Advanced Corporate Finance[1]

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0 Advanced Corporate Finance

0.1 Objective and Syllabus

This course extends the valuation of firm assets and liabilities to cases where they contain *embedded contingent claims (options)*, reconciling these option prices of assets and liabilities with the Capital Asset Pricing Model (CAPM). It also discusses the role and activity of Hedge Funds since as investment "Corporations", they often make intensive use of options.

After reviewing the traditional (static) Corporate Finance paradigms and clarifying the necessary Black Scholes and Merton technology, it goes on to explain the implications for (dynamic) corporate asset & liability management when optionality is present. It covers the valuation of Warrants & Convertibles and optimal capital structure management with costly bankruptcy. Mergers and Acquisitions are discussed in this framework and examples cited where the Modigliani Miller value additivity principle does not hold. Finally, so called "Real Options", the valuation of the embedded options in physical (not financial) asset ownership are evaluated.

0.2 Learning Outcomes

At the end of this course, students should be able to:–

- Determine expected returns on holding options
- Value firm components using simple option sharing rules
- Discuss the determinants of a firm's cost of capital
- Discuss the determinants of a firm's capital structure
- Discuss firm restructuring and the relevance of Agency Theory
- Discuss why traditional NPV rules are deficient and evaluate simple Real Option examples

0.3 Readings

There core texts is Financial Theory and Corporate Policy, 4th Edition, by Copeland, Weston and Shastri (ISBN 0–321–22353–5 Pearson/Addison Wesley). You are strongly advised to follow the readings as the lectures add to the relevant chapters (shown in the course outline next to the listing of sessions below).

0.4 Course assessment

Assessment comprises a negotiation exercise based on optimal firm capital structure. Pairs of equity and debt holders negotiate their firm structures under different firm "frictions". Each student will be required to produce an individual report on their valuation and negotiation process (3,000 word individual report describing the negotiation process, valuation and outcomes and an analysis). This is described in a separate document.

0.5 Course outline

- Readings are from Copeland, Weston and Shastri; chapters (and pages)

1. Review of static firm valuation and the CAPM
   - 14 (497–553), 5(101–120, 137–140), 6 (147–176)
2. Market efficiency and random walk representations
   - 10 (353–373), 11(377–406)
3. Options and the CAPM
   - 7 (199–250)
4. Dynamic option valuation of firm liabilities
   - 15 (557–588)
5. Equity calls, dilution, equity warrants and convertible bonds
   - 15 (596–625)
6. Tax and bankruptcy, optimal capital structure
   - 15 (588–596)
7. Hedging
   - 17 (697–747), 19 (819–856)
8. Agency Theory
   - 12 (415–461)

9. Mergers & Acquisitions
   - 18 (753–807)
10. Real (perpetual American) options
    - 9 (305–355)

0.6 Special sessions

1. Optional: "Midas Touch" Horizon Video: Black, Scholes, Merton & LTCM
2. Negotiation exercise 1: Capital structure (joint basis for the 3,000 word CW)
3. Negotiation exercise 2: Capital structure (joint basis for the 3,000 word CW).

Total Assets	**Total Liabilities**
Cash and Marketable Securities	Current Interest & Tax
Raw Materials	Creditors
Finished Stock	Senior Loans & Debt
Debtors	Junior Debt & Preferred Equity
Intangible Assets	Shareholders Equity
Fixed Assets	Retained Profit

Table 1: Typical Book Value Balance Sheet (Gross Format)

<table>
<tr><th>Net Assets</th><th>Net Liabilities</th></tr>
<tr><td rowspan="2">WCR = Cash and Marketable Securities
+ Raw Materials + Finished Stock + Debtors
− Current Interest & Tax − Creditors</td><td>Debt = Senior Loans & Debt
+ Junior Debt & Preferred Equity</td></tr>
<tr><td rowspan="2">Equity = Shareholders Equity
+ Retained Profit</td></tr>
<tr><td>Fixed & Intangible Assets</td></tr>
</table>

Table 2: Net Book Value Balance Sheet

1 Review of firm valuation & CAPM

1.1 Book value balance sheet

Book values are the lower of cost or net realisable value and can be presented in net or gross formats (see Table 1 for example of gross format). The sheet must balance due to accounting identities.

The amount of cash, materials and credit given in normal business less the amount of credit taken is called the working capital requirement (WCR) and is capital that the firm requires to operate. This is separate to the capital required to purchase operating assets, even traders who own no asset may require working capital although some firms get sufficient trade credit and give so little credit themselves (supermarkets) that they have a negative WCR. Profits must reflect a return to this capital that is tied up. Anything the firm can do to diminish the WCR will release cash and create value.

Firms sometimes hold cash as marketable securities for strategic reasons. The shareholders will be seen to be neutral to this strategy but it should not be considered value creating.

Assets and liabilities are increased or decreased over time as their economic change in value is passed through the income statement. Table 2 shows a net book value balance sheet with all current liability holders moved from right to left and now included in Working Capital. This will be generalised to a market value balance sheet.

1.2 "Off Balance Sheet" items

Often assets are contracted for via a lease where regular payments (rather than outright purchase) are made. Subcontracting & outsourcing (e.g. rent v. buy) are other examples of a firms activities that do not enter the book value balance sheet but enter firm value implicitly though the cash flows.

Intangibles & growth options, Brands, Research and Development, strategic options & joint ventures are all off the accounting balance sheet as well but on the market value balance sheet so long as their cash flows are anticipated.

Other off balance sheet, Hedge Instruments, Contingent liabilities, non recourse lending, option like investments and the government's tax slice.

1.3 Market values; Cash

The Market value is given by the discounted (present) value of future expected cashflows, i.e. (net) present value of future cashflows. Consider a cash interest account that pays an interest rate r, on an amount I invested, the cashflow at the end of the year is $(1+r)\,I$. The present value is still I because the NPV formula includes cashflows on top and discount rates beneath

$$PV\,(\text{Cash}) = I = \frac{(1+r)\,I}{(1+r)} \tag{1}$$

Thus for cash the book value equals the market value![1] This is not necessarily true of other marketable securities, it depends how the accounting treats the changes in their market value.

1.4 Debtors & Creditors

Debtors are due within the year, say at the year end (exactly when depends on the terms of trade) so the present (market) value will actually be *less than the book value*

$$PV\,(\text{Debtors or Creditors}) = \frac{D \text{ or } C}{(1+r')} < D \text{ or } C$$

[1] This is true not matter what liquidiation time is chosen, the longer the time the higher the cashflow, in just the right proportion to offset the discounting

$$\frac{(1+r)^T\,I}{(1+r)^T} = I$$

Raw materials and stock could be valued on its expected sale, invoice and final payment date less the cash flows required to get it to the point of sale.

1.5 Other assets

Intangibles assets are difficult to value on their own (valuation is included with residual equity claims) but fixed assets can be valued if their (potential) sale value or cashflow benefits are known.

1.6 Debts and Loans

Companies can issue loans or bonds as they are known. Once issued bonds with fixed terms will have a value that can fluctuate away from their issue and book value as the interest rate and credit they offer becomes more or less attractive compared to other bonds in issue. The present value is simply the sum across all future interest payments c per bond, discounted at the yield to maturity YTM, for a three year bond with principal

$$PV\left(\text{Bond}\right)=\frac{c_1}{1+YTM}+\frac{c_2}{\left(1+YTM\right)^2}+\frac{100+c_3}{\left(1+YTM\right)^3}=\sum_{t=1}^{3}\frac{c_t}{\left(1+YTM\right)^t}+\frac{100}{\left(1+YTM\right)^3} \tag{2}$$

and is sensitive to the current rate of interest. If there are m bonds in issue the total market value of the debt is $D = mB$ and the total interest rate expense is mc.

The book value however remains at 100 per bond excluding the interest due within the year (c_1, c_2 per bond etc. which goes under current liabilities), unless the bond is partly or fully repaid in which case the book value of the bond is adjusted accordingly.

There are some special cases for the valuation of bonds and some income streams.

1.6.1 Zero coupon bond

For one large "coupon" of 100 at redemption T alone, the NPV formula is

$$PV\left(\text{Zero Coupon Bond}\right)=\frac{100}{\left(1+YTM\right)^T}$$

1.6.2 Perpetuity

Some UK Government bonds pay interest for ever and have no redemption date, and thus are perpetual in nature. They are called Consols or War Loan because they are government debts consolidated after the Napoleonic Wars. Interest rates then were very low so they only carry coupons of 2–3% but now that their price has fallen to 25% of face value, their yield is around 6%. Indeed if the limit of Equation 2 is taken as the maturity becomes very large, the price of the perpetuity with constant coupon c can be shown to be

$$\begin{aligned} PV\,(\text{Perpetuity}) &= \frac{c}{YTM} \\ YTM &= \frac{c}{PV\,(\text{Perpetuity})} \end{aligned}$$

which is the same as Equation 1 for cash if the coupon from a cash holding is considered to be $c = YTM * I$ and YTM is the yield on the perpetuity.

1.6.3 Annuity

A stream of cash flows that starts now and finishes at a specific time T can be valued using two perpetuities, one that starts now and runs for ever and a second one that negates the first for all time greater than T. For a riskless perpetual government bond the required rate of return is the risk free rate r

Time 0	1	2	$\to$	T	$T+1$	$T+2$	$\to$	∞
Perpetuity 1: $=\frac{c}{r}$	c	c	$\to$	c	c	c	$\to$	
Perpetuity 2: $=\frac{c}{r}\frac{1}{(1+r)^T}$	0	0	$\to$	0	$-c$	$-c$	$\to$	
Annuity: $1-2=\frac{c}{r}\left(1-\frac{1}{(1+r)^T}\right)$	c	c	$\to$	c	0	0		

In the limit as $r \to 0$ the annuity expression tends toward $cT! \; r$

1.6.4 Discrete and continuous compounding

If a bank account pays interest at 10% p.a. at the end of the year, it doesn't take too long to realise that £110 will be left for every £100 deposited. How about if the 10% is calculated (on a pro rata basis) every 6 months and interest earns interest, you might be quick and come to the answer £110.25 which is in fact £100$*1.05^2$. It would take longer to calculate the effect of quarterly or monthly interest compounding and only a mathematician would be able to guess if you would become infinitely rich or not if interest were calculated every second of the day or even more frequently. Table 3 shows the effect on £100 of every increasingly frequent compounding, in fact the limit is $100e^{0.1} = 110.5171$ which can be shown mathematically from the limit of the series or by taking a large number of n.

