining an article in the inventory.

Formulate a dynamic programming model, provide tables with the possible states and decisions, and solve it to find an optimal solution.

Solution

$N = 4$ production periods;
State $= e_{n-1} =$ the inventory available at the end of $n-1$;
Decision $= x_n =$ units to produce in n;
$e_{n+1} = e_{n-1} + x_n - d_n$;
Recursive function: $f_n(e_{n-1}, x_n) = f_{n+1}(e_{n-1} + x_n - d_n) + M_n + V_n \cdot x_n + a_n \cdot (e_{n-1} + x_n - d_n)$

$n = 4$ $\mathrm{M_n} = 200$ $\mathrm{V_4} = 14$ $\mathrm{a_n} = 1.2$ $\mathrm{d_4} = 180$

e_4	f_4*	x_4*
0	2,720	180
180	0	0

$n = 3$ $\mathrm{M_n} = 200$ $\mathrm{V_3} = 10$ $\mathrm{a_n} = 1.2$ $\mathrm{d_3} = 210$

e_3/x_3	0	180	210	390	f_3*	x_3*
0	–	–	5,020	4,316	4,316	390
210	2,720	2,216	–	–	2,216	180
390	216	–	–	–	216	0

$n = 2$ $M_n = 200$ $V_2 = 12$ $a_n = 1.2$ $d_2 = 140$

e_2/x_2	0	140	180	210	350	390	530	f_2*	x_2*
0	–	6,196	–	–	6,868	–	7,244	6,196	140
140	4,316	–	–	5,188	–	5564	–	4,316	0
350	2,468	–	3,044	–	–	–	–	2,468	0
530	684	–	–	–	–	–	–	684	0

$n = 1$ $M_n = 200$ $V_1 = 15$ $a_n = 1.2$ $d_1 = 100$

e_1/x_1	100	240	450	630	f_2*	x_2*
0	7896	8284	9718	10970	7896	100

The optimal solution is to produce 100 units in month 1,140 units in month 2 and 390 units in month 3, with a total cost of $7,896.

9.12 Selecting Slot Machines

The person in charge of the OZONA bowling alley is considering obtaining three new types of slot machines. For this purpose, a space covering 3 m^3 has been made. The volume required for each machine type and the annual estimated profit per fitted machine in thousands of dollars are provided in Table 9.6.

Obviously, we can see that the profit obtained by each machine changes when more machines of the same game type are added. For example, this reflects the fact that the customers could occupy a trivial game machine all the time, but a second trivial game machine could be stood idle for part of this time. Moreover, experience shows that there has to be at least one machine of each type and no more than two basketball game machines.

How many machines of each type should be fitted to maximise profits?

Solution

Table 9.6 Volume and profit foreseen per fitted machine

Machine		Profit (thousands of dollars) obtained by		
Machine type	Volume (m^3)	Machine 1	Machine 2	Machine 3
Trivial	1	10	7	4
Basketball	¼	9	9	8
Boxing	½	11	10	9

$N = 3$ machine types (trivial, basketball, boxing)
Decision $= x_n =$ number of machines to fit in each n;
State $= e_n =$ the volume available in n;
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(e_n - x_n a_n) + B_n x_n$

where $a_n =$ the volume required by n; and $B_n =$ profit of n;
$n = 3$ (boxing)

e_3	x_3^*	f_3^*
0.5	1	11
1.5	3	30

$n = 2$ (basketball)

e_2/x_2	1	2	f_2^*	x_2^*
1	20	29	29	2
2	39	48	48	2

$n = 1$ (trivial)

e_1/x_1	1	2	f_1^*	x_1^*
3	58	46	58	1

The optimal solution is to fit one trivial machine, two basketball machines and three boxing machines to obtain annual profits of $58,000.

9.13 Planning an Electrical Expansion Project

Table 9.7 provides the data of a planning project to extend electricity generation:

Table 9.7 Demand and costs of the annual investment

Years	Demand (MW)	Investment cost per 1 GW generator ($/GW year)
1999	1000	50
2000	2000	55
2001	4000	60
2002	6000	65
2003	7000	45
2004	8000	40

Furthermore:

- There is an additional cost of 15$/year if at least one generator is constructed.
- It is not possible to install more than 3,000 MW of electricity generation in any year.
- The task is based on an electric system that has no generator installed.

Consider a dynamic programming model to minimise the total costs (fixed and variables) of extending the generator equipment.

Solution

$N = 6$ year ($n = 1999, \ldots, 2004$);
Decision $= x_n =$ Power to be installed each year;
State $= e_n =$ Total number of generators installed at the start of each year;
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(e_n + x_n) + CI_n x_n + CF(if\ x_n > 0)$

where CI is the investment cost per generator and CF is the fixed cost should at least one generator be constructed.

$n = 2004$

State	8000	Cost	Optimal decision
7,000	15 + 40 = 55	55	1,000
8,000	0	0	0

$n = 2003$

State	7000	8000	Cost	Optimal decision
6,000	15 + 45 + 55 = 115	15 + 90 = 105	105	2,000
7,000	55	15 + 45 = 60	55	1,000
8,000	–	0	0	0

$n = 2002$

State	6000	7000	8000	Cost	Optimal decision
4,000	15 + 130 + 105 = 250	15 + 195 + 55 = 265	–	250	2,000
5,000	15 + 65 + 105 = 185	15 + 130 + 55 = 200	15 + 195 = 210	185	1,000
6,000	105	15 + 65 + 55 = 135	15 + 130 = 145	105	0
7,000	–	55	15 + 65 = 80	55	0
8,000	–	–	0	0	0

$n = 2001$

State	4000	5000	6000	7000	8000	Cost	Optimal decision
2,000	15 + 120 + 250 = 385	15 + 180 + 185 = 380	–	–	–	380	3,000
3,000	15 + 60 + 250 = 325	15 + 120 + 185 = 320	15 + 180 + 105 = 300	–	–	300	3,000
4,000	250	15 + 60 + 185 = 260	15 + 120 + 105 = 240	15 + 180 + 55 = 250	–	240	2,000
5,000	–	185	15 + 60 + 105 = 180	15 + 120 + 55 = 190	15 + 180 = 195	180	1,000
6,000	–	–	105	15 + 60 + 55 = 130	15 + 120 = 135	105	0

$n = 2000$

State	2000	3000	4000	5000	6000	Cost	Optimal decision
1,000	15 + 55 + 380 = 445	15 + 110 + 300 = 425	15 + 165 + 240 = 420	–	–	420	3,000
2,000	–	15 + 55 + 300 = 365	15 + 110 + 240 = 365	15 + 165 + 180 = 360	–	360	3,000
3,000	–	–	15 + 55 + 240 = 305	15 + 110 + 180 = 305	15 + 165 + 105 = 285	285	3,000

$n = 1999$

State	1000	2000	3000	Cost	Optimal decision
0	15 + 50 + 420 = 485	15 + 100 + 360 = 475	15 + 150 + 285 = 450	450	3,000

Therefore, the power to be installed each year is:

Years	Power (MW)
1999	3,000
2000	3,000
2001	0
2002	0
2003	2,000
2004	0

With a total cost of \$450.

Table 9.8 Orders information

Supplier	Product	No. of pallets to order	Priority (points/pallet)	Pallet size (m)
1	A	6	2	1
2	B	5	3	2
3	C	4	2	1

9.14 Transport Planning in a Firm of the Automobile Sector

A firm that supplies automobile seats must place an order with three of its suppliers in Germany. To minimise transport costs, the transport known as grouping is employed; this entails contracting a certain space on a truck to be used by several suppliers and paying the volume (m^3) or linear metres actually employed which, in this case, is 6 linear metres. The packaging units of the products are measured in the linear metres occupying the corresponding pallet. It is assumed that pallets are not stackable. Specifically, the products shown in Table 9.8 must be ordered.

The products with more priority are those that are required the most. Besides, it necessary to receive one pallet of each product type at least. The objective is to do an order that occupies the 6 linear metres at the most by maximising the points relating to the priority of the demand that is to be met.

(a) Consider a dynamic programming model that can help the firm with its transport planning
(b) What is the optimum number of pallets to order of each product?

Solution

(a) Consider a dynamic programming model that can help the firm with its transport planning

$N = 3$ products (A, B, C)
Decision $= x_n =$ number of pallets to be ordered of each n;
State $= e_n =$ The linear metres available in n;
Recursive function: $f_n(e_n, x_n) = f^*_{n+1}(e_n - x_n a_n) + B_n x_n$

where $a_n =$ the metres required by n; and $B_n =$ the priority of n;

(b) What is the optimum number of pallets to order of each product?

$n = 3$ (C)

e_3	x_3^*	f_3^*
1	1	2
2	2	4
3	3	6

$n = 2\ (B)$

e_2/x_2	1	2	f_2^*	x_2^*
3	5	–	5	1
4	7	–	7	1
5	9	8	9	1

$n = 1\ (A)$

e_1/x_1	1	2	3	f_1^*	x_1^*
6	11	11	11	11	1, 2, 3

There are three possible solutions:

$x_A = 1; x_B = 1; x_C = 3;$
$x_A = 2; x_B = 1; x_C = 2;$
$x_A = 3; x_B = 1; x_C = 1.$

9.15 Inventories Planning in a Chemist's

A chemist's must decide how many pallets of infant food to order and have in stock in the 8 linear metres used for it in the warehouse by considering the minimum demand forecasts and the profit generated per product, which are provided in Table 9.9. It is assumed that each pallet occupies 1 linear metre and that pallets are not stackable for product safety reasons.

Table 9.9 Demand and profit forecast per product

Product	Minimum demand forecasts (no. of pallets)	Profit (hundreds of $/pallet)
A	2	5
B	1	3
C	2	7

The objective is to store the largest number of products that produce more profit in 8 linear metres, but to also fulfil the minimum demand forecasts.

(a) Consider a dynamic programming model that can help the chemist's plan its inventories
(b) What is the optimum number of pallets of each product to be stored? Obtain this number by solving the dynamic programming model set out in the previous section.

Solution

(a) Consider a dynamic programming model that can help the chemist's plan its inventories

$N = 3$ products (A, B, C)
Decision $= x_n =$ number of pallets to order of each n;
State $= e_n =$ The linear metres available in n;
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(e_n - x_n) + B_n x_n$

where $B_n =$ profit of n;

(b) What is the optimum number of pallets of each product to be stored? Obtain this number by solving the dynamic programming model set out in the previous section

$n = 3$ (C)

e_3	x_3^*	f_3^*
2	2	14
3	3	21
4	4	28
5	5	35

$n = 2$ (B)

e_2/x_2	1	2	3	4	5	6	7	8	f_2^*	x_2^*
3	17	–	–	–	–	–	–	–	17	1
4	24	20	–	–	–	–	–	–	24	1
5	31	27	23	–	–	–	–	–	31	1
6	38	34	30	26	–	–	–	–	38	1

$n = 1$ (A)

e_1/x_1	2	3	4	5	f_1^*	x_1^*
8	48	46	44	42	48	2

The optimal solution is:
$x_A = 2$; $x_B = 1$; $x_C = 5$.

9.16 Truck Load Planning

A truck can transport a total of 10 tonnes of products. Three product types are transported, whose weight and value are provided in Table 9.10. By assuming that at least one article of each class must be transported, establish the load that maximises the total value.

Solution

$N = 3$ products $(A,\ B,\ C)$
Decision $= x_n =$ units to transport of each n;
State $= e_n =$ tonnes available in n;
Recursive function: $f_n(e_n,\ x_n) = f_{n+1}^*(e_n - x_n p_n) + V_n x_n$
where $V_n =$ value of n;

$n = 3\ (C)$

e_3	x_3^*	f_3^*
2	1	6
3	1	6
4	2	12
5	2	12
6	3	18
7	3	18

$n = 2\ (B)$

e_2/x_2	1	2	3	f_2^*	x_2^*
4	11	–	–	11	1
5	11	–	–	11	1
6	17	16	–	17	1
7	17	16	–	17	1
8	23	22	21	23	1
9	23	22	21	23	1

$n = 1\ (A)$

e_1/x_1	1	2	3	4	5	6	f_1^*	x_1^*
10	25	27	23	25	21	23	27	2

The optimal solution is:
$x_A = 2$; $x_B = 1$; $x_C = 3$, which provides a total value of $27,000.

9.17 Orders Planning of Imported Products

A firm foresees starting a business activity in Valencia which involves importing three products from China to distribute them in Spain. These products are machine spare parts which, in China (specifically Hong Kong), are well-priced, and come in boxes of 100 units. So one box of product 1 is priced at 200$/box, a box of product 2 is priced at 180$/box, and a box of product 3 costs 150$/box. These three products can be distributed in Spain at the mean price of 250$/box. However, these are complementary products and at least one box of each product must be available.

The firm is considering placing a first order of five boxes. Later, if this business proves profitable, it can order larger quantities.

In principle, the main problem identified is transport costs, which is a key factor to determine if this business is feasible or not. For this reason, the firm is contemplating using maritime transport in groupings because, despite being slower, the most important point is to minimise the costs to transport the merchandise. The price of grouping, owing to volume and weight specifications, depends on the number of boxes per product to be sent. So, the cost is around 10$/box if only one box of a given product is sent, 7$/box if two boxes of the same product are sent, and 5$/box if three boxes of the same product are sent.

Consider and solve the dynamic programming model that determines the number of boxes of each product that you recommend the firm to order in order to maximise its gross profit margin.

Table 9.10 Information about each product to be transported

Class	Value (thousands of dollars)	Weight (t)
A	2	1
B	5	2
C	6	2

Solution

$N = 3$ products (1, 2, 3)
Decision $= x_n =$ number of boxes to order of each n;
State $= e_n =$ Boxes pending to be ordered in n;
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(e_n - x_n) + B_n x_n$

$n = 3$ (product 3)

e_3	f_3^*	x_3^*
1	90	1
2	186	2
3	285	3

$n = 2$ (product 2)

e_2/x_2	1	2	3	f_2^*	x_2^*
2	150	–	–	150	1
3	246	216	–	246	1
4	345	312	285	345	1

$n = 1$ (product 1)

e_1/x_1	1	2	3	f_1^*	x_1^*
5	385	332	285	385	1

There is one optimal solution:
$x_1 = 1$; $x_2 = 1$; $x_3 = 3$.