$e = 2.71828..$ is a clearly a special number in growth theory and is the basis of Napierian logarithms. Frequently in finance it is easier to assume that

Compounding Frequency	Calculation	Closing value
Annual	$1+0.10$	£110.00
Semi–Annual	$(1+0.05)^2$	£110.25
Monthly	$(1+0.10/12)^{12}$	£110.471
Daily	$(1+0.10/365)^{365}$	£110.5156
n times a year	$(1+0.10/n)^n$	
$n\to\infty$ continuous	$\lim_{n\to\infty}(1+0.10/n)^n = e^{0.1}$	£110.5171

Table 3: The effect of more frequent compounding

interest and other flows are compounded continuously rather than discretely. This is because the t period discount factor is just e^{-rt} and this form is particularly easy to sum i.e. integrate.

1.7 Equity

The book value accounts record the initial amount of equity contributed by shareholders, any further amount contributed by new issue and the amounts earned by but not distributed to shareholders over the course of the years (retained profit/loss). This contrasts starkly with the equity's market value and can often be different by a factor of 10! Using the dividend growth model of Gordon [22] with a constant growth rate for dividends of g and infinite horizon

$$P_{equity} = \frac{d_1}{1+r_e} + \frac{d_1(1+g)}{(1+r_e)^2}\cdots$$

$$\boxed{P_{equity} = \frac{d_1}{r_e - g}}$$

d is the dividend per share, if there are n shares in issue the total dividend expense is nd and the market value of Equity is $E = nP$. Book equity remains low typically, even though augmented by retained profit because it still does not look at future growth, only what has been contributed or accumulated.

In continuous time if $d(t)$ represents the expected dividend rate at time t then the present value of continuous dividends over an infinite horizon is given by the integral of the dividends at period t discounted by $e^{-r_e t}$ which is the appropriate discount rate

$$P_{equity} = \int_0^\infty e^{-r_e t} d(t)\, dt$$

Now if dividends are expected to grow exponentially at a rate g, then the integral is simple to evaluate since the dividends themselves follow a continuous compounding formula

$$\begin{aligned} d(t) &= d(0)\, e^{gt} \\ P_{equity} &= \int_0^\infty e^{(g-r_e)t} dt \\ &= -\frac{1}{r_e - g}\left[e^{(g-r_e)t}\right]_0^\infty \\ &= \frac{d(0)}{r_e - g} \end{aligned}$$

Whether or not d_1 the prospective dividend or $d(0)$ the current dividend are used depends on the choice of model, discrete or continuous dividends.

1.8 Book values and the Clean Surplus Relationship

For an all equity financed firm, under the continuous specification it is also easy to show that the price of a firm's equity can differ from its book value. Suppose that in addition to the continuous dividend process the continuous earnings process is given by $e(t)$, now the amount earned as earnings but not distributed as dividends must be reflected in an increase in book value $b'(t)$, $e(t) - d(t) = b'(t)$ alternatively $d(t) = e(t) - b'(t)$. Integrating this relationship while noting that the integral of the change in book value can be completed by parts[2] yields

$$\begin{aligned} d(t) &= e(t) - b'(t) \\ \int_0^\infty d(t)\, e^{-r_e t} dt &= \int_0^\infty e(t)\, e^{-r_e t} dt - \int_0^\infty b'(t)\, e^{-r_e t} dt \\ P_{equity} &= b(0) + \int_0^\infty (e(t) - r_e b(t))\, e^{-r_e t} dt \end{aligned}$$

$$\boxed{\text{Market Value} = \text{Book Value} + \text{PV Residual Income}}$$

This says that the price of a firms equity is equal to the current book value plus the discounted expected earnings in excess of the opportunity cost of capital on book value. If $e(t) - r_e b(t)$ which has an interpretation of value

[2]

$$\begin{aligned} \int_0^\infty b'(t)\, e^{-r_e t} dt &= \left[b(t)\, e^{-r_e t}\right]_0^\infty + \int_0^\infty r_e b(t)\, e^{-r_e t} dt \\ &= \int_0^\infty r_e b(t)\, e^{-r_e t} dt - b(0) \end{aligned}$$

Market Value of Assets	Market Value of Liabilities
	Debt: $D = mB$
Assets: A	
	Equity: $E = nP$

Table 4: Market Value Balance Sheet

added is positive, then the price of the firm will exceed the book value. If however the firms earnings are insufficient to cover the capital charge $r_e b(t)$ then the firms price will be less than its book value.

1.9 Market value balance sheet

Now the market values of Equity and Debt can be added to infer the market value of the assets (net of creditors, i.e. fixed assets and working capital requirement).

1.10 Value additivity

Book values add up because of accounting rules, market values add up because they are based on cashflows and they add up. Thus the fact that the cash that the assets produce over their life adds up to the cash available for distribution to liability holders means that the NPV's and market values must also add up. Cash Flows CF_t, Discount rate r, horizon T, Investment I

$$(N)PV = (-I) + \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t}$$

$$\begin{aligned} CF_t &= CF_t^A - CF_t^L \\ NPV(A-L) &= \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} = \sum_{t=1}^{T} \frac{CF_t^A - CF_t^L}{(1+r)^t} = NPV(A) - NPV(L) \end{aligned}$$

and thus for the firm as a whole $CF_t = 0$ and $NPV(A) = NPV(L)$ and the market value of assets is equal to the market value of liabilities (see Table 4)

$$\boxed{A = D + E}$$

1.11 Putting asset and liabilities returns together; one asset

Consider operation of a firm with assets of value A over one period followed by liquidation. For a given current asset value

$$A = 100$$

and an *expected* operating cash flow (or profit ignoring investment, cash flow timings etc. in a one period world),

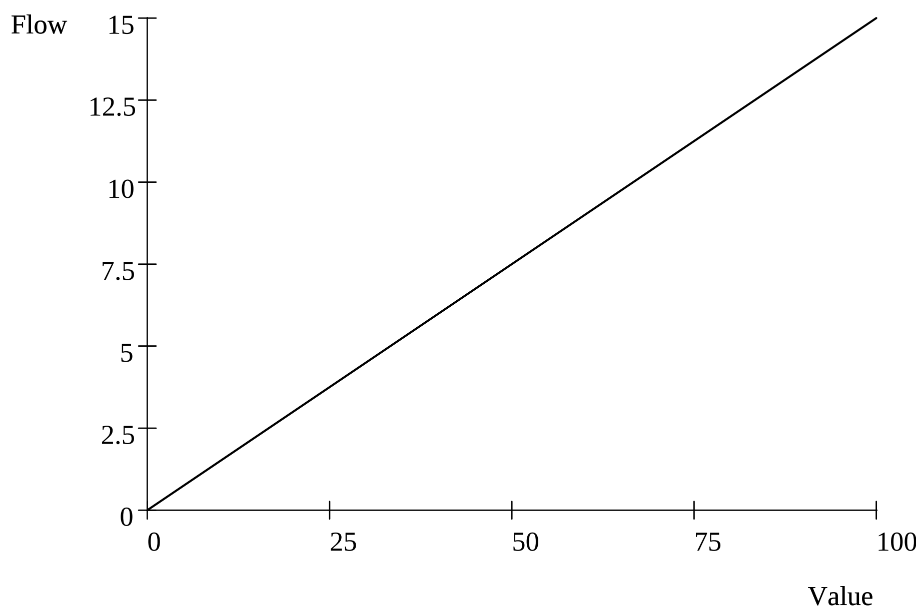
$$\pi_a = 15$$

the *expected* return on assets is

$$R_a = \frac{\pi_a}{A} = 0.15$$

N.B. this is consistent with the NPV rule

$$\begin{aligned} \text{NPV (Assets)} &= \text{Discounted liquidation value} \\ A &= NPV(A) = \frac{A + \pi_a}{1 + R_a} \\ R_a &= \frac{\pi_a}{A} \end{aligned}$$



Flow on y against stock on x (yield as slope)

1.12 Two assets

π_a could be the sum of flows (profits) from two different subassets B and C with individual flows $\pi_b, \pi_c$and current values B, C

$$\begin{aligned} \pi_a &= \pi_b + \pi_c \\ A &= B + C \end{aligned}$$

Assuming

$$B = 20$$

$$\pi_b = 5$$

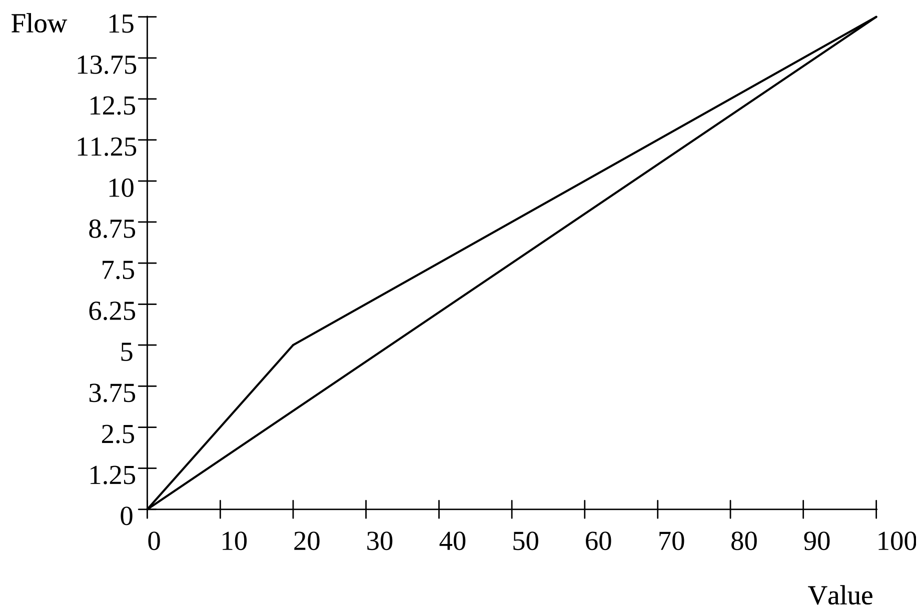
$$C = 80$$

$$\pi_c = 10$$

then the expected rates of return on the individual assets themselves can be defined & calculated

$$\begin{aligned} R_b &= \frac{\pi_b}{B} = \frac{5}{20} = 0.25 \\ R_c &= \frac{\pi_c}{C} = \frac{10}{80} = 0.125 \end{aligned}$$

so that graphically.