9.18 Selecting Investment Strategies

ADESA has \$4 million to invest in three businesses located in emerging countries. The profitability expected from these businesses depends on the quantity invested, as given in Table 9.11.

Table 9.11 Profitability expected depending on the quantities invested

Quantity invested (millions of dollars)	Expected profitability (millions of dollars)		
	Business 1	Business 2	Business 3
0	4	3	3
1	7	6	7
2	8	10	8
3	9	12	13
4	11	14	15

Assume that the quantity invested in each business must be an exact multiple of \$1 million, determine an investment strategy by dynamic programming that maximises ADESA's expected profitability.

Solution

$N = 3$ business in which to invest (business 1, business 2, business 3)
Decision $= x_n =$ millions of euros to invest in each n;
State $= e_n =$ millions of euros available to invest in n;
Recursive function: $f_n(e_n,\ x_n) = f^*_{n+1}(e_n - x_n) + B_n x_n$

$n = 3$ (business 3)

e_3	f^*_3	x^*_3
0	0	3
1	1	7
2	2	8
3	3	13
4	4	15

$n = 2$ (business 2)

e_2/x_2	0	1	2	3	4	f^*_2	x^*_2
0	6	–	–	–	–	6	0
1	10	9	–	–	–	10	0
2	11	13	13	–	–	13	1, 2
3	16	14	17	15	–	17	2
4	18	19	18	19	17	19	1, 3

$n = 1$ (business 1)

e_1/x_1	0	1	2	3	4	f_1^*	x_1^*
4	23	24	21	19	17	24	1

There is one optimal solution: investing \$1 million in business 1, \$2 million in business 2, and \$1 million in business 3. The total expected profitability is \$24 million.

9.19 Replacing a Car

John Smith faces the following problem: the date is January 1999, his car is 2 years old and he is considering what the best replacement policy must be.

He needs a car for the next 4 years, after which time he will sell because he will move to live in a town on the coast. As part of his replacement policy, Alfonso wishes to continue with a car in the same range as the one he now has. Based on the information acquired in new and second-hand car magazines, he knows: for each age i, the purchase prices C_i and the sales prices V_i, plus the car maintenance cost, including a comprehensive insurance policy E_i (the cost during the year by which the car exceeds age i to $i + 1$), are provided in Table 9.12.

Alfonso would never replace his current car for one that is of the same age or older.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how to calculate e_{n+1} and the possible states of each stage, plus the feasible decisions that can be made
(b) Solve the problem.

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how to calculate e_{n+1} and the possible states of each stage, plus the feasible decisions that can be made

Table 9.12 Information about the purchase, sales and maintenance costs

i	0	1	2	3	4	5	6	7	8
C_i	1,000	900	775	700	625	550	400	350	250
V_i	750	700	600	550	500	450	350	300	200
E_i	100	125	150	175	225	275	325	375	–

Phase = year, $N = 4$
Decision = x_n = years the car has when starting year n;
State = e_n = years the car has when finishing year $n - 1$;
Recursive function: $f_n(e_n, x_n) = f^*_{n+1}(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = x_{n+1}$;
$c_n(e_n, x_n)$: according to Table 9.12

(b) Solve the problem.

$n = 4$

e_4	$x_4 = 0$	$x_4 = 1$	$x_4 = 2$	$x_4 = 3$	$x_4 = 4$	$x_4 = 5$	f_4*	x_4*
1	−300	−475	–	–	–	–	−475	1
2	−200	−175	−400	–	–	–	−400	2
3	−150	−125	−175	−325	–	–	−325	3
4	−100	−75	−125	−125	−225	–	−225	4
5	−50	−25	−75	−75	−50	−75	−75	5

$n = 3$

e_3	$x_3 = 0$	$x_3 = 1$	$x_3 = 2$	$x_3 = 3$	$x_3 = 4$	f_3*	x_3*
1	−75	−275	–	–	–	−275	1
2	25	25	−175	–	–	−175	2
3	75	75	50	−50	–	−50	3
4	125	125	100	150	150	100	2

$n = 2$

e_2	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	f_2*	x_2*
1	125	−50	–	–	−50	1
2	225	250	100	–	100	2
3	275	300	325	275	275	0–3

$n = 1$

e_1	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	f_1*	x_1*
2	450	525	425	425	2

Solution: 2 – 3 – 2 – 3 with an end result of 425 of the total cost.

9.20 Operation Programming of Electric Generators

The engineer in charge of an electric substation has to decide on a daily basis which of the two generators available must operate, and at what times, given the demand forecast for that day. The quantity required during 1 day fluctuates along four periods which 1 day is divided into for this purpose. Table 9.13 offers the demand forecast for the following day.

When each period commences, the generators can be connected or disconnected, and the maximum capacities during each period are 6,000 and 10,000 MW. The costs of starting up the generators are \$100 and \$75, respectively, while the operating costs (costs to use the generator, but they are fixed, are not proportional to the demand they meet) per period of usage are \$175 and \$210, respectively.

The two generators are disconnected for 30 min at midnight. The aim is to obtain the operation programming for these generators which minimises the total cost.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and which states are feasible in each phase as the possible decisions that can be made
(b) Solve the problem.

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and which states are feasible in each phase as the possible decisions that can be made

Phase = Period (4)
Decision = x_n = power available in n;
State = e_n = power available in $n-1$;
Recursive function: $f_n(e_n, x_n) = f^*_{n+1}(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = x_n$;
$c_n(e_n, x_n)$ = starting up cost + operation cost (see above)
Possible states and feasible decisions in each phase:

Table 9.13 Electric demand forecast

Period	Demand in MW
00:30–06:00	4,000
06:00–12:00	9,000
12:00–18:00	12,000
18:00–24:00	5,000

No.	e_n	x_n
1	0	6, 10, 16
2	6, 10, 16	10, 16
3	10, 16	16
4	16	6, 10, 16

(b) Solve the problem.

$n = 4$

e_4	f_4*	x_4*
16	175	6

$n = 3$

e_3	$x_3 = 16$	f_3*	x_3*
10	660	660	16
16	560	560	16

$n = 2$

e_2	$x_2 = 10$	$x_2 = 16$	$f_2^*(e_2)$	x_2*
6	945	1,020	945	10
10	870	1,045	870	10
16	870	945	870	10

$n = 1$

e_1	$x_1 = 6$	$x_1 = 10$	$x_1 = 16$	f_1*	x_1*
0	1,220	1,155	1,430	1,155	10

Solution: 10, 10, 16, 6 with a total cost of $1,155.

9.21 Summer Cakes Production Planning

With the arrival of summer, the craft firm Tartel S. A. wants to establish the frozen cakes production it must produce for the summer season. The firm's maximum monthly production is 5,000 cakes. After conducting some market research, the firm estimates that demand will be 2,000 cakes in June, 2,000 cakes in July and 3,000 cakes in August.

Production must be launched in boxes of 1,000 units. In order to simplify costs calculations, it will be considered that each cake stored at the end of the month will entail a cost of $10. The production costs per cake depend on the number of boxes with 1,000 units produced monthly (See Table 9.14).

The expiry dates of the cakes are assumed to be 1 month, so unsold cakes will serve only for the next month. The firm begins with an inventory of zero units and it does not want any inventory at the end of August.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made
(b) Solve the problem.

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made

Phase = month, $N = 3$
Decision = x_n = boxes of cakes produced each month;
State = e_n = boxes of cakes in the inventory at the beginning of the month;
Recursive function: $f_n(e_n, x_n) = f^*_{n+1}(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = e_n + x_n - d_n$;
$c_n(e_n, x_n) = x_n \cdot \text{CostPro}(x_n) + e_{n+1} \cdot 10$ (in thousands of dollars);
Maximum production per month = 5,000 cakes;

Table 9.14 Production costs per cake

Boxes	Cost (dollars)
0	0
1	10
2	10
3	5
4	5
5	4

Cakes would never be produced for them to expire as this decision is only an added cost.

The possible states and feasible decisions in each stage:

N	e_n	x_n
1	0	2, 3, 4
2	0, 1, 2	0, 1, 2, 3, 4, 5
3	0, 1, 2, 3	0, 1, 2, 3

(b) Solve the problem.

$n = 3$

e_3	$f_3^*(e_3)$	x_3^*
0	15	3
1	20	2
2	10	1
3	0	0

$n = 2$

e_2	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$	$f_2^*(e_2)$	x_2^*
0	–	–	35	45	50	50	35	2
1	–	25	50	45	50	–	25	1
2	15	40	50	45	–	–	15	0

$n = 1$

e_1	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$f_1^*(e_2)$	x_1^*
0	55	50	55	50	3

Solution: the production programme must include three boxes in June, one box in July and three boxes in August with a total cost of $5,000.

9.22 Selecting Investments

After being on a postgraduate training course at EPSA on Quantitative Methods applied to investments, Anne wants to apply her knowledge on dynamic programming to decide on the best investment to be made with the \$10,000 she has saved.

Anne is considering investing in four different investment funds which offer a suitable return and acceptable security. In order to manage the accounts more easily, Anne is considering investing multiples of \$1,000 in each fund.

The expected return after 1 year that each fund offers her is that indicated in thousands of dollars in Table 9.15.

In order to minimise the risk by diversifying the investment made, Anne has considered investing at least \$1,000 in each fund and does not wish to invest more than \$4,000 in any fund.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made
(b) Solve the problem.

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made

Phase = fund, $N = 4$
Decision = x_n = thousands of dollars to invest in the fund;
State = e_n = thousands of dollars that remain to be invested;
Recursive function: $f_n(e_n, x_n) = f^*_{n+1}(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = e_n - x_n$;
$c_n(e_n, x_n)$ = table of return(x_n)
The possible states and feasible decisions in each stage:

Table 9.15 Return from investment funds

Thousands of invested \$	1	2	3	4	5	6	7	8	9	10
Fund 1	0.15	0.30	0.45	0.60	0.75	0.90	1.07	1.23	1.40	1.56
Fund 2	0.20	0.40	0.60	0.80	0.93	1.05	1.18	1.30	1.43	1.55
Fund 3	0.25	0.50	0.75	0.85	0.95	1.05	1.15	1.25	1.35	1.45
Fund 4	0.30	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40

No.	e_n	x_n
1	10	1, 2, 3, 4
2	6, 7, 8, 9	1, 2, 3, 4
3	2, 3, 4, 5, 6, 7, 8	1, 2, 3, 4
4	1, 2, 3, 4	1, 2, 3, 4

(b) Solve the problem

$n = 4$

e_4	$f_4^*(e_4)$	x_4^*
1	0.30	1
2	0.60	2
3	0.70	3
4	0.80	4

$n = 3$

e_3	$x_3 = 1$	$x_3 = 2$	$x_3 = 3$	$x_3 = 4$	$f_3^*(e_3)$	x_3^*
2	0.55	–	–	–	0.55	1
3	0.85	0.8	–	–	0.85	1
4	0.95	1.1	1.05	–	1.1	2
5	1.05	1.2	1.35	1.15	1.35	3
6	–	1.3	1.45	1.45	1.45	3–4
7	–	–	1.55	1.55	1.55	3–4
8	–	–	–	1.65	1.65	4

$n = 2$

e_2	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_2^*(e_2)$	x_2^*
6	1.55	1.5	1.45	1.35	1.55	1
7	1.65	1.75	1.7	1.65	1.75	2
8	1.75	1.85	1.95	1.9	1.95	3
9	1.85	1.95	2.05	2.15	2.15	4

$n = 1$

e_1	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$f_2^*(e_2)$	x_1^*
10	2.3	2.25	2.2	2.15	2.3	1

Solution: a profit of \$2.3 thousand is obtained by making the following investment:

	Thousands of \$ invested
Fund 1	1
Fund 2	4
Fund 3	3
Fund 4	2

9.23 Marketing Mix in a Mobile Phone Company

The firm iMobile, a manufacturer of mobile phones, is planning its 3-year marketing policy and it is considering moving from the mixed strategy it is currently using to targeted strategies. It can choose from among four targeted strategies: (1) low prices; (2) design; (3) novelties; (4) good technical specifications.

Table 9.16 provides the annual profits (in millions of dollars) that each strategy is estimated to generate in every year of the 3-year plan.

iMobile can decide to change strategy at the beginning of every year. Each change in strategy generates a cost of \$3 million.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made
(b) Solve the problem

Solution

Table 9.16 The annual profits forecast for each strategy

Strategy	Profit for year 1	Profit for year 2	Profit for year 3
1	8	8	3
2	9	5	4
3	7	7	7
4	4	7	8

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made

Phase = Year, N = 3
Decision = x_n = active strategy in year n
State = en = a strategy that continues throughout year $n-1$
Recursive function: $f_n(e_n, x_n) = f^*_{n+1}(e_{n+1}) + c_n(e_n, x_n)$
$e_{n+1} = x_n$
$c_n(e_n, x_n)$ = Profit(x_n) − CostChange(x_n, e_n)

The possible states and feasible decisions in each stage:

N	e_n	x_n
1	0	1, 2, 3, 4
2	1, 2, 3, 4	1, 2, 3, 4
3	1, 2, 3, 4	1, 2, 3, 4

(b) Solve the problem

$n = 3$

e_3	$x_3 = 1$	$x_3 = 2$	$x_3 = 3$	$x_3 = 4$	$f_3^*(e_3)$	x_3^*
1	3	1	4	5	5	4
2	0	4	4	5	5	4
3	0	1	7	5	7	3
4	0	1	4	8	8	4

$n = 2$

e_2	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_2^*(e_2)$	x_2^*
1	13	7	11	12	13	1
2	10	10	11	12	12	4
3	10	7	14	12	14	3
4	10	7	11	15	15	4

$n = 1$

e_1	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$f_1^*(e_1)$	x_1^*
0	18	18	18	16	18	1–2–3

There are three possible solutions giving a maximum profit of 18:
Solution 1: $1 - 1 - 4$
Solution 2: $2 - 4 - 4$
Solution 3: $3 - 3 - 3$.