Flow against stock for the sum of two assets

Mathematically this relationship gives the total asset return as a weighted average of the two subasset returns

$$\begin{aligned} \pi_a &= \pi_b + \pi_c \\ R_a A &= R_b B + R_c C \end{aligned} \tag{3}$$

$$\boxed{R_a = \tfrac{B}{A} R_b + \tfrac{C}{A} R_c}$$

where $\frac{B}{A}$ and $\frac{C}{A}$ are the weights attached to the two individual expected rates of return R_b, R_c. This relationship holds for portfolios of stocks and any other (long lived) investments as well as physical asset, i.e. the expected return operator is *linear* in its components.

1.13 Return to liability holders; debt

If the asset flow is not entirely attributable to one class of owners, how is the operational flow divided up between these liability holders? If the debt holders are promised interest payments of

$$\pi_d = 5$$

on their (current) face value of Debt

$$D = 50$$

their expected return is

$$R_d = \frac{\pi_d}{D} = \frac{5}{50} = 0.10$$

1.14 Equity

This leaves an expected return attributable to (equity) shareholders

$$\pi_e = \pi_a - \pi_d = 15 - 5 = 10$$

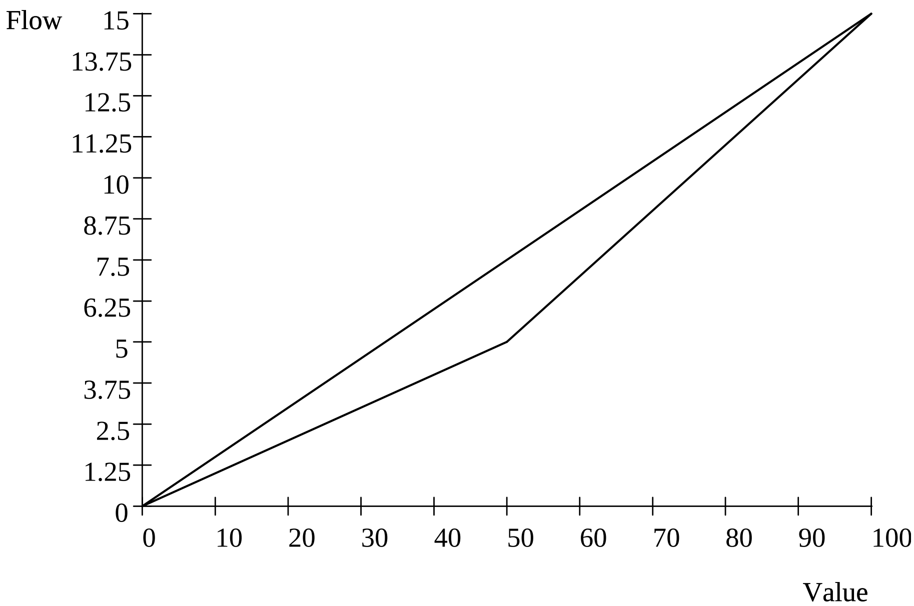
on a (current) value of

$$E = A - D = 50$$

which corresponds to an expected return of

$$R_e = \frac{\pi_e}{E} = \frac{10}{50} = 0.20$$

The return to equity holders is less certain than the return to debtholders (almost certain) so $R_e > R_d$.



Flow against stock for a two liability case

The capital structure is 50% debt and 50% equity ($\frac{D}{A} = \frac{E}{A} = \frac{1}{2}$) so the weighted expected return (asset return) is half that of the return to debt holders and half that of the return to equity holders

$$\begin{aligned} \pi_a &= \pi_d + \pi_e \\ 0.15 &= \frac{1}{2}0.10 + \frac{1}{2}0.20 \end{aligned}$$

Thus the *weighted average cost of capital* (across debt and equity) is given by WACC

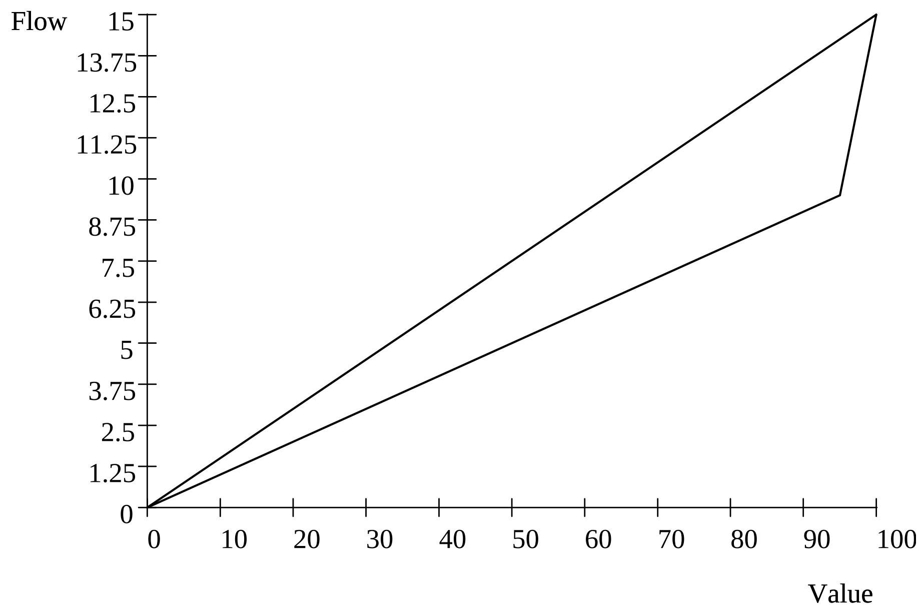
$$R_a A = R_d D + R_e E$$

$$\boxed{R_a = \tfrac{D}{A} R_d + \tfrac{E}{A} R_e}$$

where it is the *market value (not book value)* of debt and equity that must be used to weight the expected returns (since expected returns are returns on market investments). Again the expected returns operator is seen to be linear in its components and the return on equity is a linear (but leveraged) function of return on assets less return to debt where new weights are used $\frac{A}{E}, -\frac{D}{E}$ (N.B. these still sum to one)

$$\begin{aligned} R_e &= \frac{A}{E} R_a - \frac{D}{E} R_d \\ 0.20 &= 2 \times 0.15 - 1 \times 0.10 \end{aligned}$$

(N.B. The weighted average asset return - Equation 3 was also the same WACC!) Thus it can be seen that since leverage should not affect asset return R_a and if leverage does not affect the cost of debt (return to debtholders) R_d, $R_e$can be made arbitrarily large through the choice of extreme leverage!



Flow against stock for the case of extreme leverage

Of course at some extreme point of leverage, the debtholders own virtually of the firm and therefore their claim is as risky as that of the 100% equity financed firm so that

$$E \to 0 \Longleftrightarrow R_d \to R_a$$

Thus is absence of taxes, market imperfections (information asymmetries etc.), other frictions (transaction costs), the firm value must be independent of the chosen level of leverage. If it were not, investors could put together a portfolio of some of the firms debt and equity and expect to earn excess returns, but it is the Asset Value and return that is the fixed point from which liability values are derived.

1.15 Other liability holders, e.g. Preferred Equity

What happens if a third tranche of liability holders are present? Denoting R_p for the expected return to preferred equity holders, who own a claim currently worth P and expected cashflow $\pi_p = R_p P$ the weighted average cost of capital formula generalises to

$$\begin{aligned}\pi_a &= \pi_d + \pi_p + \pi_e \\ R_a A &= R_d D + R_p P + R_e E \\ R_a &= \frac{D}{A} R_d + \frac{P}{A} R_p + \frac{E}{A} R_e\end{aligned}$$

where again market weights (not book) must be used. If the preferred claim ranks above equity holders but below debt holders we would expect

$$R_d < R_p < R_e$$

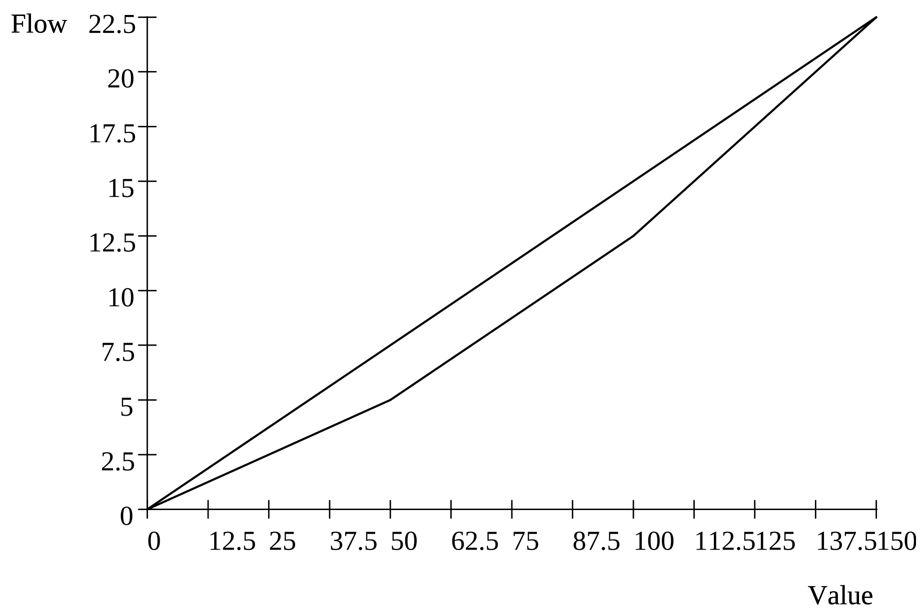
else risk is not rewarded. Selling 50 units of preferred capital to buy 50 units more of existing (like) assets since

$$P = 50$$

and if

$$R_p = .15$$

yields



Flow against stock for the three liability case

Of course any number of asset or liability splits can be implemented this way.