9.24 Location of Production Plants

A firm is considering a location problem for its production plant over the next 3 years. There are four feasible locations where there are adequate premises in each one to carry out the firm's production operations. Table 9.17 shows the annual renting costs (thousands of dollars) of all four premises for the next 3 years.

Preparing a new plant at the beginning of the year costs \$2,000 and closing it at the end of the year costs \$1,000. Annual maintenance costs are cited in \$10,000. Currently, the firm has no production plant operating.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made
(b) Solve the problem

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made

Phase = Year, $N = 3$
Decision = x_n = warehouse that remains open throughout year n;

Table 9.17 Annual costs for renting premises

Plant	Cost in year 1	Cost in year 2	Cost in year 3
1	1	8	8
2	1	9	7
3	8	2	8
4	7	2	6

State = e_n = warehouse that has opened in year $n - 1$;
Recursive function: $f_n(e_n, x_n) = f_{n+1}{}^*(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = x_n$;
$c_n(e_n, x_n)$ = CostToRent(x_n) + CostChange(x_n, e_n);
CostChange (x_n, e_n) = CostClosure(x_n, e_n) + CostOpening(x_n, e_n)
The possible states and feasible decisions in each stage:

No.	e_n	x_n
1	0	1, 2, 3, 4
2	1, 2, 3, 4	1, 2, 3, 4
3	1, 2, 3, 4	1, 2, 3, 4

(b) Solve the problem

$n = 3$

e_3	$x_3 = 1$	$x_3 = 2$	$x_3 = 3$	$x_3 = 4$	$f_3{}^*(e_3)$	$x_3{}^*$
1	8	10	11	9	8	1
2	11	7	11	9	7	2
3	11	10	8	9	8	3
4	11	10	11	6	6	4

$n = 2$

e_2	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_2{}^*(e_2)$	$x_2{}^*$
1	16	19	13	11	11	4
2	19	16	13	11	11	4
3	19	19	10	11	10	3
4	19	19	13	8	8	4

$n = 1$

e_1	$x_1 = 1$	$x_1 = 2$	$x_1 = 3$	$x_1 = 4$	$f_2{}^*(e_2)$	$x_1{}^*$
0	14	14	20	17	14	1 o 2

There are two possible solutions:
Solution 1: 1 − 4 − 4
Solution 2: 2 − 4 − 4

9.25 Reducing Time to Develop Software

The firm ALCSOFT develops software for telecommunication companies. It is currently working on developing software for which 54 weeks of work have been estimated. Its customer is willing to increase the project budget by \$50,000 if ALCSOFT manages to cut this time by at least 30 % with this increase. The manager of the project in question has analysed the possibilities of cutting the time to develop the software according to the increase in expense in the three fundamental project resources: personnel, software and hardware. The Project Manager has concluded that for every \$10,000 invested in personnel, the time is cut by 5 weeks, and if the same amount is invested in software, it is cut by 3 weeks, and if it is invested in hardware, the time is reduced by 2 weeks. Nonetheless, according to the current combination of resources, it will not be possible to invest more than \$20,000 in personnel, and the same limit applies for software.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how the e_{n+1} is calculated and what the possible states in each phase and the feasible decisions to be made are
(b) Solve the problem.

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how the e_{n+1} is calculated and what the possible states in each phase and the feasible decisions to be made are

Phase = Resources, $N = 3$
Decision = x_n = money to invest in resource n;
State = e_n = money remaining at the beginning of stage n;
Recursive function: $f_n(e_n, x_n) = f_{n+1}{}^*(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = e_n - x_n$;
$c_n(e_n, x_n)$ = weeks saved when investing x_n money in resource n.
The possible states and feasible decisions in each stage:

No.	e_n	x_n
1	50	0, 10, 20
2	30, 40, 50	0, 10, 20
3	0, 10, 20, 30, 40, 50	0, 10, 20, 30, 40, 50

(b) Solve the problem

$n = 3$

e_3	$f_3{}^*$	$x_3{}^*$
0	0	0
10	0	10
20	0	20
30	0	30
40	0	40
50	0	50

$n = 2$

e_2	$x_2 = 0$	$x_2 = 10$	$x_2 = 20$	$f_2{}^*$	$x_2{}^*$
30	6	7	8	8	20
40	–	9	10	10	20
50	–	–	12	12	20

$n = 1$

e_1	$x_1 = 0$	$x_1 = 10$	$x_1 = 20$	$f_1{}^*$	$x_1{}^*$
50	12	15	18	18	20

2 Solution

Personnel 20
Software 20
Hardware 10
Total weeks saved 18.

9.26 Planning Holidays

The Simsons are planning their summer holidays when they will spend 7 days visiting three European cities: Rome, Paris and Berlin. Liza has studied the characteristics of each city (and each family member's preferences) and has given a satisfaction score that the family obtains depending on the days they stay in each city (see Table 9.18). They all agree that they want to visit these three cities. The minimum number of visiting days is 1, which includes air travel.

Table 9.18 Satisfaction values per city and days stayed

Days	Satisfaction		
	Rome	Paris	Berlin
1	1	1	1
2	2	4	3
3	3	6	3
4	4	8	2
5	5	8	1
6	4	6	1
7	3	5	1

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made

(b) Solve the problem by obtaining the days used to visit each city so that family satisfaction is maximised.

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made

Phase = City, $N = 3$
Decision = x_n = no. of days to remain in city n;
State = e_n = no. of remaining days to use;
Recursive function: $f_n(e_n, x_n) = f_{n+1}{}^*(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = e_n - x_n$;
$c_n(e_n, x_n)$ = Satisfaction according to the table.
The possible states and feasible decisions in each stage:

No.	e_n	x_n
a	7	1, 2, 3, 4, 5
b	6, 5, 4, 3, 2	1, 2, 3, 4, 5
c	5, 4, 3, 2, 1	1, 2, 3, 4, 5

(b) Solve the problem by obtaining the days used to visit each city so that family satisfaction is maximised

$n = \mathrm{c}$

e_c	f_c*	x_c*
1	1	1
2	3	2
3	3	2 o 3
4	3	2 o 3
5	3	2 o 3

$n = \mathrm{b}$

e_b	$x_b = 1$	$x_b = 2$	$x_b = 3$	$x_b = 4$	$x_b = 5$	f_b*	x_b*
2	2	–	–	–	–	2	1
3	4	5	–	–	–	5	2
4	4	7	7	–	–	7	2, 3
5	4	7	9	9	–	9	3, 4
6	4	7	9	11	9	11	4

$n = \mathrm{a}$

e_a	$x_a = 1$	$x_a = 2$	$x_a = 3$	$x_a = 4$	$x_a = 5$	f_a*	x_a*
7	12	11	10	9	7	12	1

Solution

a = 1
b = 4
c = 2
Satisfaction = 12.

9.27 Personnel Planning in Industry with Seasonal Demand

The labour requirements in each season vary vastly with strong seasonal demand in industry, as Table 9.19 shows:

If labour availabilities exceed requirements, a cost of $120 per surplus operator is estimated, whereas if there are fewer operators than requirements, the costs in

Table 9.19 Operators required

	Spring	Summer	Autumn	Winter
Requirements (operators)	6	4	5	3

overtime and special measures equal 100$ per lacking operator. Changing the personnel level when moving from one season to another represents a cost of $90 for every contracted operator, and of $80 for each operator dismissed. Last Winter, there were two operators.

The ideal situation sought is that there is never more than one excess operator or one operator short. Besides, contracting or dismissing more than three operators during one 3-monthly period goes against the company policy.

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made
(b) Solve the problem.

Solution

(a) Consider this problem with dynamic programming to describe the phase, the state, the decision variable and the recursive function. Specify how e_{n+1} is calculated and what the possible states are in each phase, as well as the feasible decisions that can be made

Phase = 3-monthly period $N = 4$
Decision = x_n = operators to be contracted;
State = e_n = operators on the payroll;
Recursive function: $f_n(e_n, x_n) = f_{n+1}{}^*(e_{n+1}) + c_n(e_n, x_n)$;
$e_{n+1} = e_n + x_n$;

$$c_n(e_n,\ x_n) = co_n(e_n) + cv_n(x_n);$$

$$co_n(e_n) = \begin{cases} (e_n - N_n) * 120, e_n \geq N_n \\ (N_n - n_n) * 100, e_n < N_n \end{cases}$$

$$cv_n(x_n) = \begin{cases} x_n * 90, x_n \geq 0 \\ x_n * 80, e_n < 0 \end{cases}$$

(b) Solve the problem.

$n = 4$

e_4	$x_4 = -3$	$x_4 = -2$	$x_4 = -1$	$x_4 = 0$	$f_4^*(e_4)$	x_4^*
4	–	260	80	120	80	−1
5	340	160	200	–	160	−2
6	240	280	–	–	240	−3

$n = 3$

e_3	$x_3 = -1$	$x_3 = 0$	$x_3 = 1$	$x_3 = 2$	$x_3 = 3$	$f_3^*(e_3)$	x_3^*
3	–	–	270	340	630	270	1
4	–	180	250	540	–	180	0
5	260	160	450	–	–	160	0

$n = 2$

e_2	$x_2 = -3$	$x_2 = -2$	$x_2 = -1$	$x_2 = 0$	$f_2^*(e_2)$	x_2^*
5	–	530	260	280	260	−1
6	610	340	360	–	340	−2

$n = 1$

e_1	$x_1 = 2$	$x_1 = 3$	$f_1^*(e_1)$	x_1^*
3	540	610	540	2

Final solution :
$x_1^* = 2$
$x_2^* = -1$
$x_3^* = 0$
$x_4^* = -1$
Total cost = 540.

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Chapter 10
Markov Processes

Abstract This chapter begins with an introduction to Markov chains in which different calculations to characterise and analyse a system which has been modelled by a Markov chain are described. Then a varied set of Markov chain problems is proposed and their corresponding solutions are provided. The objective of this chapter is to provide a better understanding of modelling stochastic systems with Markov chains. Problems are proposed in which the several step transition probabilities, long-term stationary probabilities, first passage times and mean operation costs should be calculated to be able to compare the various system configuration options to select the most suitable.

10.1 Introduction

A Markov chain is a special type of stochastic process in which a system alters according to the different possible states with transition stability probabilities, which do not depend on the state in which the system is found at any time. This memorylessness characteristic is known as a Markov property (Markov 1971). Modelling a system by means of a Markov chain allows us to analyse its performance. Markov chains do not permit the calculation of the optimal decision, but allow instead easy comparisons of various alternatives (Howard 1960).

A Markov chain is a sequence of aleatory variables $\{X_t\} = X_1, X_2, X_3, \ldots$ that fulfil the Markov property, which can be expressed as follows:

$$P\left(X_{t+1} = j / X_0 = k_0, X_1 = k_1, \ldots, X_t = i\right) = P\left(X_{t+1} = j / X_t = i\right)$$

In other words, the probability that the system is in state j in instant $t + 1$ only depends on state i in which the system was in the previous instant t, and does not depend on the states in which it was before. This probability is known as a one-step transition, and if this probability does not change with time, it can be considered stationary:

DOI: 10.1007/978-1-4471-5577-5_10,

$$P\left(X_{t+1} = j/_{X_t = i}\right) = P\left(X_1 = j/_{X_0 = i}\right) \forall t = 0, 1, \ldots$$

The following compact notation tends to be used for one-step transition probabilities:

$$p_{ij} = P\left(X_{t+1} = j/_{X_t = i}\right)$$

In general, n step transition probabilities are denoted as:

$$p_{ij}^{(n)} = P\left(X_{t+n} = j/_{X_t = i}\right)$$

It is necessary to calculate all the one-step transition probabilities between each state by a Markov chain for a system model. The matrix notation facilitates modelling and subsequent calculations; for example, the one-step transition probabilities of a Markov chain with three states $\{s_1, s_2, s_3\}$ can be expressed by this matrix:

$$P^{(1)} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

Given a probabilities vector with which the system is found in all the states in instant t, the probabilities in $t + 1$ are calculated by applying the one-step probabilities matrix:

$$v_{t+1} = v_t \cdot P^{(1)}$$

Chapman-Kolmogorov equations allow the calculation of n-step transition probabilities by means of the matrix product; that is, to calculate the two-step transition probabilities, we generate the following product:

$$P^{(2)} = P^{(1)} \cdot P^{(1)}$$

In general to calculate n-step transition probabilities:

$$P^{(n)} = \left(P^{(1)}\right)^n$$

So based on the probabilities that the system is found to be in all the states in instant t, the probabilities in $t + n$ are calculated by applying the n-step probabilities matrix:

$$v_{t+n} = v_t \cdot P^{(n)}$$

When n is large enough, the transition probabilities stabilise so that the probability that the system is found in a given state after many steps does not depend on the state in which it began. These probabilities are called stationary probabilities:

$$\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$$

Stationary probabilities are calculated by the following equations system:

$$\pi_j = \sum_{i=1}^{M} \pi_i \cdot p_{ij};\ j = 0, 1, \ldots, M$$

$$\sum_{j=1}^{M} \pi_j = 1$$

where M is the total number of Markov chain states. Note that the first equation gives way to M equations, of which $M - 1$ are linearly independent. So one of them is left over, which is substituted for the second equation, which determines that the stationary probabilities must sum 1.

In the matrix notation, the first equation is as follows:

$$\pi = \pi \cdot P^{(1)}$$

What this means is that the probabilities vector of the different states must be such that it is not affected by the application of the one-step transition probabilities matrix.

Stationary probabilities are an interesting characteristic of a system which can be modelled by a Markov chain because it allows us to know the percentage of time that the system is in each state. If each state means a different cost (or profit), the mean cost (or profit) can be easily calculated by:

$$\bar{C} = \sum_{j=1}^{M} \pi_j \cdot C(j)$$

The mean first passage times are another interesting long-term property in a Markov chain because they inform about the steps that must be taken on average to move from one state to another. These times are calculated by this equation:

$$\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \cdot \mu_{kj}$$

When developing the equation, we realise that to calculate μ_{ij}, it is necessary to know μ_{kj}, so equations are added until an equations system is formed which suffices to calculate the first passage time from i to j, and some others.

The basic concepts described in this introduction can be extended in several references as so: Markov games (Van Der Wal 1981) an extension of the Games Theory to Markov chains environments; stochastic games (Owen 1982); sequential decisions (Derman 1962); statistical methods (Billingsley 1961; absorbing markov chains (Novotny, 1995)).

After reading this chapter, readers should be able to: understand the nature of stochastic systems that can be modelled by a Markov chain; calculate the one-step transition probabilities between the various system states; know the different calculation formulae of the several step transition probabilities, stationary probabilities and the mean first passage times; calculate the mean system operation costs; employ the analyses which provide Markov chains for decision making.

Selected books for further reading can be found in the References section.

10.2 How Social Classes Evolve

Sometimes sociologists consider that the social mobility of successive generations can be modelled like a Markov chain: it is assumed that a son's occupation depends only on his father's occupation and not on his grandfather's occupation. Let us assume that 1,000 people are being interviewed of whom 50 % belong to the low social class and 20 % to the high social class, that the model is appropriate and that the social mobility between one generation and the next is determined as indicated below:

- The low social class retains 40 % of the study population, and loses 50 % in favour of the middle class and 10 % in favour of the high class.
- The middle class retains 70 % of the population and loses 25 % in favour of the high class and 5 % in favour of the low class.
- The high class retains 45 % of the population and loses 50 % in favour of the middle class and 5 % in favour of the low class.

(a) After two generations, what proportion of people is likely to be retained in each social class?
(b) Could the middle class disappear in the long term? Explain and justify your answer with calculations.
(c) If someone belongs to the low social class, how many generations (average value) have to pass before this person belongs to the middle social class for the first time?

Solution

(a) After two generations, what proportion of people is likely to be retained in each social class?

If we take:

s_1 = the proportion of people belonging to the low class.
s_2 = the proportion of people belonging to the middle class.
s_3 = the proportion of people belonging to the high class.

The one-step transition probabilities matrix

	s_1	s_2	s_3
s_1	0.4	0.5	0.1
s_2	0.05	0.7	0.25
s_3	0.05	0.5	0.45

The two-step transition probabilities matrix

	s_1	s_2	s_3
s_1	0.19	0.6	0.21
s_2	0.0675	0.64	0.2925
s_3	0.0675	0.6	0.3325

After two generations:

The low class will retain 19 %.
The middle class will retain 64 %.
The high class will retain 33.25 %.

(b) Could the middle class disappear in the long term? Explain and justify your answer with calculations.

This is an irreducible Markov chain because for $n = 1$, all the states interrelate. All the states of an irreducible Markov chain are recurrent; that is, the probability of returning to state i from itself at some time is 1. Therefore, it is not possible that the middle class disappears in the long term.

Stationary transition probabilities:

$\pi_1 = 0.0769231$
$\pi_2 = 0.625$
$\pi_3 = 0.298077$

The probability of belonging to the middle class (e_2) in the long term is 62.5 %.

(c) If someone belongs to the low social class, how many generations (average value) have to pass before this person belongs to the middle social class for the first time?

The expected first passage times are:

	s_1	s_2	s_3
s_1	13	2	5.161
s_2	20	1.6	4.194
s_3	20	2	3.355

Two generations must pass for someone who belongs to the low social class (s_1) to move on to belong to the middle social class (s_2) for the first time.

10.3 Replacing a Car

At the beginning of each year, my car is in good, regular or bad state. There is a 90 % probability that a good car will be in a good state at the beginning of next year, a 5 % probability that it will be in a regular state, and a 5 % probability that it will be in bad state. There is a 70 % probability that a car in a regular state will continue to be in a regular state at the beginning of next year, and a 30 % probability that it will be in a bad state. It costs \$12,000 to buy a good car, but a regular one can be found for \$5,000. A bad car has no sale value and must be immediately replaced with a good one. It costs \$1,000 a year for a good car and \$2,000 for a regular car to function properly. Let us assume that the cost to keep a car functioning properly throughout 1 year depends on the type of car that one has at the *beginning* of the year (after any new car arriving, if this were the case).

Let us assume that the bad state occurs only at the *end* of a year, and then (at the beginning of the next year) the car in a bad state "must be replaced immediately".

Must I replace my car as soon as it is in a regular state, or must I wait until it is in a bad state? To overcome this question:

(a) Define the different Markov chains with the three states (Good, Regular and Bad at the *beginning* of the year) for each replacement policy (Policy A: Replace the car when it is in a bad state; Policy B: Replace the car when it is in a regular state).
(b) Obtain the stationary probabilities.
(c) Determine the mean cost per year for each replacement policy.
(d) What replacement policy do you recommend?

Solution

(a) Define the different Markov chains with the three states (good, regular and bad at the *beginning* of the year) for each replacement policy (Policy A: Replace the car when it is in a bad state; Policy B: Replace the car when it is in a regular state).

Given the following states:

s_1 = The car is in a good state at the end of the year
s_2 = The car is in a regular state at the end of the year
s_3 = The car is in a bad state at the end of the year

The following policies are evaluated:

Policy A: Replace the car when it is in a bad state.

The one-step transition probabilities matrix

	s_1	s_2	s_3
s_1	0.9	0.05	0.05
s_2	0	0.7	0.3
s_3	0.9	0.05	0.05

Policy B: Replace the car when it is in a regular state.
The one-step transition probabilities matrix

	s_1	s_2	s_3
s_1	0.9	0.05	0.05
s_2	0.9	0.05	0.05
s_3	0.9	0.05	0.05

(a) Obtain the stationary probabilities.

Policy A: Replace the car when it is in a bad state.
The stationary transition probabilities

$\pi_1 = 0.771429$
$\pi_2 = 0.142857$
$\pi_3 = 0.0857143$

Policy B: Replace the car when it is in a regular state.
The stationary transition probabilities

$\pi_1 = 0.9$
$\pi_2 = 0.05$
$\pi_3 = 0.05$

(b) Determine the mean cost per year for each replacement policy.

Policy A: Replace the car when it is in a bad state.

i	State	p_i	C_i	$p_i x C_i$
1	Good	0.7714	1000	771.4
2	Regular	0.1429	2000	285.7
3	Bad	0.08571	13000	1114

The mean cost per year is \$2,171.
Policy B: Replace the car when it is in a regular state.
The mean cost per year is \$1,950.

i	State	p_i	C_i	$p_i x C_i$
1	Good	0.9	1000	900
2	Regular	0.05	8000	400
3	Bad	0.05	13000	650

(c) What replacement policy do you recommend?

The "Replace the car when it is in a regular state" policy as it has a lower cost than the "Replace the car when it is in a bad state" policy. The difference is $221 a year.

10.4 Forecasting Evaluation Tasks for Subject Matters

A Business Studies teacher, not wishing to be predictable, decides to send works to her pupils for their continuous evaluation, based on probabilities. On the first day of class, she drew Table 10.1 on the blackboard to indicate to students that they should expect an unannounced end-of-term exam (E), teacher collects proposed problems (R) or no type of continuous evaluation (N).

(a) If students were presented with an unannounced end-of-term exam today, what would the probability be of them having another unannounced end-of-term exam tomorrow?
(b) Today is Wednesday and the teacher collects students' proposed problems. What is the probability that they have no evaluation task type on Friday?
(c) If students have had an unannounced end-of-term exam today, what is the probability of them having another unannounced end-of-term exam on the next two days?
(d) Obtain the matrix in which the initial transition matrix would concur after a considerable number of days. Explain the meaning of the matrix obtained.
(e) How much time as a mean value must pass for the students who have had an unannounced end-of-term exam to not be presented with another evaluation type task? And how much time must pass until they have another unannounced end-of-term exam?

Table 10.1 Probabilities of evaluation task types

	E	R	N
E	0.4	0.35	0.25
R	0.45	0.4	0.15
N	0.8	0.15	0.05

Solution

(a) If students were presented with an unannounced end-of-term exam today, what would the probability be of them having another unannounced end-of-term exam tomorrow?

Based on the initial transition probabilities matrix, the probability is 0.4.

(b) Today is Wednesday and the teacher collects students' proposed problems. What is the probability that they have no evaluation task type on Friday?

Given that s_1 = unannounced end-of-term exam (E); s_2 = teacher collects proposed problems (R); and s_3 = No continuous evaluation-type task (N), and based on the initial two-step transition probabilities matrix, obtained by multiplying the initial matrix by itself, we obtain:

	s_1	s_2	s_3
s_1	0.5175	0.3175	0.165
s_2	0.48	0.34	0.18
s_3	0.4275	0.3475	0.225

The obtained probability is 0.18.

(c) If students have had an unannounced end-of-term exam today, what is the probability of them having another unannounced end-of-term exam on the next two days?

The probability of having an unannounced end-of-term exam twice on the run is 0.4. The probability of there being an unannounced end-of-term exam, then the teacher collects proposed problems, and next another unannounced end-of-term exam is $0.35 \cdot 0.45 = 0.1575$. The probability of there being an unannounced end-of-term exam, then no exam and afterwards another unannounced end-of-term exam is $0.25 \cdot 0.8 = 0.2$. Therefore, the requested probability is 0.7575.

(d) Obtain the matrix in which the initial transition matrix would concur after a considerable number of days. Explain the meaning of the matrix obtained.

	E	R	N
E	0.49	0.33	0.18
R	0.49	0.33	0.18
N	0.49	0.33	0.18

This matrix indicates that if we are thinking about the future (in 2 week's time, or more), it does not matter what today's evaluation type has been for there to be a 49 % probability of having an unannounced end-of-term exam, 33 % probability of collecting proposed problems and an 18 % probability of having no type of continuous evaluation.

(e) How much time as a mean value must pass for the students who have had an unannounced end-of-term exam to not be presented with another evaluation type task? And how much time must pass until they have another unannounced end-of-term exam?

Using the expected first passages times matrix:

	s_1	s_2	s_3
s_1	2.046	3.243	4.691
s_2	2.009	3.027	5.185
s_3	1.37	3.784	5.531

We can see that 4.691 days must pass for students who have had an unannounced end-of-term exam, have no type of continuous evaluation, but 2.046 days must pass for these students to have another unannounced end-of-term exam.

10.5 Market Research in Ice-Cream Parlours

A Business Studies student at EPSA is interested in analysing the impact on the market and the loyalty of the customers of La Jijonenca (LJ) and La Rubia (LR), two competing ice-cream parlours in Alicante. It is assumed that one customer consumes once a week at La Jijonenca or at La Rubia, but not at both.

The student has collected information from 100 consumers over a 10-week period. When checking the data, it is discovered that of all the customers who consumed at La Jijonenca in a given week, 90 % returned to La Jijonenca the following week, while 10 % switched to La Rubia. Of all the consumers who consumed at La Rubia in a given week, 80 % returned to La Rubia the following week, but 20 % changed to La Jijonenca.

By assuming that the transition probabilities are the same for any customer and that they do not change with time, it is a Markov process:

(a) If the last consumption of a customer (weekly period 0) was at La Jijonenca, determine the probability of this customer returning to La Jijonenca during the following period (weekly period 1).
(b) What is the probability that one customer goes to La Jijonenca 2 weeks running?
(c) What is the probability that one customer changes to La Rubia after a first consumption at La Jijonenca and then returns to La Jijonenca?
(d) Consider the customer in problem a), what is the probability that this customer purchases at La Jijonenca after two weekly periods (weekly period 2)? And what is it if the customer buys at La Rubia?
(e) What market impact will La Nueva (a new third ice-cream parlour) achieve?

(f) Based on 1,000 consumers, the original Markov process used in questions (a), (b), (c) and (d) implies 667 weekly consumptions at La Jijonenca and 333 at La Rubia, What impact will La Nueva have on the consumers going to La Jijonenca and La Rubia? Explain your results.

Solution

(a) If the last consumption of a customer (weekly period 0) was at La Jijonenca, determine the probability of this customer returning to La Jijonenca during the following period (weekly period 1).

The initial transition probabilities matrix is:

		La Jijonenca	La Rubia
		s_1	s_2
La Jijonenca	s_1	0.9	0.1
La Rubia	s_2	0.2	0.8

Based on the initial transition probabilities matrix, 90 % of those who buy at the La Jijonenca return during the following period.

(b) What is the probability that one customer goes to La Jijonenca 2 weeks running?

The probability is $0.9 \cdot 0.9 = 0.81$.

(c) What is the probability that one customer changes to La Rubia after a first consumption at La Jijonenca and then returns to La Jijonenca?

The probability of a customer purchasing at La Jijonenca and then going to La Rubia is 0.1, and of then buying at La Jijonenca is de 0.2. Therefore, the requested probability is $0.1 \cdot 0.2 = 0.02$.

(d) Consider the customer in question a), what is the probability that this customer purchases at La Jijonenca after two weekly periods (weekly period 2)? And what is it if the customer buys at La Rubia?

Based on the two-step transition probabilities matrix:

	s_1	s_2
s_1	0.83	0.17
s_2	0.34	0.66

It can be deduced that the probability of a customer buying at La Jijonenca after two weekly periods is 0.83, and 0.66 at La Rubia.

Let us assume that a third ice-cream parlour, La Nueva (LN), starts to compete with the two already existing ice-cream parlours in Alicante. La Nueva has one competitive advantage in that it serves bigger ice-creams than those sold at the La

Jijonenca and La Rubia. So it expects to attract some consumers who currently buy their weekly ice-creams at La Jijonenca or at La Rubia. By assuming that the transaction probabilities are those shown below:

		La Jijonenca	La Rubia	La Nueva
		s_1	s_2	s_2
La Jijonenca	s_1	0.85	0.1	0.05
La Rubia	s_2	0.2	0.75	0.05
La Nueva	s_3	0.15	0.1	0.75

(e) What market impact will La Nueva achieve?