50/50 Firm	Flow	Value	Yield
Asset Flow & Value	15	100	15%
Debt Interest & Value	5	50	10%
Equity Dividend & Value	10	50	20%
50 shares in issue	0.2 per share	1.0 per share	

95/5 Firm	Flow	Value	Yield
Asset Flow & Value	15	100	15%
Debt Interest & Value	9.5	95	10%
Equity Dividend & Value	5.5	5	110%
5 shares in issue	1.1 per share	1.0 per share	

Table 5: Effect of leverage on Earnings per Share

1.16 Earnings per share and other measures

How is the fact that the liabilities can be partitioned anyway without affecting the value of the assets reconciled with the fact that higher earnings per share is favoured? Consider two possible firm structures from the previous case (identical assets worth 100).

Refinancing 45 units of equity by issuing 45 units of debt seems to have greatly increased the per share figure from 0.2 per share to 1.1, a factor of more than five! Is this good? Well the asset value has not changed because the asset cashflows remain unaltered, secondly the share price has not changed because the required yield has also gone up! All that has changed for the share is its risk return profile and now there is more chance that the firm value will fall below the debt value (equivalently that the profit will fall below the interest expense).

What is really required to add value to a firms equity is to increase the per share dividend while keeping the gearing constant, funnily enough because this usually involves hard work on the asset side (trimming costs, increasing revenues etc.) slick financial managers are less keen on it as a quick fix, it is easy to increase the gearing and hope no one will notice!

1.17 Modigliani Miller

This statement that the firm value is independent of the capital structure is the seminal work of Modigliani and Miller (1958) [45] and (1961) [44]. It is also a statement of *linear valuation,* the Modigliani Miller proposition and is true for NPV valuation because the NPV rule is a *linear* operation on it components.

$$\boxed{NPV\left(A+B\right)=NPV\left(A\right)+NPV\left(B\right)}$$

1.18 Risky cashflows & returns

We haven't yet said where discount rates come from so this section tells us. It results from work on Portfolio Theory by Markowitz (1959) [37] and the Capital Asset Pricing Model (CAPM) of Sharpe [61] (1964), Lintner [32] (1963) et al.

1.19 Market returns & risk

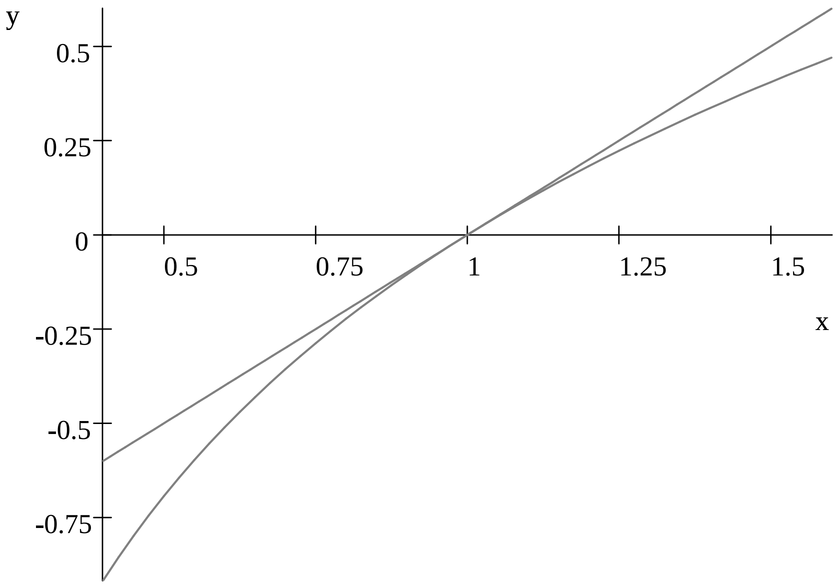
(Log) Market returns are risky, normally distributed with mean/variance statistics

$$\boxed{\%\text{ return } r_1=\frac{P_1+D_1-P_0}{P_0}=\frac{P_1+D_1}{P_0}-1}$$

$$\boxed{=\%\text{ capital gain }+\ \%\text{ dividend yield}}$$

$$\boxed{\%\text{ log return } lr_1=\log_e\left(\frac{P_1+D_1}{P_0}\right)=\ln\left(1+r_1\right)\approx r_1}$$

$$\boxed{\%\text{ log return } lr_1=\ln\left(P_1+D_1\right)-\ln\left(P_0\right)}$$

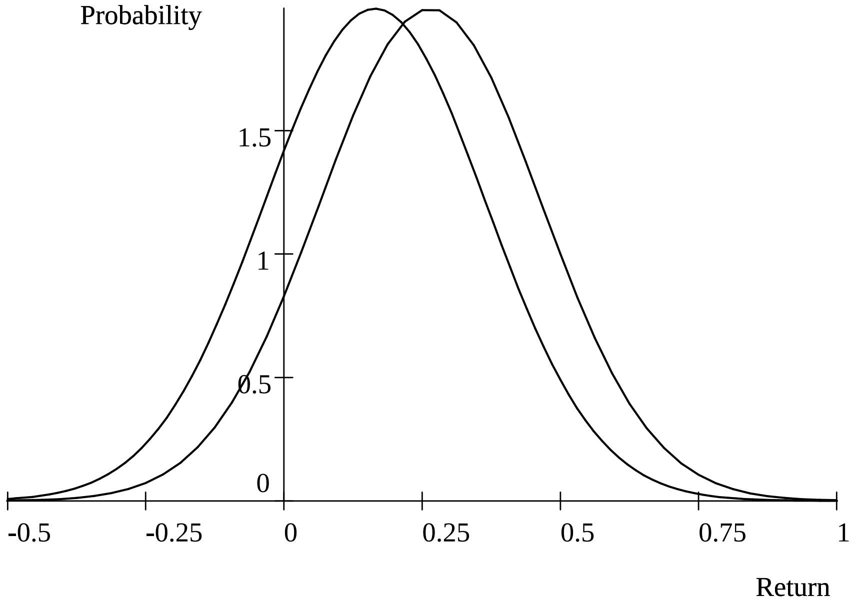


Log versus linear returns

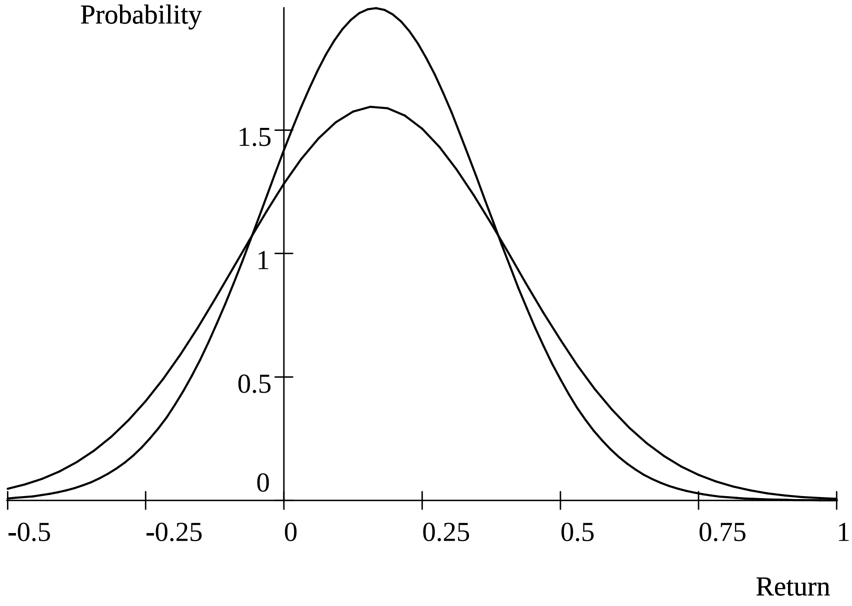
$$\begin{aligned} Mean\left(r_i\right) &= \overline{r_i} = \frac{1}{N}\sum_{i=1}^{N} r_i \\ \overline{lr_i} &= \frac{1}{N}\sum_{i=1}^{N}\ln\left(1+r_i\right) = \frac{1}{N}\ln\prod_{i=1}^{N}\left(1+r_i\right) \\ Var\left(r_i\right) &= \frac{1}{N-1}\sum_{i=1}^{N}\left(r_i-\overline{r_i}\right)^2 \qquad \text{etc.} \end{aligned}$$

$$n\left(x;\overline{x},\sigma\right) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2}\left(\frac{x-\overline{x}}{\sigma}\right)^2}$$

$$\begin{aligned} \overline{x} &= 0.16, \sigma = 0.20 \\ \overline{x} &= 0.26, \sigma = 0.25 \end{aligned}$$

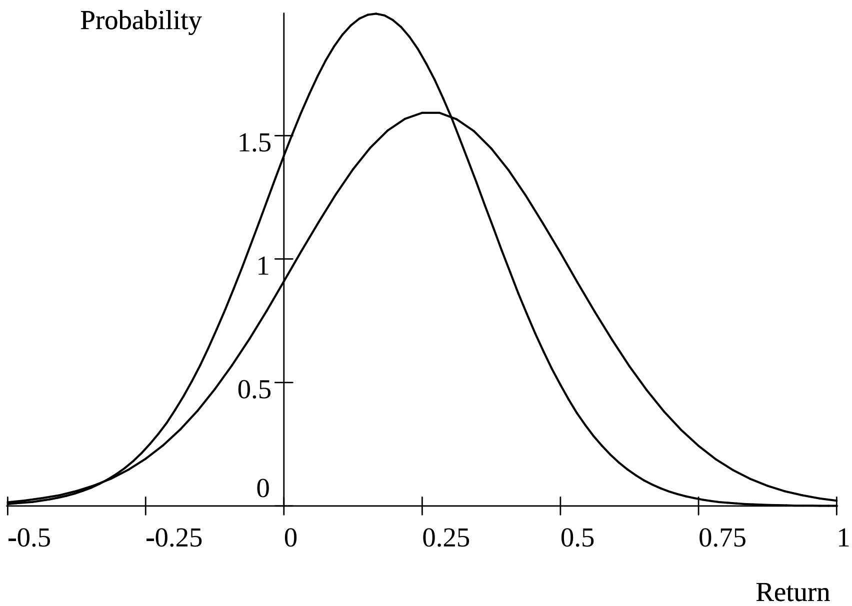


Two normal distributions for returns with different means



Two normal distributions with different variances

Investors prefer higher expected return and lower risk but what about combinations?