To determine this, the stationary probabilities are calculated as so:

$\pi_1 = 0.548$
$\pi_2 = 0.286$
$\pi_3 = 0.166$
16.6 % as a result of π_3.

(f) Based on 1,000 consumers, the original Markov process used in questions (a), (b), (c) and (d) implies 667 weekly consumptions at La Jijonenca and 333 at La Rubia, What impact will La Nueva have on the consumers going to La Jijonenca and La Rubia? Explain your results.

La Nueva must consider:

$667 - 0.548 \cdot (1000) = 119$ La Jijonenca customers
$333 - 0.286 \cdot (1000) = 47$ La Rubia customers
Total: 166 La Nueva customers, who will be taken from La Jijonenca and La Rubia.

10.6 Evolution of Breakfast Cereal Purchases

A retailer provides three breakfast cereal brand names. After analysing purchasing records, this retailer discovers that every week: *Kallogs* cereals lose 12 % of its customers who change to *Cranch* cereals, and that 19 % purchase *Pascal* cereals. *Cranch* cereals lose 16 % of its customers who buy *Kallogs* cereals, and 10 % of its customers who buy *Pascal* cereals; *Pascal* cereals lose 20 % of its customers who buy *Kallogs* and 14 % who purchase *Cranch*.

Let us assume that this week 1,500 people buy *Kallogs*, 1,400 consume *Cranch* and 2,000 purchase *Pascal*.

(a) How many people will buy *Kallogs, Cranch* and *Pascal* next week?
(b) How many customers will buy *Pascal* cereals in 2 week's time?

(c) After a given time, what proportion of people will buy the *Kallogs*, *Cranch* and *Pascal* cereals?
(d) On average, what time must pass for one person who bought *Kallogs* to buy this brand name again? And on average, what how long before he/she buys *Pascal*?

Solution

(a) How many people will buy *Kallogs, Cranch* and *Pascal* next week?

The analysed Markov chain transition matrix is:

	Kallogs	*Cranch*	*Pascal*
Kallogs	0.69	0.12	0.19
Cranch	0.16	0.74	0.10
Pascal	0.20	0.14	0.66

It is an irreducible Markov chain. So:
$(0.69{\cdot}1500) + (0.16{\cdot}1400) + (0.2{\cdot}2000) = 1659$ people will buy *Kallogs*
$(0.12{\cdot}1500) + (0.74{\cdot}1400) + (0.14{\cdot}2000) = 1496$ people will buy *Cranch*
$(0.19{\cdot}1500) + (0.10{\cdot}1400) + (0.66{\cdot}2000) = 1745$ people will buy *Pascal*

(b) How many customers will buy *Pascal* cereals in 2 week's time?

Given that s_1 = Buy *Kallogs,* s_2 = Buy *Cranch,* and s_3 = Buy *Pascal,* and based on the two-step transition probabilities matrix, which is obtained by multiplying the initial matrix by itself, we find that:
The two-step transition probabilities matrix is:

	s_1	s_2	s_3
s_1	0.5333	0.1982	0.2685
s_2	0.2488	0.5808	0.1704
s_3	0.2924	0.22	0.4876

$(0.2685 \cdot 1500) + (0.1704 \cdot 1400) + (0.4876 \cdot 2000) = 1617$ people will buy *Pascal* within 2 week's time

(c) After a given time, what proportion of people will buy the *Kallogs*, *Cranch* and *Pascal* cereals?

The stationary transition probabilities are:

$\pi_1 = 0.366142$ (will buy *Kallogs*)
$\pi_2 = 0.331693$ (will buy *Crunch*)
$\pi_3 = 0.302165$ (will buy *Pascal*)

(d) On average, what time must pass for one person who bought *Kallogs* to buy this brand name again? And on average, what how long before he/she buys *Pascal*?

The expected first passage times are:

	s_1	s_2	s_3
s_1	2.731	7.864	6.189
s_2	5.914	3.015	7.655
s_3	5.376	7.567	3.309

The average time that must pass for one person who bought *Kallogs* cereals to buy them again is 2.731 weeks, while it is 6.189 weeks for *Pascal* cereals.

10.7 Car Insurance

The Straight Line insurance company charges its customers according to their accidents record. A customer who had no accident in the past 2 years pays \$100 for an annual premium. Whoever has had an accident in each of the past 2 years pays \$400 for an annual premium. The people who have had an accident in only one of the past 2 years pay \$300. A customer who had an accident in the last year has a 10 % probability of having an accident this year. If a customer had no accident in the last year, he/she has a 3 % probability of having an accident this year.

(a) In a given year, what is the average premium that a Straight Line customer pays?
(b) What is the probability that a customer who had no accidents in the past 2 years reports at least one accident in the next 2 years?
(c) What is the average time that passes since one customer who had an accident in each of the past 2 years pays an annual premium of \$100?

Solution

(a) In a given year, what is the average premium that a Straight Line customer pays?

	$s_1 = 00$	$s_2 = 01$	$s_3 = 10$	$s_4 = 11$
$s_1 = 00$	0.97	0.03	0	0
$s_2 = 01$	0	0	0.9	0.1
$s_3 = 10$	0.97	0.03	0	0
$s_4 = 11$	0	0	0.9	0.1

where

s_1 represents the customer who had no accident in the past 2 years.
s_2 represents the customer who had an accident in the last year.
s_3 represents the customer who had an accident in the last year but one.
s_4 represents the customer who had an accident in each of the past 2 years.

It is an irreducible Markov chain.
The stationary transition probabilities are:

$\pi_1 = 0.93871$
$\pi_2 = 0.0290323$
$\pi_3 = 0.0290323$
$\pi_4 = 0.00322581$

The premium that a Straight Line customer pays on average is:
The premium that a Straight Line customer pays on average is:
$0.93871 \cdot (100) + (0.0290323 + 0.0290323) \cdot (300) + (0.00322581) \cdot 400 =$ 112.58 dollars

(b) What is the probability that a customer who had no accidents in the past 2 years reports at least one accident in the next 2 years?

The probability that a customer who had no accidents has one in the next year is 0.03, and the probability that a driver has one in the next 2 years is 0.0291. Therefore, the probability that a customer who had no accidents in the past 2 years reports at least one accident in the next 2 years is 0.0591.

(c) What is the average time that passes since one customer who had an accident in each of the past 2 years pays an annual premium of $100?

The expected first passage times are:

	s_1	s_2	s_3	s_4
s_1	1.065	33.33	34.44	343.3
s_2	2.176	34.44	1.111	310
s_3	1.065	33.33	34.44	343.3
s_4	2.176	34.44	1.111	310

The average time which passes since a customer who had an accident in each of the past 2 years pays an annual premium of $100 is 2.176 years.

10.8 Market Research into Car Purchases

Automotor, a national cars magazine, has conducted some market research into the percentage of firms repeating a purchase with three automobile brand names. It is assumed that the surveyed customers change their car every 6 years. The

Table 10.2 Probabilities of buying cars according to brand name

	Ford	Renault	Citroën
Ford	0.6	0.3	0.1
Renault	0.2	0.8	0
Citroën	0.1	0.2	0.7

simplified results of this study can be considered a Markov process whose transition probabilities matrix is provided in Table 10.2.

(a) What is the probability that a customer who has bought a Ford next buys a Renault and then another Renault?
(b) What is the probability that a customer who has bought a Ford next buys at least another Ford in his/her next two car purchases?
(c) What percentage of customers will buy a Ford in the long term? What is it for Renault? And for Citroën?
(d) What will these percentages be in 6 year's time?
(e) How much time on average must pass before a buyer of a Citroën decides to buy a Ford? And another Citroën?

Solution

(a) What is the probability that a customer who has bought a Ford next buys a Renault and then another Renault?

The probability that someone who has bought a Ford then buys a Renault is 0.3, and the probability that this operation is repeated is 0.8. Thus, the requested probability is $0.3 \cdot 0.8 = 0.24$.

(b) What is the probability that a customer who has bought a Ford next buys at least another Ford in his/her next two car purchases?

The probability that someone who has bought a Ford will do so again is 0.6.

The probability that someone who has bought a Ford, then a Renault and then another Ford is $0.3 \cdot 0.2 = 0.06$. The probability that someone who has bought a Ford, then a Citroën and afterwards a Ford is $0.1 \cdot 0.1 = 0.01$. Hence, the requested probability is $0.6 + 0.06 + 0.01 = 0.67$.

(c) What percentage of customers will buy a Ford in the long term? What is it for Renault? And for Citroën?

The stationary transition probabilities are:

$\pi_1 = 0.315789$
$\pi_2 = 0.578947$
$\pi_3 = 0.105263$

(d) What will these percentages be in 6 year's time?

The percentages in 6 year's time are given by the initial matrix and the initial purchase is taken into account.

(e) How much time on average must pass before a buyer of a Citroën decides to buy a Ford? And another Citroën?

An average of 40.02 years will pass before a buyer of a Citroën decides to purchase a Ford, and this average becomes 57 years before the buyer purchases another Citroën. These values are obtained from the expected first passage times matrix, expressed in 6-year periods, where s1, s2 and e3 represent the purchase of a Ford, a Renault and a Citroën, respectively.

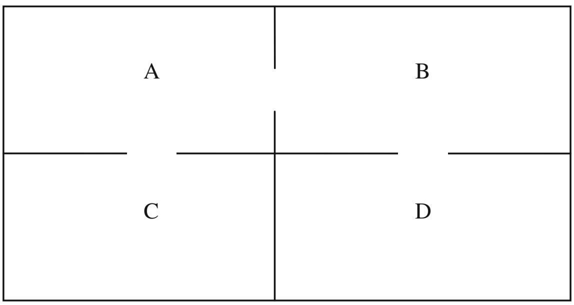
	s_1	s_2	s_3
s_1	3.167	3.636	25
s_2	5	1.727	30
s_3	6.667	4.545	9.5

10.9 Experiment Done with Mice

In a research line with mice, a "labyrinth" is employed with four cells, A, B, C and D, as shown in Fig. 10.1. During each minute, there is a 50 % probability that a mouse remains in the same cell it was in during the previous minute or that it moves to one of the cells adjoining that cell where it was. If there is only one adjacent cell, there is a 50 % probability that the mouse moves to it, while for two adjacent cells the probability is 25 % for each cell.

(a) Find the Markov chain transition matrix which models this system.
(b) If a mouse is found in cell B at a given time, what is the probability that it is in cell D 2 min later?
(c) If a mouse stays inside the labyrinth for a long time, what is the probability of finding it in all the cells?

Fig. 10.1 Labyrinth used in the experiments



Solution

(a) Find the Markov chain transition matrix which models this system.

The set of states is s = {s_1, s_2, s_3, s_4} depending on whether the mouse is found in cell A, B, C or D. The transition matrix is:

	s_1	s_2	s_3	s_4
s_1	0.5	0.25	0.25	0
s_2	0.25	0.5	0	0.25
s_3	0.5	0	0.5	0
s_4	0	0.5	0	0.5

(b) If a mouse is found in cell B at a given time, what is the probability that it is in cell D 2 min later?

The intention of the Chapman-Kolmogorov equations is try to calculate one of the p^2 elements, as so:

$$p_{24}^{(2)} = 0.25 \cdot 0 + 0.5 \cdot 0.25 + 0 \cdot 0 + 0.25 \cdot 0.5 = 0.25$$

(c) If a mouse stays inside the labyrinth for a long time, what is the probability of finding it in all the cells?

As the Markov chain is finite and ergodic, it can be stated that a stationary distribution exists, which is exactly what is requested. To calculate it, the following equations system is considered:

$$\pi_1 = 0.5\pi_1 + 0.25\pi_2 + 0.5\pi_3 + 0\pi_4$$

$$\pi_2 = 0.25\pi_1 + 0.5\pi_2 + 0\pi_3 + 0.5\pi_4$$

$$\pi_3 = 0.25\pi_1 + 0\pi_2 + 0.5\pi_3 + 0\pi_4$$

$$\pi_4 = 0\pi_1 + 0.25\pi_2 + 0\pi_3 + 0.5\pi_4$$

$$1 = \pi_1 + \pi_2 + \pi_3 + \pi_4$$

By solving the linear equations system, the following stationary transition probabilities are obtained:

$\pi_1 = 0.333333$
$\pi_2 = 0.333333$
$\pi_3 = 0.166667$
$\pi_4 = 0.166667$

This is the distribution of the mouse's long-term location probability.

10.10 Evolution of Financial Products

Anne, who has graduated in Business Studies, works for a private banking firm and is trying to model the annual evolution of a set of 150 financial products by means of a Markov chain. The financial products are classified into three categories:

- 80 products are considered appealing: highly profitable in the short term, but uncertain in the mid and long terms.
- 30 products are considered secure: mean estimated profitability for the short, mid and long terms.
- 40 products are considered insecure: their profitability varies, even in the short term.

Anne estimates that an appealing product still has the same probability of remaining appealing the next year or becomes a secure or an insecure product. Conversely, the secure product becomes an appealing one with a ½ probability, whereas its probability of remaining secure is ¼. Insecure products have a 3/5 probability of being considered secure products and of 1/5 of still being considered insecure products.

(a) Find the mean recurrence time for each state of the Markov chain proposed by Anne. Interpret its meaning.
(b) In the long term, how many products will there be of each type? Interpret the result by bearing in mind the initial distribution.
(c) What is the probability that an appealing product continues to be appealing after 2 years?
(d) Determine the mean time required for an insecure product to become an appealing product.

Solution

(a) Find the mean recurrence time for each state of the Markov chain proposed by Anne. Interpret its meaning.

Given the following states:

s_1 represents appealing products.
s_2 represents secure products.
s_3 represents insecure products.

The transition probabilities matrix is as follows:

	s_1	s_2	s_3
s_1	1/3	1/3	1/3
s_2	0.5	0.25	0.25
s_3	0.2	0.6	0.2

The expected first passage times are:

	s_1	s_2	s_3
s_1	2.778	2.428	3.25
s_2	2.333	2.679	3.5
s_3	3	1.857	3.75

A mean of 2.778 years is expected for an appealing product to be once again considered appealing, 2.679 years is needed for a secure product to be once again considered secure, and a time of 3.75 years is required for an insecure product to be once again considered insecure.

(b) In the long term, how many products will there be of each type? Interpret the result by bearing in mind the initial distribution.

The stationary transition probabilities are:

$\pi_1 = 9/25 = 0.36 \rightarrow 150 \cdot \pi_1 = 54$
$\pi_2 = 28/75 = 0.3733 \rightarrow 150 \cdot \pi_2 = 56$
$\pi_3 = 4/15 = 0.2667 \rightarrow 150 \cdot \pi_3 = 40$

In the long term, there are less appealing products and more secure ones. There is the same number of insecure products.

(c) What is the probability that an appealing product continues to be appealing after 2 years?

The probability of going from s_1 to s_1 in 2 steps $= 31/90 = 0.3444$

(d) Determine the mean time required for an insecure product to become an appealing product.

Based on the expected first passage times matrix in section (a), we see that 3 years are required.

10.11 Market Research into Cars for Firms

An automobile manufacturer is investigating the behaviour of the market in which firms buy cars for their fleets. Preliminary research work has revealed that firms buy cars of several brand names, but tend to maintain a percentage assigned to each automobile manufacturer owing to discounts per fleet; that is, they have at least one car of a given brand name. Every year when new models are announced, firms amend the percentages assigned to each brand name.

For the purpose of performing a preliminary analysis, the automobile manufacturer considers that the set of the values of the percentages that these firms assign to their car brand name is discrete, and is: 100, 70, 50 or 20 %. An in-depth study resulted in a 1-year transition probabilities matrix, as shown in Table 10.3.

Table 10.3 A one-year transition probabilities matrix

From %	To %			
	100	70	50	20
100	0.6	?	0.1	0
70	?	0.7	0.3	0
50	0	0.4	0.4	?
20	0	0.2	?	0.3

(a) Fill in the ? in the matrix
(b) What percentage of firms has, in the long term, a percentage assigned to the analysed automobile manufacturer by analysing 100 %? And the percentage for 70 %? And for 50 %? And finally for 20 %?
(c) The current situation shows that 30 % of the firms maintain the assigned percentage of 70 %, 45 % have an assigned percentage of 50 %, and 20 % have an assigned percentage of 20 %. What will these percentages be next year?

Solution

(a) Fill in the ? in the matrix

From %	To %			
	100	70	50	20
100	0.6	0.3	0.1	0
70	0	0.7	0.3	0
50	0	0.4	0.4	0.2
20	0	0.2	0.5	0.3

(b) What percentage of firm has, in the long term, a percentage assigned to the analysed automobile manufacturer by analysing 100 %? And the percentage for 70 %? And for 50 %? And finally for 20 %?

The stationary transition probabilities are:

From %	To %			
	100	70	50	20
100	0	0.5424	0.3559	0.1017
70	0	0.5424	0.3559	0.1017
50	0	0.5424	0.3559	0.1017
20	0	0.5424	0.3559	0.1017

In the long term, 54.24 % will assign 70 % of their fleet, whereas 35.59 and 10.17 % will assign 50 and 20 % of their fleets, respectively.

(c) The current situation shows that 30 % of the firms maintain the assigned percentage of 70 %, 45 % have an assigned percentage of 50 % and 20 % have an assigned percentage of 20 %. What will these percentages be next year?

The probabilities next year will be:

Initial vector			
100	70	50	20
0.05	0.3	0.45	0.2
Vector after 1 year:			
100	70	50	20
0.03	0.445	0.375	0.15

Therefore:

$(0.6 \cdot 0.05) = 0.03 = 3$ % of firms will maintain 100 %
$(0.3 \cdot 0.05) + (0.7 \cdot 0.3) + (0.4 \cdot 0.45) + (0.2 \cdot 0.2) = 0.445 = 44.5$ % of firms will maintain 70 %
$(0.1 \cdot 0.05) + (0.3 \cdot 0.3) + (0.4 \cdot 0.45) + (0.5 \cdot 0.2) = 0.375 = 37.5$ % of firms will maintain 50 %
$(0.2 \cdot 0.45) + (0.3 \cdot 0.2) = 0.15 = 15$ % of firms will maintain 20 %

10.12 Natural Gas Repairs Technician

The work of a repairs technician in a natural gas supply firm involves travelling to the homes of those customers who have a contract maintenance service. The states that describe this technician's activity are: state 0: idle and awaiting a customer call; state 1: travelling to a customer's home; state 2: working on a repair at a customer's home. After carrying out the repair, it is assumed that the technician returns to state 0 immediately. The possible modelling of how these states evolve is given by a Markov chain, as represented in Fig. 10.2.

Each change is the equivalent to a 1-day time period.

The following is requested:

(a) The fraction of the total time spent on the repair task
(b) The first passage time before starting the next repair since the time when the last repair finished.

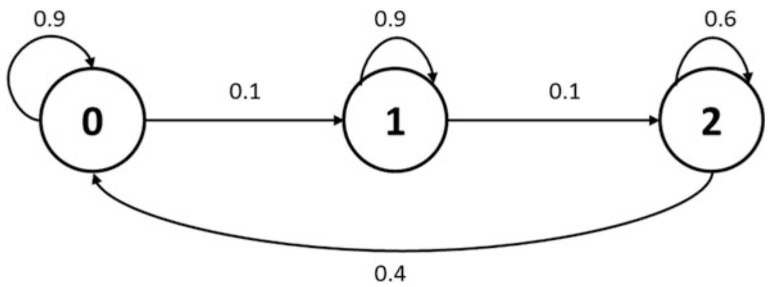


Fig. 10.2 Evolution of the repair technician's states

(c) Given the difficulty of the repair tasks it is known that one in ten technicians leaves a job half finished for other more expert technicians to come, while the first technicians await new customer calls, although a negative mark may appear in their file. Calculate the expected passage time for a new technician who starts to work for the firm to have a negative mark when there is no repair waiting to be done when he/she enters.

Solution

(a) The fraction of the total time spent on the repair task.

Given that the following states are:

s_0 represents idle and awaiting a customer call.
s_1 represents travelling to a customer's home.
s_2 represents working on a repair at a customer's home.

By obtaining the stationary probabilities, we find that $\pi_2 = 0.44$.

(b) The first passage time which passes before starting the next repair since the time when the last repair finished.

By obtaining the first passage times, we find that $\mu_{02} = 20$ days.

(c) Given the difficulty of the repair tasks it is known that one in ten technicians leaves a job half finished for other more expert technicians to come, while the first technicians await new customer calls, although a negative mark may appear in their file. Calculate the expected passage time for a new technician who starts to work for the firm to have a negative mark when there is no repair waiting to be done when he/she enters.

The process is modelled as shown in Fig. 10.3.

Given state s_3, which represents a technician who leaves a repair task half finished, and by obtaining the first passage times, it is determined that $\mu_{03} = 90$ days.

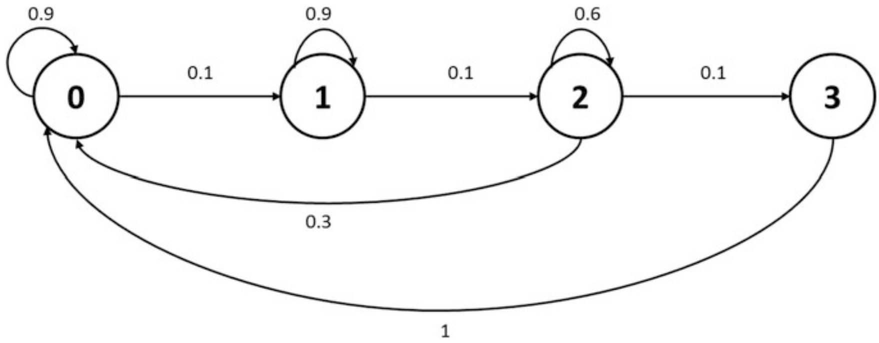


Fig. 10.3 Evolution of the repair technician states

10.13 Weather Forecasts

According to the meteorological statistics of the month of April, Paris never has two sunny days on the run. After one sunny day, the probability is that it is followed by a rainy or a cloudy day. Likewise if it is cloudy (or rainy), the probability of the next day being cloudy (or rainy) is 0.5. If the weather changes however, there is a sunny day only for half the times.

(a) Model this process like a Markov chain. Classify it.
(b) How many rainy days are there in 30 days on average?
(c) If today is rainy, how many days on average must we wait before we have a sunny day?

Solution

(a) Model this process like a Markov chain. Classify it.

It is a three-step Markov chain: Sunny (S or s_1), Rainy (Ll or s_2) and Cloudy (N or s_3). The transition probabilities matrix is defined as:

		S	Ll	N
		s_1	s_2	s_3
S	s_1	0	0.5	0.5
Ll	s_2	0.25	0.5	0.25
N	s_3	0.25	0.25	0.5

It is an irreductible Markov chain.

(b) How many rainy days are there in 30 days on average?

In the long term and, on average, 1 in every 5 days is sunny ($\pi_1 = 0.2$), 2 of every 5 days is rainy ($\pi_2 = 0.4$) and 2 of every 5 days is cloudy ($\pi_3 = 0.4$). Thus in 30 days, 12 days are rainy on average.

(c) If today is rainy, how many days on average must we wait before we have a sunny day?

4 days.

10.14 Students Evolution

Let us assume that, on average, 50 % of the year 1 students move on to year 2 each year, 30 % remain in year 1 and 20 % drop out. Of the year 2 students, 50 % move on to year 3, 40 % remain in year 2 and 10 % drop out. Of the year 3 students, 60 % finish or drop out, while 40 % repeat this academic year. The following is requested:

(a) Write the transition matrix by previously describing the states of the process.
(b) If 600 students start year 1, calculate how many there will be of these 600 in each year at the beginning of year 3.
(c) What states interrelate? Is the Markov chain irreducible? Provide reasons to indicate which states are recurrent and which are transitory.

Solution

(a) Write the transition matrix by previously describing the states of the process.

Four states can be defined:

s_1 year 1 students;
s_2 year 2 students;
s_3 year 3 students;
s_4 students who drop out or finish;

The transition matrix is shown below:

	s_1	s_2	s_3	s_4
s_1	0.3	0.5	0	0.2
s_2	0	0.4	0.5	0.1
s_3	0	0	0.4	0.6
s_4	0	0	0	1

(b) If 600 students start year 1, calculate how many there will be of these 600 in each year at the beginning of year 3.

Fifty-four year 1 students, 210 year 2 students, 150 year 3 students, and 186 will have dropped out.

(c) What states interrelate? Is the Markov chain irreducible? Provide reasons to indicate which states are recurrent and which are transitory.

$\{s_1\}$,$\{s_2\}$, $\{s_3\}$, $\{s_4\}$; only these four states interrelate. There are four classes, so, it is not an irreductible Markov chain. All the states are transitory, except for s_4 which is an absorbent state.

10.15 How a Squirrel Population Evolves

In a mountainous area of Toledo, two classes of squirrels have been sighted: brown and black. At the beginning of each year, which of the following statements is true?:

- There are only brown squirrels in the mountainous area (s_1).
- There are only black squirrels in the mountainous area (s_2).
- There are brown and black squirrels in the mountainous area (s_3).

As the years pass, a transition matrix like that indicated in Table 10.4 has been estimated:

(a) In what fraction of the years will brown squirrels live in the mountains?
(b) In what fraction of the years will black squirrels live in the mountains?
(c) If both squirrel classes are sighted in the same year, how many years on average must pass before they are both spotted again? And how many years must pass to spot only brown squirrels?

Solution

(a) In what fraction of the years will brown squirrels live in the mountains?

By calculating the stationary probabilities, we obtain $\pi_1 = 0.32$

(b) In what fraction of the years will black squirrels live in the mountains?

By calculating the stationary probabilities, we obtain $\pi_2 = 0.26$

(c) If both squirrel classes are sighted in the same year, how many years on average must pass before they are both spotted again? And how many years must pass to spot only brown squirrels?

By calculating the first passage times, 2.375 years must pass before we spot both classes again, and 8.333 years must pass before we spot only brown squirrels.

Table 10.4 The transition matrix of the squirrel population

	s_1	s_2	s_3
s_1	0.7	0.2	0.1
s_2	0.2	0.6	0.2
s_3	0.1	0.1	0.8

10.16 How Trees in a Nature Reserve Evolve

The trees in the Font Roja Nature Reserve have been classified into three age groups: young trees (0–15 years old); adult trees (16–30 years old); old trees (over 30 years old).

The model contemplates the following assumptions:

- A given percentage of trees in each age group dies.
- Surviving trees move on to the next age group; all the old trees remain in the old trees age group.
- Any lost trees are replaced with young trees.
- The total trees population does not change with time.

When considering a 15-year time period, the distribution of age from one period to the next is given by the matrix provided in Table 10.5.

Let us assume a total population of 50,000 trees and that this nature reserve has recently undergone reforestation:

(a) How many trees will there be in each age group after 15 years?
(b) How many trees will there be in each age group after 30 years?
(c) After a long time period, what proportion of trees will there be in each age group?

Solution

(a) How many trees will there be in each age group after 15 years?

Young trees: 50.000·0.1 = 5,000 trees
Adult trees: 50.000·0.9 = 45,000 trees
Old trees: 50.000 ·0 = 0 trees

(b) How many trees will there be in each age group after 30 years?

Young trees: 50.000·0.19 = 9,500 trees
Adult trees: 50.000·0.09 = 4,500 trees
Old trees: 50.000 ·0.72 = 36,000 trees

(c) After a long time period, what proportion of trees will there be in each age group?

Young trees: 0.2326
Adult trees: 0.2093
Old trees: 0.5581

Table 10.5 The transitions matrix according to age of trees

	Young	Adult	Old
Young	0.1	0.9	0
Adult	0.2	0	0.8
Old	0.3	0	0.7

10.17 Study into a Radioactive Element

A laboratory is analysing the probabilities that atoms from a radioactive element become atoms of another element. With every minute that passes, there is a probability of 0.003 that one ^{212}Bi atom is converted into ^{208}Tl, and a probability of 0.007 of it being converted into ^{208}Pb, whereas one ^{208}Tl atom has a probability of 0.2 of being converted into ^{208}Pb (and no probability of being converted into ^{212}Bi). ^{208}Pb is stable.