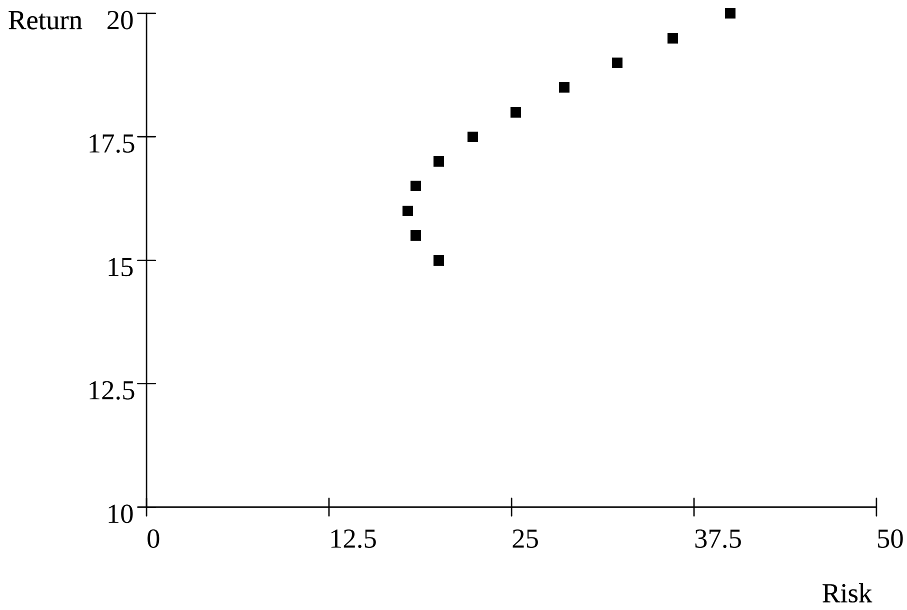
Two normal distributions with different means and variances

1.20 Portfolio theory & risk return space

Formation of portfolios, covariance & Pythagoras

$$\left(\frac{1}{2}\sigma_a + \frac{1}{2}\sigma_b\right)^2 = \frac{1}{4}\sigma_a^2 + \frac{1}{2}\rho_{ab}\sigma_a\sigma_b + \frac{1}{4}\sigma_b^2$$

$$\boxed{\text{Portfolio Variance} = \tfrac{1}{4}\text{Var}_\text{a} + \tfrac{1}{2}\text{Cov}_\text{ab} + \tfrac{1}{4}\text{Var}_\text{b}}$$



The risk (x) return (y) fontier

This frontier can be calculated for many combination from many assets and the path of minimum risk for a given return is known as the efficient frontier.

1.21 Covariance & regression

Covariance is defined by

$$Cov\left(a,b\right) = \frac{1}{N-1}\sum_{i=1}^{N}\left(a_i-\overline{a}\right)\left(b_i-\overline{b}\right)$$

$$Cov\left(a,a\right) = \frac{1}{N-1}\sum_{i=1}^{N}\left(a_i-\overline{a}\right)^2 = Var\left(a\right)$$

and the correlation coefficient

$$\boxed{\rho_{ab} = r_{ab}\ (\text{C\&W}) = \tfrac{Cov(a,b)}{\sigma_a\sigma_b}}$$

the final variable from regression that is needed is the slope

$$\boxed{\text{slope of b on a } \beta_{ab} = \tfrac{Cov(a,b)}{Var(a)} = \tfrac{\rho_{ab}\sigma_b}{\sigma_a}}$$

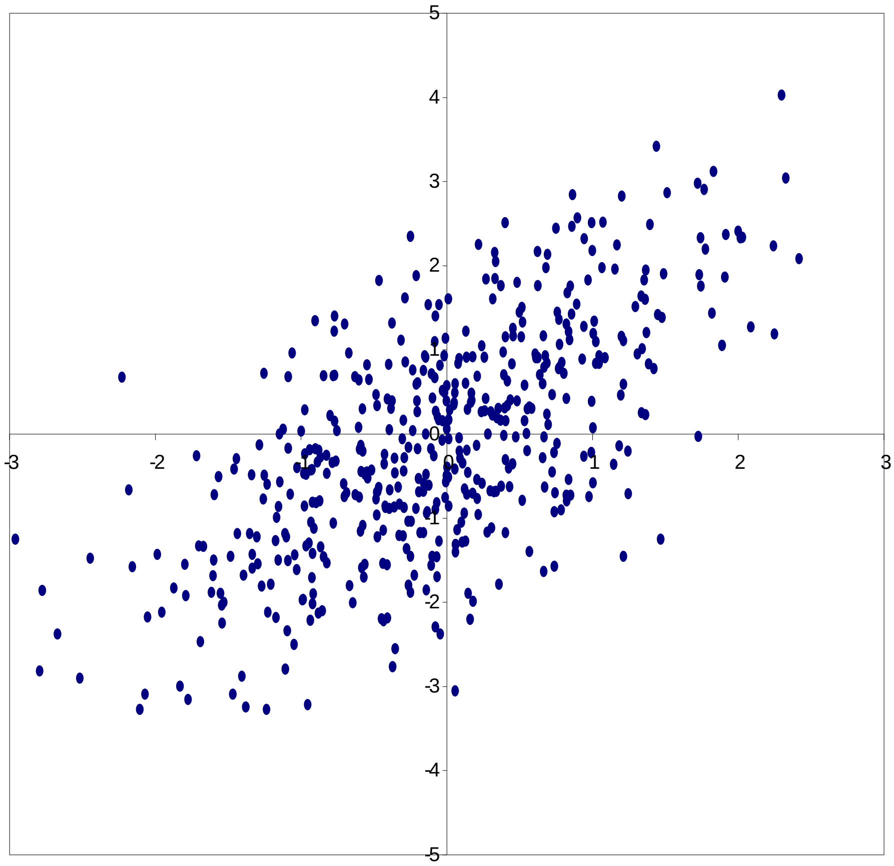


Figure 1: Scatter plot from regression model; market return on x and asset return on y axes.

1.22 Example of regression

For example consider 500 points generated by a regression model with unit slope

$$\begin{aligned} b_t &= a_t + \epsilon_t \\ a_t &\sim n(0,1) \\ \epsilon_t &\sim n(0,1) \\ Cov(a_t, \epsilon_t) &= 0 \end{aligned}$$

$$\begin{aligned} Var\,(a) &= 0.97\ (1.00) \\ Var\,(b) &= 1.83\ (2.00) \\ Cov\,(a,b) &= 0.89\ (1.00) \\ \rho_{ab} &= 0.67\ (0.71) \\ \text{slope } \beta_{ab} &= 0.92\ (1.00) \end{aligned}$$

For stocks, we need to regress the % stock return on the % return of "the market", typically we use the (entire) Equity Market (Index) as the regressor but theoretically we need the total return *across all assets* (bonds, gold, property, works of art etc.). The slope coefficient of this regression is said to be the beta of the company β, (what happened to alpha?)

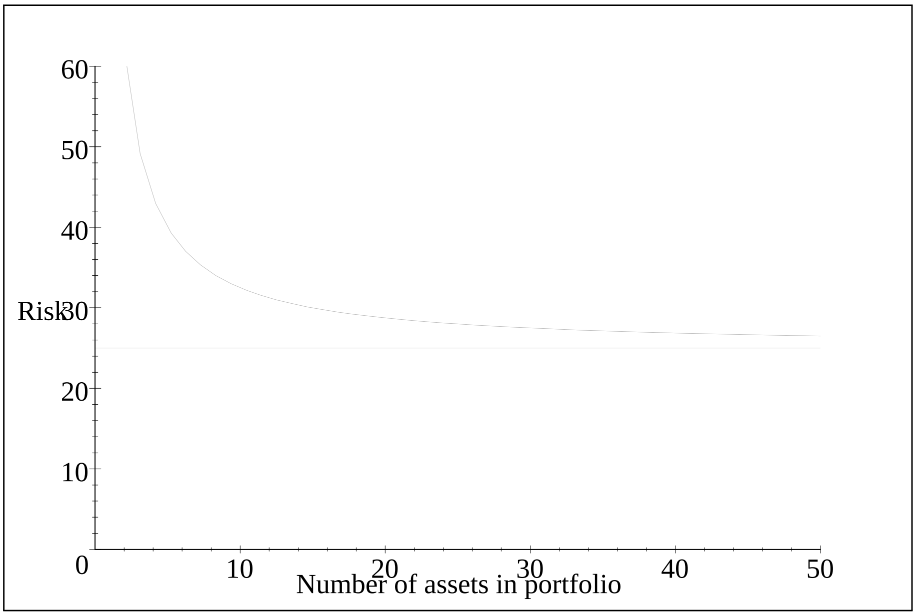
1.23 Portfolio of many assets

Extension to N assets

$$\begin{array}{ccccc} \times & \sigma_1 & \sigma_2 & \cdots & \sigma_N \\ \sigma_1 & \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \sigma_2 & \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{2N}\sigma_2\sigma_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_N & \rho_{N1}\sigma_N\sigma_1 & \rho_{N2}\sigma_N\sigma_2 & \cdots & \sigma_N^2 \end{array}$$

$$\begin{aligned} \left(\sum_{n=1}^{N}\sigma_n\right)^2 &= (\sigma_1+\sigma_2+\ldots\sigma_N)\times(\sigma_1+\sigma_2+\ldots\sigma_N) \\ &= \frac{1}{N^2}\sum_{n=1}^{N}\sigma_n^2+\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1,\neq n}^{N}\rho_{nm}\sigma_n\sigma_m \\ &= \frac{1}{N^2}N\overline{\sigma^2}+\frac{1}{N^2}\left(N^2-N\right)\overline{\rho\sigma^2} \\ &= \frac{\overline{\sigma^2}}{N}+\left(1-\frac{1}{N}\right)\overline{\rho\sigma^2}\rightarrow\overline{\rho\sigma^2} \qquad \text{as } N\rightarrow\infty \end{aligned}$$