(a) Consider the described system according to a Markov Chain
(b) How many minutes will pass on average for one ^{212}Bi atom to be converted into ^{208}Pb? And how many for one ^{208}Tl to be converted into ^{208}Pb?
(c) What are the probabilities of there being a stable state?

Solution

(a) Consider the described system according to a Markov Chain

		Bi	Tl	Pb
		s_1	s_2	s_3
Bi	s_1	0.99	0.003	0.007
Tl	s_2	0	0.8	0.2
Pb	s_3	0	0	1

(b) How many minutes will pass on average for one ^{212}Bi atom to be converted into ^{208}Pb? And how many for one ^{208}Tl to be converted into ^{208}Pb?

Stationary transition probabilities allow us to calculate the system's average cost:

$$\mu_{ij} = 1 + \sum_{k \neq j} p_{ik} \cdot \mu_{kj}$$

$\mu_{13} = 1 + \sum_{k \neq 3} p_{1k} \cdot \mu_{k3} \rightarrow \mu_{13} = 1 + p_{11} \cdot \mu_{13+} p_{12} \cdot \mu_{23} \rightarrow \mu_{13} = 1 + 0.99 \cdot \mu_{13+}$ $0.003 \cdot \mu_{23}$

$\mu_{23} = 1 + \sum_{k \neq 3} p_{2k} \cdot \mu_{k3} \rightarrow \mu_{23} = 1 + p_{21} \cdot \mu_{13+} p_{22} \cdot \mu_{13} \rightarrow \mu_{23} =$ $1 + 0 \cdot \mu_{13+} 0.8 \cdot \mu_{23}$

By solving, $\mu_{23} = 5, \mu_{13} = 101.5$

On average, 101.5 min must pass for one ^{212}Bi atom to be converted into ^{208}Pb

On average, 5 min must pass before one ^{208}Tl atom is converted into ^{208}Pb

(c) What are the probabilities of there being a stable state?

State s_3 is absorbent, so:

$$\pi_1 = 0$$

$$\pi_2 = 0$$

$$\pi_3 = 1$$

10.18 Files Proceedings

John is the supervisor of a team of workers in the TRAMEX firm's Files Proceedings Department. One new file is assigned to each worker on a daily basis for it to be processed. TRAMEX works according to a workflow system, so files reach each worker's PC and they are processed in the PC itself, and John has access to each worker's efficiency statistics. The files have different levels of complexity so, on occasion, they cannot be fully processed on the same day, but must be finished the next day (or on successive days). In normal circumstances, each worker serially works the assigned files, and does not begin the next file while the current one is unfinished.

John has estimated that the service cost for leaving pending files unfinished at the end of each day is not linear: that is, if one file is pending, the cost is \$3; if there are two pending, the cost is \$7; if there are three pending, the cost is \$11; if there are four pending, the cost comes to \$16. With this information, John has calculated that the department's average service cost is \$3.80 per worker and day.

Nevertheless in the past few months, John has noticed the peculiar way of working that one of his workers (Peter) has shown: when more than one file is pending, he begins to work on them in a parallel fashion so that he leaves all the files pending at the end of the working day or he finishes them all. John has calculated that the probability of Peter finishing all the files he has pending on a given day is 1/3. Nevertheless the strange thing is that, whenever Peter has four pending files, he finishes them all on this same day. John wants to calculate certain indicators of the service level that Peter offers according to his peculiar way of working since it does not match the way all the other employers work.

(a) Consider the problem (Peter's way of working) according to a Markov Chain
(b) What is the service cost that Peter generates?
(c) How many days must pass on average since Peter left no files pending until the same situation occurs again?
(d) What should John think about the way Peter works?

Solution

(a) Consider the problem (Peter's way of working) according to a Markov Chain

Given the following states:

s_1 begins the day with 0 pending files
s_2 begins the day with 1 pending file
s_3 begins the day with 2 pending files
s_4 begins the day with 3 pending files

The one-step transition probabilities matrix is as follows:

	s_1	s_2	s_3	s_4
s_1	1/3	2/3	0	0
s_2	1/3	0	2/3	0
s_3	1/3	0	0	2/3
s_4	1	0	0	0

(b) What is the service cost that Peter generates?

The stationary transition probabilities and the costs associated with each state are:

		Cost
π_1	0.415385	0
π_2	0.276923	3
π_3	0.184615	7
π_4	0.123077	11

Therefore, the total service cost is:
Total cost $= 0.276923 \cdot 3 + 0.184615 \cdot 7 + 0.123077 \cdot 11 = 3.476921$ dollars

(c) How many days must pass on average since Peter left no files pending until the same situation occurs again?

The expected first passage times are:

	s_1	s_2	s_3	s_4
s_1	2.407	1.5	3.75	7.125
s_2	2.111	3.611	2.25	5.625
s_3	1.667	3.167	5.417	3.375
s_4	1	2.5	4.75	8.125

Therefore, 2.407 days go by, on average, since Peter has no pending files until the same situation occurs again.

(d) What should John think about the way Peter works?

Peter generates a daily saving of $3.8 - 3.476921 = 0.323079$

Therefore, John should think that Peter's way of working is more efficient than the standard way of working.

10.19 Controlling Machinery

The control system for a certain machine comprises two independent controls, A and B. This system is activated for every hour the machine operates to check that the machine works according to suitable parameters. The controls have a limited life. Indeed, control A has been replaced 52 times in the last 1,300 h of operation, whereas control B has been substituted 39 times in the same time.

Every time the system is activated, apart from controlling the machine, it warns that one (or both) of the controls has failed. The faulty control is replaced before the system is next activated. When one of the two controls is replaced, its reliability is lesser because it has not been verified. Specifically, it is considered that when the system is next activated, the reliability of the substituted control is half its normal value.

Replacing control A entails a cost of \$70 and of \$50 for replacing control B.

(a) Consider the problem according to a Markov chain
(b) What is the hourly cost of the control system?
(c) How many hours must pass on average since both controls are replaced at the same time until both fail again at the same time?
(d) If 10 h have passed since both controls were changed at the same time and neither has failed since then, with what probability will the system in each state be in after 2 h's time?

Solution

(a) Consider the problem according to a Markov chain

Given the following states:

s_1 A and B are replaced
s_2 Only A is replaced
s_3 Only B is replaced
s_4 Neither A nor B is replaced

If the reliability of A is 0.96 and that of B is 0.97, the one-step transition probabilities matrix is:

	s_1	s_2	s_3	s_4
s_1	0.9312	0.0288	0.0388	0.0012
s_2	0.4656	0.4944	0.0194	0.0206
s_3	0.4656	0.0144	0.5044	0.0156
s_4	0.2328	0.2472	0.2522	0.2678

(b) What is the hourly cost of the control system?

The stationary transition probabilities and the costs associated with each state are:

		cost
π_1	0.869305	
π_2	0.0537715	50
π_3	0.0724421	70
π_4	0.00448096	120

Therefore, the total cost is:

Total cost $= 0.053771\ 5 \cdot 50 + 0.0724421 \cdot 70 + 0.00448096 \cdot 120 = 8.297$ 2372 dollars

(c) How many hours must pass on average since both controls are replaced at the same time until both fail again at the same time?

The expected recurrence times in hours are:

μ_{11} 1.15
μ_{22} 18.60
μ_{33} 13.80
μ_{44} 223.17

Thus, on average, 223.167 h must pass since both controls are replaced at the same time until both fail at the same time.

(d) If 10 h have passed since both controls were changed at the same time and neither has failed since then, with what probability will the system in each state be in after 2 h's time?

The two-step transition probabilities are:

	s_1	s_2	s_3	s_4
s_1	0.8989	0.0419	0.0566	0.0026
s_2	0.6776	0.2632	0.0426	0.0166
s_3	0.6788	0.0316	0.2767	0.0129
s_4	0.5116	0.1988	0.2086	0.081

Therefore, the probabilities of being in each state, beginning with state e_1, are:

$$p_{11}^{(2)} 0.8989$$

$$p_{12}^{(2)} 0.0419$$

$$p_{13}^{(2)} 0.0566$$

$$p_{14}^{(2)} 0.0026$$

10.20 Analysing the Effects of a New Virus

Frank is the head biologist in a laboratory that is currently analysing the effects of mortality that a new and dangerous virus has on one type of cells. Frank has prepared a culture of healthy cells and has introduced the virus into the culture to observe it every 10 h. After the first 10-hour period, 40 % of the cells are infected, but none is dead; after 20 h, 24 % of the cells are dead; after, 30 h, 45.6 % of the cells are dead. Frank estimates that the behaviour of this phenomenon can be modelled by means of a Markov chain.

(a) Determine the one-step transition probabilities matrix
(b) How many hours does one healthy cell take to die?
(c) What are the long-term ratios?

Solution

(a) Determine the one-step transition probabilities matrix

Given these states:

s_1 healthy cell
s_2 infected cell
s_3 dead cell

and the state probabilities vector:

	s_1	s_2	s_3
0 h	1	0	0
10 h	0.6	0.4	0
20 h	a1	a2	0.24
30 h	b1	b2	0.456

It is stated that the one-step transition probabilities matrix is:

	s_1	s_2	s_3
s_1	x_1	x_2	x_3
s_2	y_1	y_2	y_3
s_3	0	0	1

(one dead cell does not revive)

Calculating the 10-hour vector from the transition matrix:

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.4 & 0 \end{pmatrix}$$

By performing the operation, we can deduce that:

$x_1 = 0.6$
$x_2 = 0.4$
$x_3 = 0$

Calculating the 20-hour vector from the transition matrix:

$$\begin{pmatrix} 0.6 & 0.4 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.4 & 0 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & 0.24 \end{pmatrix}$$

By performing the operation, we can deduce that:

$y_3 = 0.6$
$0.36 + 0.4 \cdot y_1 = a_1$ (1)
$0.24 + 0.4 \cdot y_2 = a_2$ (2)

Calculating the 30-hour vector from the transition matrix:

$$\begin{pmatrix} a_1 & a_2 & 0.24 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.4 & 0 \\ y_1 & y_2 & 0.6 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & 0.456 \end{pmatrix}$$

By performing the operation, we can deduce that:

$a_2 = 0.36$

With the relation:

$a_1 + a_2 + 0.24 = 1$

It is deduced that:

$a_1 = 0.4$

By substituting in Equation (1):

$y_1 = 0.1$

By substituting in Equation (2):

$y_2 = 0.3$

Finally, the one-step transition probabilities matrix is:

	s_1	s_2	s_3
s_1	0.6	0.4	0
s_2	0.1	0.3	0.6
s_3	0	0	1

(b) How many hours does one healthy cell take to die?

Calculating the expected first passage time from state 1 to 3:

$\mu_{13} = 1 + p_{11}\ \mu_{13} + p_{12}\ \mu_{23}$
$\mu_{23} = 1 + p_{21}\ \mu_{13} + p_{22}\ \mu_{23}$
$\mu_{13} = 4.58$

Therefore, it takes 45.8 h

(d) What are the long-term ratios?

State 3 is absorbent, so the long-term ratios are 0 for s_1 and s_2, and 1 for s_3.

10.21 Market Research into Cars for Fleets of Vehicles

FORD is investigating how the market in which firms buying cars for their fleets behaves. Preliminary research work has revealed that firms buy cars of several brand names, but that they tend to maintain a percentage assigned to each automobile manufacturer. Every year when new models are announced, firms amend the percentages assigned to each brand name.

For the purpose of performing a preliminary analysis, FORD considers that the set of the values of the percentages that these firms assign to the FORD car brand name is discrete, and is: 100, 70, 50 or 20 %.

An in-depth study resulted in a 1-year transition probabilities matrix, as shown in Table 10.6.

The current situation is that 30 % of the firms maintain an assigned percentage of 70 %, 45 % maintain an assigned percentage of 50 %, and 20 % maintain an assigned percentage of 20 %.

(a) Fill in the ? in the matrix
(b) What percentage of firms has, in the long term, a percentage assigned to FORD of 100 %? And the percentage for 70 %? And for 50 %? And finally for 20 %?
(c) What are these percentages for the following year?

Solution

Table 10.6 The transition probabilities obtained after the study

		To			
		100 %	70 %	50 %	20 %
From	100 %	0.6	?	0.1	–
	70 %	?	0.7	0.3	–
	50 %	–	0.4	0.4	?
	20 %	–	0.2	?	0.3

(a) Fill in the ? in the matrix

		To			
		100 %	70 %	50 %	20 %
From	100 %	0.6	0.3	0.1	0
	70 %	0	0.7	0.3	0
	50 %	0	0.4	0.4	0.2
	20 %	0	0.2	0.5	0.3

(b) What percentage of firms has, in the long term, a percentage assigned to FORD of 100 %? And the percentage for 70 %? And for 50 %? And finally for 20 %?

The stationary transition probabilities are:

		To			
		100 %	70 %	50 %	20 %
From	100 %	0	0.5424	0.3559	0.1017
	70 %	0	0.5424	0.3559	0.1017
	50 %	0	0.5424	0.3559	0.1017
	20 %	0	0.5424	0.3559	0.1017

(c) What are these percentages for the following year?

The probabilities for the following year are:

Initial vector			
100 %	70 %	50 %	20 %
0.05	0.3	0.45	0.2

Year 1 vector			
100	70	50	20
0.03	0.445	0.375	0.15

10.22 Industrial Machinery Maintenance

A technician in charge of the maintenance for a certain firm checks the regulations of an automatic machine producing rivets for the air-sea industry on a daily basis. After checking a certain number of rivets, there are four regulation states: no need for regulations (A or s_1); a slight deregulation (B or s_2); a certain degree of deregulation (C or s_3); deregulation (D or s_4).

The technician has checked that if the machine is found to be in A, there is an 80 % probability that it will continue this way the next day, and a 20 % probability that it moves on to state B. If the machine is in state B, there is a 70 % probability that it will continue this way the next day, and a 5 % probability of a complete deregulation occurring. If the machine is in state C, there is a 50 % probability of a complete deregulation occurring, and it is certain that its state will not improve. If a complete deregulation is found in the machine, it will remain permanently.

However, the technician's work does not imply simply observing, but regulating the machine if it is in state D. Nevertheless, the technician is not sure, from an economic point of view, what is more convenient for the firm that the machine is in any of the other states. A regulation costs $200. If the machine begins to work at the start of the day and is in state A, no extra costs through loss of quality are generated. If it starts working in state B, it costs the firm $50 that day, and $80 if it is in state C.

The maintenance technician needs to economically evaluate if it is less costly to regulate the machine when it is also in state C.

(a) Model the problem according to each Markov chains (one for each maintenance policy)
(b) Which is the least costly policy?
(c) What should the regulation cost be if the answer to question b was the opposite?
(d) For the original maintenance policy, what is the expected first passage time from state D to A? What is the recurrence time for state D?

Solution

(a) Model the problem according to each Markov chains (one for each maintenance policy)

Maintenance policy 1
One-step transition probabilities matrix

		A	B	C	D
		s_1	s_2	s_3	s_4
A	s_1	0.8	0.2	0	0
B	s_2	0	0.7	0.25	0.05
C	s_3	0	0	0.5	0.5
D	s_4	1	0	0	0

Maintenance policy 2
One-step transition probabilities matrix

		A	B	C	D
		s_1	s_2	s_3	s_4
A	s_1	0.8	0.2	0	0
B	s_2	0	0.7	0.25	0.05
C	s_3	1	0	0	0
D	s_4	1	0	0	0

(b) Which is the least costly policy?

The stationary transition probabilities and their associated costs are:
Maintenance policy 1:

		cost
π_1	0.454545	00
π_2	0.30303	50.00
π_3	0.151515	80.00
π_4	0.0909091	200.00

Thus, the mean daily cost for maintenance policy 1 is \$45.45
Maintenance policy 2:

		cost
π_1	0.535714	0.00
π_2	0.357143	50.00
π_3	0.0892857	200.00
π_4	0.0178571	200.00

Therefore, the mean daily cost for maintenance policy 2 is 39.29
The least costly policy is maintenance policy 2.

(c) What should the regulation cost be if the answer to question b was the opposite?

By solving the equation below:

$$\pi_1 \cdot 0 + \pi_2 \cdot 50 + \pi_3 \cdot X + \pi_4 \cdot X = 45.45\$$$

we obtain X = \$580. Therefore, if the regulation cost exceeds this amount, the least costly maintenance policy is 1.

(d) For the original maintenance policy, what is the expected first passage time from state D to A? What is the recurrence time for state D?

The expected first passage time from state D to A is 1 as the machine is always regulated whenever it is deregulated.
The recurrence time of state D is the inverse of $\pi_4 \rightarrow 11$.

10.23 Distribution of Operators in Workshop Areas

CUSTOM S. L. is a small workshop that specialises in manufacturing special parts for the metallurgical industry. The firms' personnel comprises the Director (an Industrial Organisation Engineer), the Designer (an Industrial Design Engineer) and three highly specialised Operators. The workshop has two very different areas (A and B) given the type of machinery in them and the kind of works that can be done with it. The three Operators can move between the two areas as they prefer because they can all work on any of the workshop machines. The Director has noticed that the Operators tend to gather in the same area because their performance is enhanced if they help each other. Nevertheless, leaving an area without Operators for several days is not a good option because the tasks in this area are considerably delayed. By analysing last year's manufacturing reports, the Director has calculated that the two areas were not occupied by the Operators for 30 % of the days. However, the most worrying aspect is that one area can be void of Operators for several days running. Therefore, the Director has decided to set up the following work plan: at the beginning of the working day, the three Operators will enter a draw to see which of them changes area so no area will be empty for more than 1 day running, or so the Director believes.

(a) Consider the problem according to a Markov chain
(b) For what percentage of time will one of the areas be void of Operators?
(c) On average, how much time passes from there being no Operator in one zone to there being one Operator in it? And the time from an area with no Operator until this same situation occurs again?

Solution

(a) Consider the problem according to a Markov chain

Given that:

Ax When there are *x* Operators in area A
By When there are *y* Operators in area B

The one-step transition probabilities matrix is:

		A0B3	*A1B2*	*A2B1*	*A3B0*
		s_1	s_2	s_3	s_4
A0B3	s_1	0	1	0	0
A1B2	s_2	1/3	0	2/3	0
A2B1	s_3	0	2/3	0	1/3
A3B0	s_4	0	0	1	0

(b) For what percentage of time will one of the areas be void of Operators?

The stationary transition probabilities are:

$\pi_1 = 0.124906$
$\pi_2 = 0.375094$
$\pi_3 = 0.375094$
$\pi_4 = 0.124906$

Therefore, the percentage of time in which an area is void of Operators corresponds to
$\pi_1 + \pi_4 = 0.2498$

(c) On average, how much time passes from there being no Operator in one zone to there being one Operator in it? And the time from an area with no Operator until this same situation occurs again?

The expected first passage times are:

	s_1	s_2	s_3	s_4
s_1	8.006	1	2.999	10
s_2	7.006	2.666	1.999	9.005
s_3	9.005	1.999	2.666	7.006
s_4	10	2.999	1	8.006

On average, 1 day passes from one area being without an Operator until there is one Operator occupying it, and 8 days pass from one area being without an Operator until this same situation occurs again.

10.24 Behaviour of Cells

Bonny is a biochemist who is studying the behaviour of some cells which present three variants (coded z1, z2 and z3). Bonny has read a scientific article which states that these cells tend to change from variant z1 to z2, from z2 to z3 and from z3 to z1. Nonetheless, the article did not quantify these transitions. To be able to obtain the probabilities and the transition times, Bonny carried out the following experiments:

E1→ Beginning with a solution of 400 cells of variant z1, after 1 h, half the cells had become variant z2 (and there was none of variant z3)

E2→ Beginning with a solution of 400 cells of variant z2, after 1 h, 40 % of the cells had become variant z3 (and there was none of variant z1)

E3→ Beginning with a solution of 200 z1 cells, 200 z2 cells and 200 z3 cells, after 1 h, they had become 280, 220 and 100, respectively.

(a) Consider the problem according to a Markov chain
(b) If Bonny left a solution of 1,000 variant z1 cells for a long time, how many cells would change to each variant?
(c) On average, how much time passes since one cell of each variant goes back to being this variant for the first time after having changed? Which variant becomes another variant in the shortest time? What is the value of this time?

Solution

(a) Consider the problem according to a Markov chain

If we consider z1, z2 and z3 to be the possible states, according to Experiments E1 and E2, the *P* one-step transition probabilities matrix (1 h) is:

	z1	z2	z3
z1	0.5	0.5	0
z2	0	0.6	0.4
z3	x	0	1-x

According to Experiment E3, the vector of the initial situation is transformed through P into:

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix} \cdot P = \begin{pmatrix} 280/600 & 220/600 & 100/600 \end{pmatrix}$$

By performing the operation, we find the value of $x = 9/10$

(b) If Bonny left a solution of 1,000 variant z1 cells for a long time, how many cells would change to each variant?

The stationary transition probabilities are:

π_1 0.356436
π_2 0.445545
π_3 0.19802

Thus, the expected number of cells in each state is:

n_1 0.356436 356.4
n_2 0.445545 445.5
n_3 0.19802 198.0

(c) On average, how much time passes since one cell of each variant goes back to being this variant for the first time after having changed? Which variant becomes another variant in the shortest time? What is the value of this time?

Given the inverses of π:

$\mu_{11} = 2.80555275$
$\mu_{22} = 2.2444422$
$\mu_{33} = 5.04999495$

It is concluded that, on average, 2.80, 2.24 and 5.05 h pass since one cell of variant z1, z2 and z3, respectively, becomes this variant again for the first time after having changed.

The change from z3 to z1 occurs in the shortest time and, specifically from the first passage times, we obtain $\mu_{31} = 1.11$ h.

10.25 How the Staff in a Consultancy Firm Evolve

ABC Consulting is a consultancy firm in which employees move through three levels: Junior Consultant, Senior Consultant and Manager. Changes of level are determined by the Human Resources (HR) Director based on an annual evaluation done for each employee. Only one change of level can be done per evaluation and employees never go down one level. After doing a statistical analysis, the HR Director has determined that the Junior Consultant has a 20 % probability of moving up one level after the evaluation, whereas this probability lowers to 5 % for a Senior Consultant. He has also established that there is a probability of 15, 10 and 5 % that a Junior Consultant, a Senior Consultant or a Manager, respectively, leaves the firm (to work for another firm or to retire) after knowing the result of his or her annual evaluation. For every employee who leaves the firm, a new Junior Consultant is contracted.

(a) Consider the problem according to a Markov chain
(b) What percentage of employees at leach level does the firm currently have?
(c) On average, how much time passes since one employee moves up a given level until he or she leaves the firm?

Solution

(a) Consider the problem according to a Markov chain

The one-step transition probabilities matrix is:

		Junior	Senior	Manager	Leaves
		s_1	s_2	s_3	s_4
Junior	s_1	0.65	0.2	0	0.15
Senior	s_2	0	0.85	0.05	0.1
Manager	s_3	0	0	0.95	0.05
Leaves	s_4	1	0	0	0

(b) What percentage of employees at each level does the firm currently have?

The stationary transition probabilities are:

$\pi_1 =$	0.249	27 %	Junior
$\pi_2 =$	0.332	36 %	Senior
$\pi_3 =$	0.332	36 %	Manager
$\pi_4 =$	0.0871	100 %	

(c) On average, how much time passes since one employee moves up a given level until he or she leaves the firm?

The expected first passage times are:

		Junior	Senior	Manager	Leaves
		s_1	s_2	s_3	s_4
Junior	s_1	4.017	5.75	39.25	10.48
Senior	s_2	14.33	3.012	33.5	13.33
Manager	s_3	21	26.75	3.013	20
Leaves	s_4	1	6.75	40.25	11.48

On average, therefore, 10.48 years will pass since one employee reaches the Junior Consultant level and then leaves, 13.33 years for those who reach the Senior Consultant level and 20 years for those who become Manager.

10.26 How the Effects of a New Drug Evolve

A multinational pharmaceutical company successfully commercialises a drug to fight obesity called SLENDER. The firm is currently doing an experiment to test an evolved version of this drug, SLENDER+, using an aleatory sample of patients being treated with SLENDER. Nowadays, the population taking SLENDER is divided into: obese; overweight; normal weight and similar proportions. The first group takes a 3-pill dose a day, the second group takes 2 pills a day and the dose of the third group is 1 pill daily (for whom the purpose is to maintain their normal weight).

The follow-up of the experiment is done monthly. The technicians have observed that with SLENDER+, 60 % of those who were found to be obese in a check-up continued to be so in the next check-up. This only occurred in 20 % of those who had overweight problems, and only in 10 % of those whose weight was normal.

Moreover after the first month of SLENDER+ treatment, the overweight patients had lowered to 20 %, but the obese ones increased by 50 %.

(a) Consider the problem according to a Markov chain
(b) On average, how much time does an obese patient need to reach normal weight taking the SLENDER+ treatment? And vice versa?
(c) Is the new SLENDER+ formula better than SLENDER?

Solution

(a) Consider the problem according to a Markov chain

Given that:
s_1 obese state
s_2 overweight state
s_3 normal weight state

and the current situation (0.333 0.333 0.333) and after 1 month according to the above (0.500 0.200 0.300).

It can be stated that the one-step transition probabilities matrix is as follows:

	s_1	s_2	s_3
s_1	0.6	x_1	x_2
s_2	y_1	0.2	y_2
s_3	z_1	z_2	0.1

In order to solve the unknown quantities of the one-step transition probabilities matrix, we can consider this product:

$$\begin{pmatrix} 0.333 & 0.333 & 0.333 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & x_1 & x_2 \\ y_1 & 0.2 & y_2 \\ z_1 & z_2 & 0.1 \end{pmatrix} = \begin{pmatrix} 0.500 & 0.200 & 0.300 \end{pmatrix}$$

From the equations on the sample's evolution:

$0.333 \cdot (0.6 + y_1 + z_1) = 0.5$
$0.333 \cdot (x_1 + 0.2 + z_2) = 0.2$
$0.333 \cdot (x_2 + y_2 + 0.1) = 0.3$

From the equations on the one-step transition probabilities matrix:

$0.6 + x_1 + x_2 = 1$
$y_1 + 0.2 + y_2 = 1$
$z_1 + z_2 + 0.1 = 1$

By solving them, we obtain:

$x_1 = 0.4$
$x_2 = 0$
$y_1 = 0$
$y_2 = 0.8$
$z_1 = 0.9$
$z_2 = 0$

Therefore, the one-step transition probabilities matrix is:

	s_1	s_2	s_3
s_1	0.6	0.4	0
s_2	0	0.2	0.8
s_3	0.9	0	0.1

(b) On average, how much time does an obese patient need to reach normal weight taking the SLENDER + treatment? And vice versa?

The expected first passage times are:

	s_1	s_2	s_3
s_1	1.944	2.5	3.75
s_2	2.361	3.889	1.25
s_3	1.111	3.611	4.375

Thus, one obese patient needs, on average, 3.75 months to reach normal weight on the SLENDER+ treatment, while one normal weight patient can become obese in an average time of 1.11 months.

(c) Is the new SLENDER+ formula better than SLENDER?

The stationary transition probabilities are:

$\pi_1 = 0.514286$
$\pi_2 = 0.257143$
$\pi_3 = 0.228571$

Therefore, the long-term ratios with SLENDER+ are worse than the current ones with SLENDER (33 % in each state).

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R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5, © Springer-Verlag London 2014

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R. Poler et al., *Operations Research Problems*,
DOI: 10.1007/978-1-4471-5577-5,