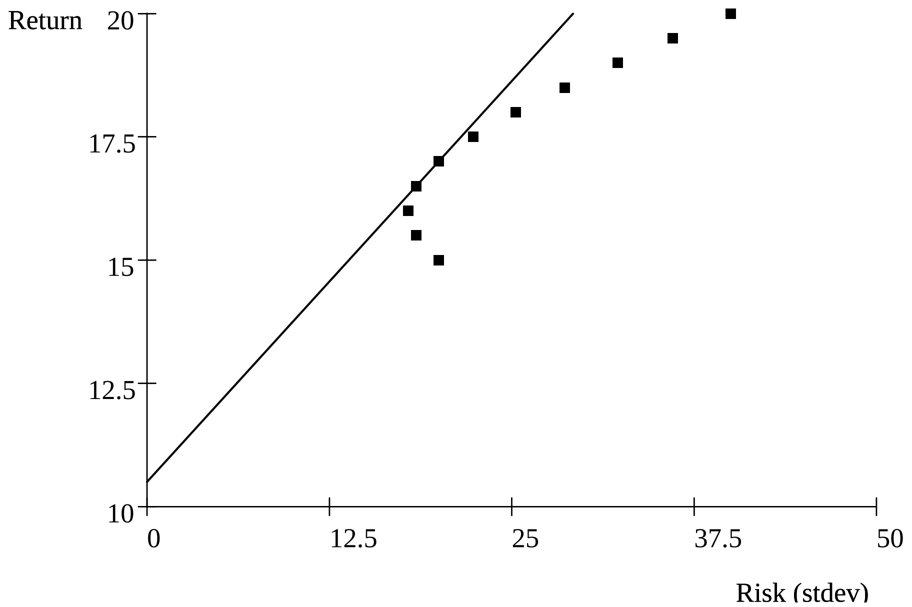
(C&W $\rho_{nm}\sigma_n\sigma_m=\sigma_{nm}$)



Diversifiable and systematic risk

1.24 Capital Asset Pricing Model (CAPM)

All identical, risk averse investors end up holding a (linear) combination of the risk free asset and the market portfolio (two fund separation) and therefore reside on a security market line (SML). Slopes of this tangency line from the risk free and of the capital market line (CML) efficient frontier must be equal at the point of contact *moreover* this point of intersection must represent the aggregate market portfolio if all assets are to be held (by someone).



Risk return frontier and tangent

Considering a portfolio with a fraction a in a risky asset i and $(1-a)$ in the market asset the expected return and risk are given respectively by

$$\begin{aligned} R_p(a) &= aR_i + (1-a)R_m \\ \sigma_p(a) &= \sqrt{a^2\sigma_i^2 + 2a(1-a)\rho_{im}\sigma_i\sigma_m + (1-a)^2\sigma_m^2} \end{aligned}$$

The rates of change of these variables w.r.t. the fraction a are

$$\begin{aligned} \frac{\partial R_p(a)}{\partial a} &= R_i - R_m \\ \frac{\partial \sigma_p(a)}{\partial a} &= \frac{2a\sigma_i^2 + (2-4a)\rho_{im}\sigma_i\sigma_m - 2(1-a)\sigma_m^2}{2\sqrt{a^2\sigma_i^2 + 2a(1-a)\rho_{im}\sigma_i\sigma_m + (1-a)^2\sigma_m^2}} \end{aligned}$$

and their values at $a = 0$ are

$$\begin{aligned} \left.\frac{\partial R_p}{\partial a}\right|_{a=0} &= R_i - R_m \\ \left.\frac{\partial \sigma_p}{\partial a}\right|_{a=0} &= \frac{\rho_{im}\sigma_i\sigma_m - \sigma_m^2}{\sigma_m} \end{aligned}$$

If all the assets are to be held in equilibrium there must be no excess or shortfall in demand for each asset i so that the slope of the risk return trade off at $a = 0$ must be equal to the slope of the CML from the risk free to the market portfolio

$$\begin{aligned}
\left.\frac{\partial R_p/\partial a}{\partial \sigma_p/\partial a}\right|_{a=0} &= \frac{R_i - R_m}{\left(\rho_{im}\sigma_i\sigma_m - \sigma_m^2\right)/\sigma_m} = \frac{R_m - R_f}{\sigma_m} \\
R_i - R_f &= \left(R_m - R_f\right)\frac{\left(\rho_{im}\sigma_i\sigma_m - \sigma_m^2\right)}{\sigma_m^2} + R_m - R_f \\
R_i - R_f &= \left(R_m - R_f\right)\frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2}
\end{aligned}$$

We have been writing R_i for the expected return $E[R_i]$ on asset i so we now (finally) have

$$\begin{aligned}
E[R_i] - R_f &= \frac{Cov(R_i, R_m)}{Var(R_m)}\left(E[R_m] - R_f\right) \\
E[R_i] - R_f &= \frac{\rho_{im}\sigma_i}{\sigma_m}\left(E[R_m] - R_f\right) \\
&= \beta_{im}\left(E[R_m] - R_f\right) \\
\boxed{\text{excess exp. return}} &= \boxed{\text{beta } \times \text{exp. risk premium}}
\end{aligned}$$

and expected excess returns are driven by the individual asset's covariance of returns with the market and a market risk premium

1.25 Examples of beta

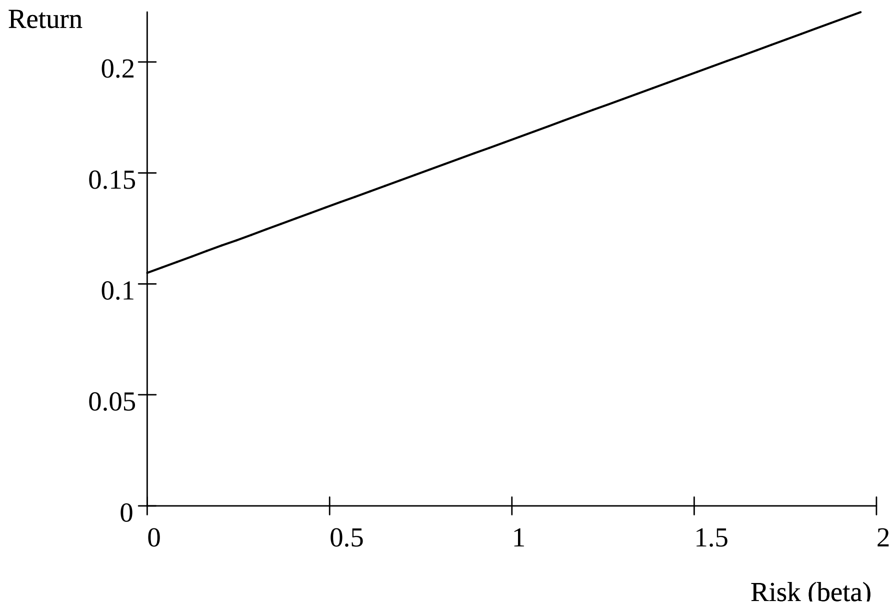
What beta to use for the following?

- The average industrial firm
- A firm of higher than average leverage
- Oil or Gold
- Works of art
- Your future wages
- Options?

1.26 The risk premium

Historically = 8%, 6%, 4%?, but we care about the *expected (future) premium*? Some debate over level although direction is considered to be falling, state your assumption!

$$\begin{aligned} R_f &= 10.5\%, RP = 6\%, E\left[R_m\right] = 16.5\% \\ E\left[R_i\right] &= 0.105 + 0.06\beta_i \end{aligned}$$



The Security Market Line

1.27 Linearity in R_i and β_i

Since returns are linearly related to betas, the (asset or liability) WACC can be re-expressed in terms of betas from the CAPM

$$\begin{aligned} R_i &= R_f + \beta_i RP \\ R_a &= R_f + \beta_a RP \quad etc \end{aligned}$$

$$\frac{B}{A}R_b + \frac{C}{A}R_c = R_a = \frac{D}{A}R_d + \frac{E}{A}R_e$$

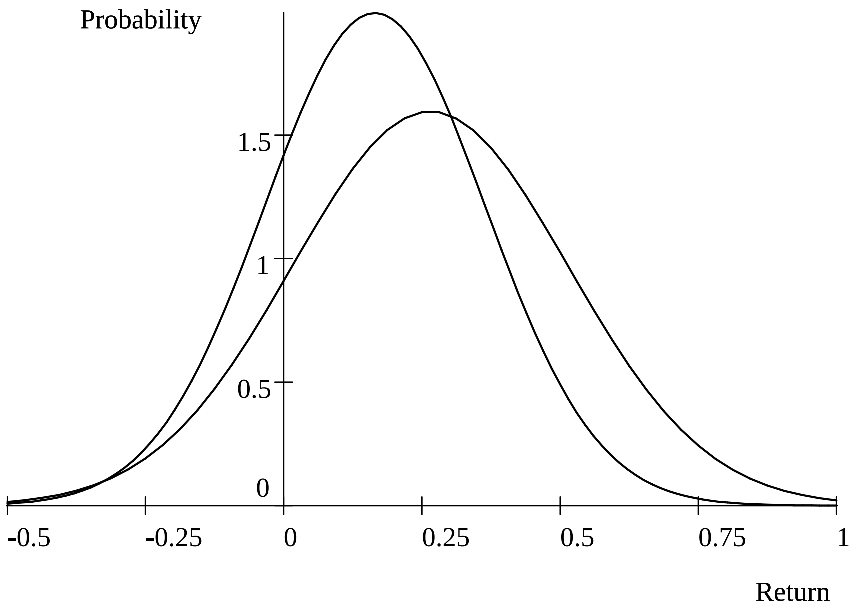
$$\boxed{\tfrac{B}{A}\beta_b + \tfrac{C}{A}\beta_c = \beta_a = \tfrac{D}{A}\beta_d + \tfrac{E}{A}\beta_e}$$

and therefore each of the asset or liability components should also lie on the SML with the weighted average residing in the (weighted) middle.

1.28 Is it possible to beat the market?

Sure! Somebody is doing it every day, it is just that it is not the same guy every day and we can't say in advance who will do it (i.e. it is like winning the lottery)! The existence of big winners (and losers) does not contravene efficiency, only if there were systematic winners might we get interested. Darwinian selection takes care of the systematic losers.

Efficient market say that *on average* you will get a return that compensates you for the risks you take and your expected return is on the capital market line and therefore you will neither beat nor fall behind the market, you can chose the level by constructing a portfolio with the appropriate beta, so another way to beat the market is to take more risk than the market portfolio and this will lead to higher expected returns! (but also a higher chance of large losses).



High expected return, high risk agianst low expected return, low risk

$$
\begin{aligned}
R_f &= 10.5\%, RP = 6\%, E[R_m] = 16.5\% \\
E[R_i] &= R_f + \beta_i E[R_m - R_f] = 0.105 + 0.06\beta_i
\end{aligned}
$$

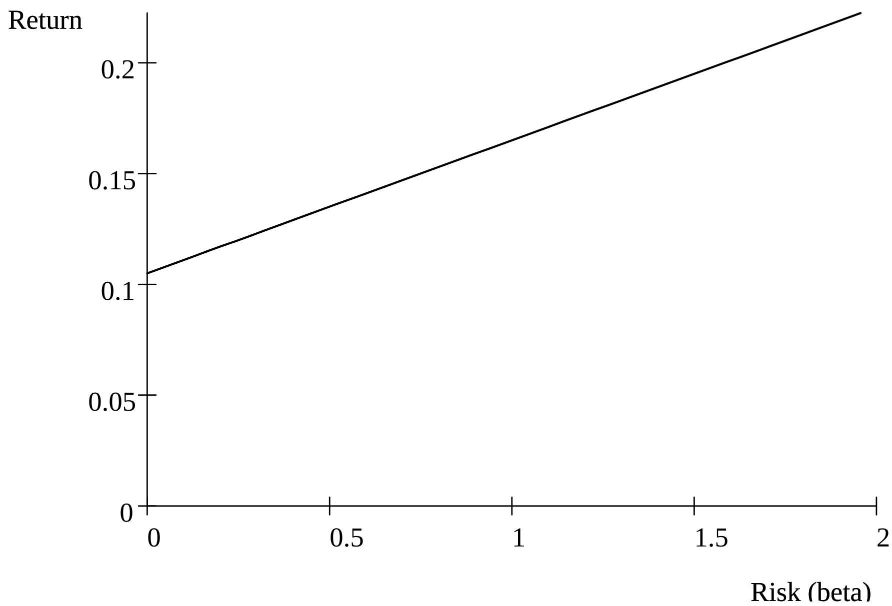
$$
\begin{aligned}
V_p E[R_p] &= V_a E[R_a] + V_b E[R_b] \\
&= (V_a + V_b) R_f + (E[R_m] - R_f)(V_a \beta_a + V_b \beta_b) \\
&= V_p R_f + (E[R_m] - R_f) V_p \beta_p
\end{aligned}
$$

Therefore a better test of real outperformance for a portfolio is to construct the risk adjusted return and to compare this to the risk premium

$$E[R_p] = R_f + \beta_p (E[R_m] - R_f)$$

$$\boxed{\frac{\text{realised excess returns}}{\text{risk measured by } \beta_p} = \frac{R_p - R_f}{\beta_p} \lesseqgtr (E[R_m] - R_f)?}$$

If it is significantly greater than the risk premium, then real risk adjusted returns may have been in excess of the SML level



Expected return against beta

1.29 Tests of Market Efficiency and the CAPM

Roll's critique (1977) [53] says that Market Efficiency and the CAPM cannot be tested separately, i.e. tests of CAPM is always a joint test of ME & CAPM.

Fama and MacBeth (1973) [18] was one of the earlier papers attempting to test the CAPM, in practice more attention is focussed on the composition of the market portfolio than the CAPM per se which can always be considered correct in a theoretical world. The practical conclusion is that you always need to account for risk; larger excess returns can always be produced by the larger risk associated with a position further up the SML.

1.30 If β became important what happened to α?

Alpha α corresponds to the intercept of the slope line of a regression of excess stock returns against the excess market return. From the CAPM alpha should be zero

$$\begin{aligned} R_i - R_f &= \alpha_i + \beta_i (R_m - R_f) + \epsilon \\ \alpha_i &= 0 \end{aligned}$$

$$\boxed{E[R_i - R_f] = \beta_i (R_m - R_f)}$$

This can be tested by looking at sample α's for systematic deviation from 0 since in an efficient market the CAPM tells us they should be zero.

1.31 Other factor models & the APT

Ross (1976) [54] formulated another multifactor models of asset returns. It is important to note that it is not of the same theoretical standing as the CAPM (which has one factor alone; that of the market return), however many empiricists seek other factors that explain asset returns. The sort of factors that are sought are things like price changes (or returns) in oil and other commodities, inflation and interest rate variables. Labelling the factors with indices i the return on one security can be explained through the return (or expected return) on all of the N factors

$$\begin{aligned} E[R_a - R_f] &= \beta_1 (R_1 - R_f) + \beta_2 (R_2 - R_f) + \beta_3 (R_3 - R_f) \ldots \qquad (4) \\ &= \boxed{\textstyle\sum_{i=1}^{N} \beta_i (R_i - R_f)} \end{aligned}$$

There is another model of asset returns that has the same theoretical support as the CAPM and nests it while including another factor. Merton's Inter–temporal CAPM (1973) [40] considers reinvestment risk (or exposure to interest rates) as a second factor distinct from the market risk that participants may wish to hedge.

Finally, currency risk between economies may or may not be priced, if it is then there is a role for exchange rates as global factors that affect asset returns (in the base currency say dollars). Solnik (1974) [62] and others describe how the currency exchange rates (if these risks are priced) may play the role of pricing factors in a similar fashion to Equation 4.

2 Market Efficiency & Random Walks

2.1 Asset valuation

- In economics the *law of one price* says that competition will drive the price of identical goods to the same level
- In finance this law is enforced through *arbitrage* an activity pursued by speculators or other investors seeking to purchase something for less than its intrinsic value and sell it for more
- Physical goods (foods, equipment etc.) can be difficult to arbitrage (buy in one location and sell in another) because of transaction costs
- Financial assets (stocks, bonds, currencies) are easy to arbitrage, if they sold for different prices in different locales, it would be easy to exploit price differentials

2.2 Measures of value

- Accounting Book Value involves *conservatism* so understate value
  - Lower of cost or net realisable value plus accrued profits less distributions
- Earnings multiples, price/earnings (P/E) and earnings per share (EPS)
  - $P/E = \frac{P}{EPS}$ with $EPS = \$2$ and an industry $P/E = 10$ one estimate of the share price would be \$20
- Cashflow models from company and public information, analysts forecasts and all other available information
  - Lay out all future cash flows and take an NPV using a required rate of return (interest rate)
- Ask around others for their opinion! Guess!
- Look at the price at which others are prepared to trade, this is the *Market Value*
- Market to Book values are normally greater than one and can easily exceed 5!

2.3 Efficient Markets

- The *efficient markets hypothesis (EMH)* says that prices must reflect the amount of information know by the market participants, if this were not the case, someone with better information would be able to exploit his arbitrage opportunity! There are three levels of market efficiency attributable to Fama (1970 [16], 1991 [17]).

- **Weak form:** No investor can earn excess returns by developing trading rules based on *historical prices* or return information. In other words, the information in past prices is not useful or relevant in achieving excess returns. Those with *superior public or private information* can still potentially profit.

- **Semi-strong form:** No investor can earn excess returns from trading rules based on *publicly available information.* (e.g. investment advice, annual reports and accounts and of course past prices). Those with *superior private information* (insiders) can still potentially profit.

- **Strong form:** No investor can earn excess returns using *any information (i.e. including private insiders)*, whether publicly or not.

- **Weak** $\subset$ **Semistrong** $\subset$ **Strong**. The weak form is contained in the semi–strong form which is it itself contained in the strong form.

2.4 Random walks

- One consequence of the EMH is that when markets fully anticipate all available information they appear to follow a random walk, Samuelson (1965) [56] was one of the first academics to show this.

- New (*unanticipated*) information arrives randomly so that in efficient markets prices follow **random walks**. e.g.

- Asset prices go up if unexpected news is good and down if unexpected news is bad (e.g. *unexpected* change in interest rates, fall in *seasonally adjusted* employment, change in *anticipated* monetary policy and inflation etc.).

- Go up if a head is tossed, go down if a tail is tossed in a repeated game (if the coin is unbiased, no matter how far ahead we look we *expect* – on average – to be where we are now).

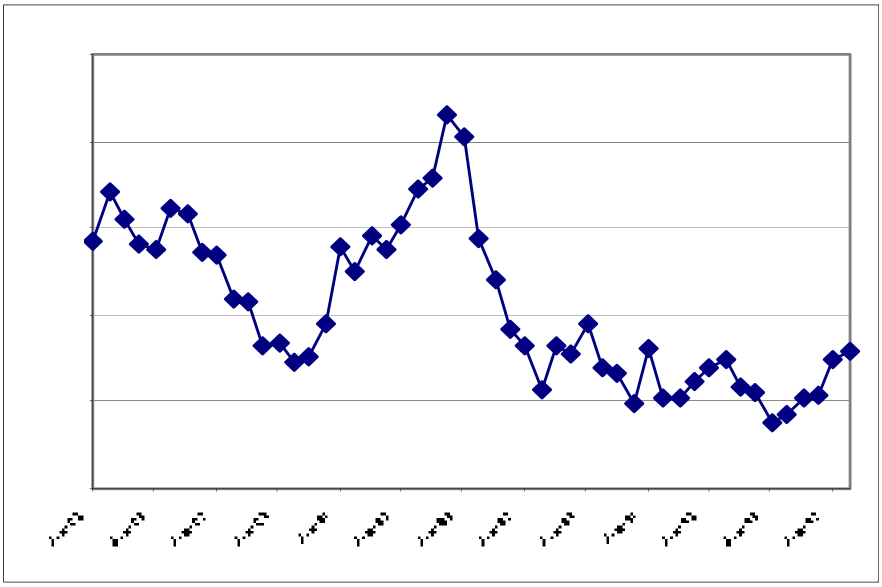


Figure 2: First series

- Spread bet on point differential in a basketball game, I pay you £1 for each point your team score and you pay me £1 for each that my team score (unbiased?).

2.5 50 half–years (25 years), quarters, months, weeks, days and hours of foreign exchange returns

If you don't believe that markets could behave randomly or in this way at all, then try to say how you could distinguish the graphs from a simulated random walk and also try to discern which of the following six graphs corresponds to 50 Deutsche Mark/U.S.Dollar exchange rates over 25 (half) years, 50 quarters, 50 months, 50 weeks, 50 days, 50 hours! Mandelbrot has written about the *fractal nature* of asset returns [35] (self similarity of scaled variance $\frac{\sigma_T^2}{T}$ over any time horizon).

2.5.1 DEM/USD Time Series

2.6 Is it possible to beat the market?

- Sure! Somebody is doing it every day, it is just that it is not the same guy every day and we can't say in advance who will do it (i.e. it is like

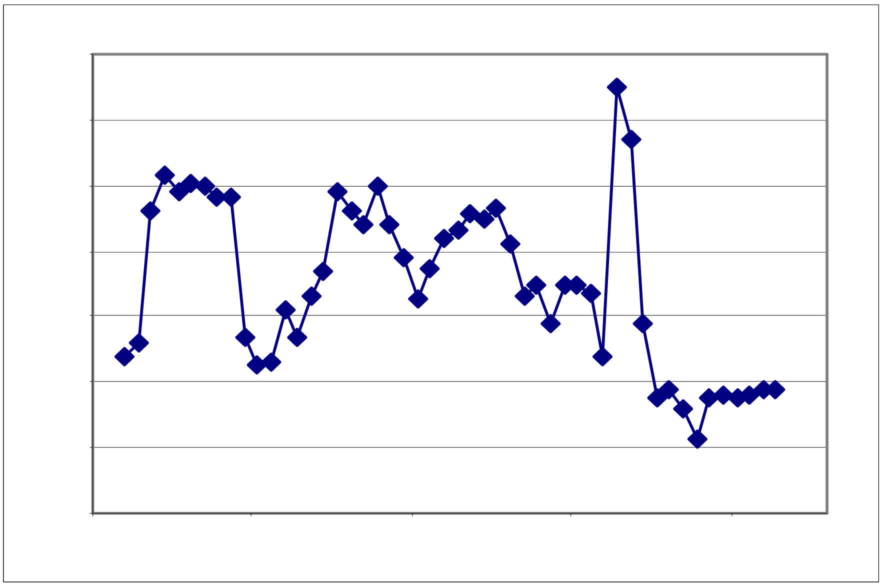


Figure 3: Second series

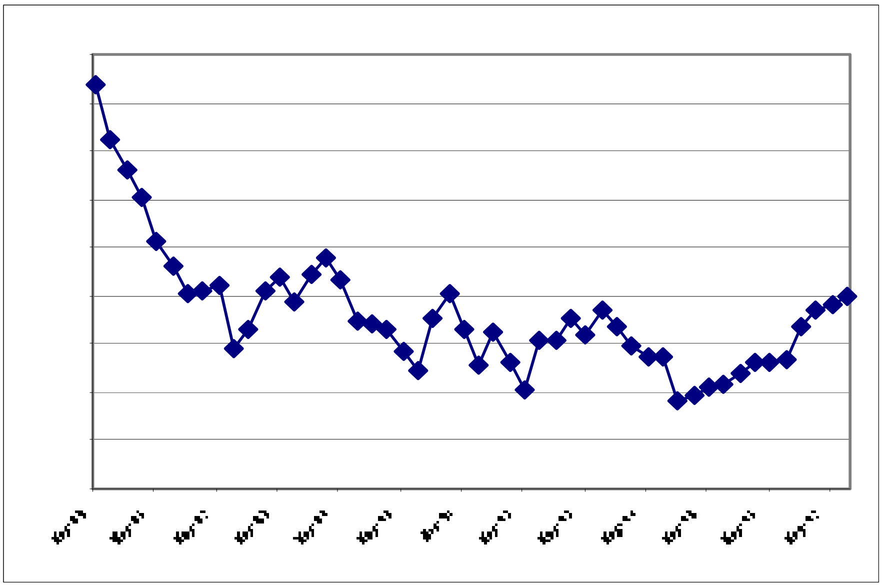


Figure 4: Third series

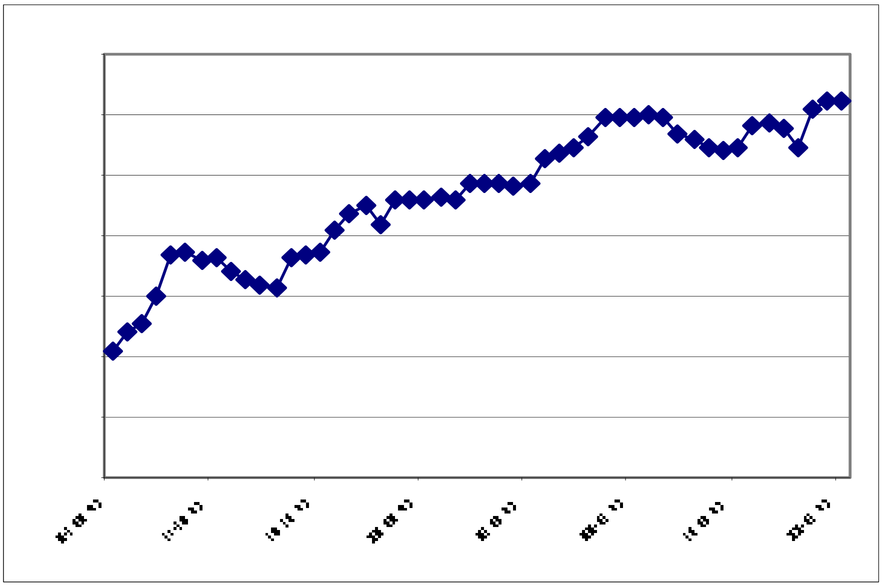


Figure 5: Fourth series

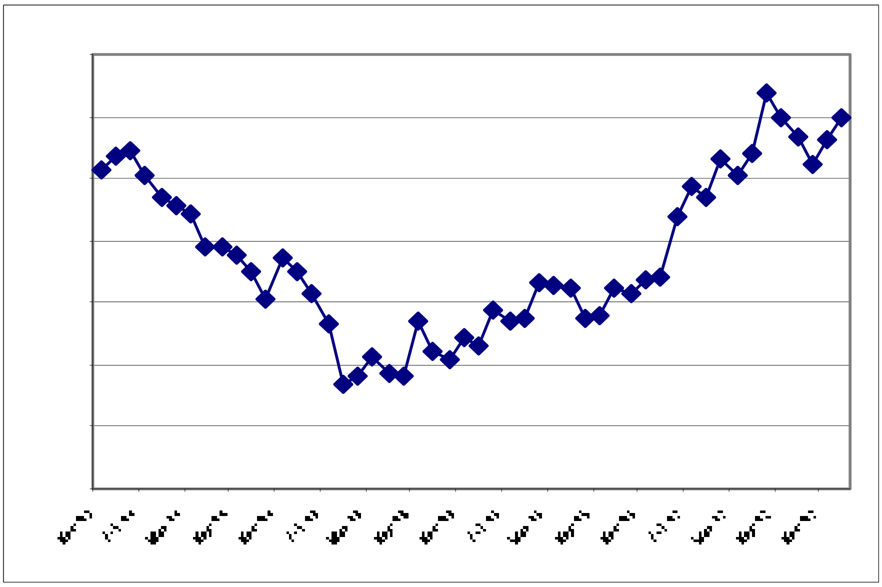


Figure 6: Fifth series

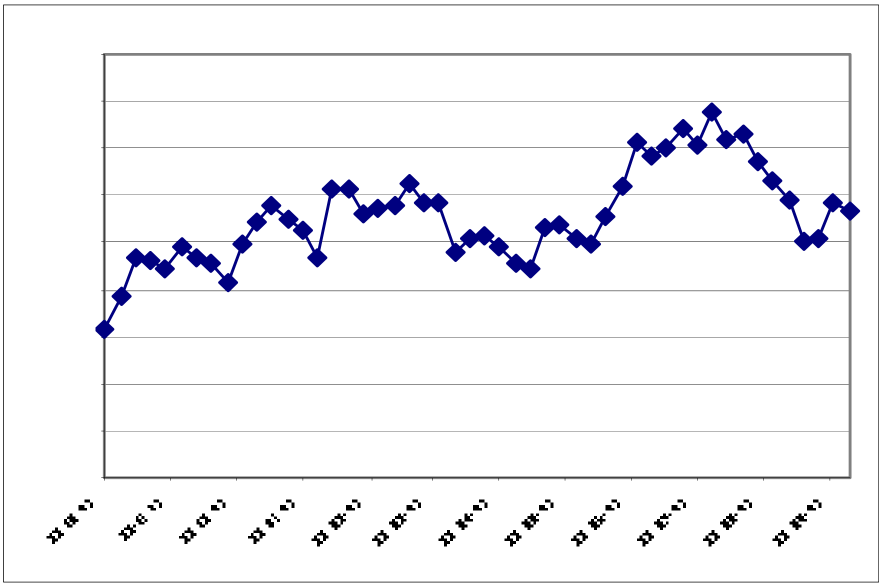


Figure 7: Sixth series

winning the lottery)! The existence of big winners (and losers) does not contravene efficiency, only if there were systematic winners might we get interested. Darwinian selection takes care of the systematic losers.

- Efficient market say that *on average* you will get a return that compensates you for the risks (we have yet to define which risk matters) you take and your **expected** return is fair

- After the risks are realised, you may or may not have been lucky and out or underperformed the market, this does not mean that it was not a fair bet when you invested.

- In efficient markets, it should not be possible to **consistently** beat the market.

2.7 Tests of Market Efficiency and the CAPM

Roll's critique (1977) [53] says that Market Efficiency and the CAPM cannot be tested separately, i.e. tests of CAPM is always a joint test of ME & CAPM. Fama MacBeth (1973) [18] was one of the earlier papers attempting to test the CAPM, in practice more attention is focussed on the composition of the market portfolio than the CAPM per se which can always be considered correct in a theoretical world. The practical conclusion is that you always need to account for risk; larger excess returns can always be produced by the larger risks.

2.8 Market Efficiency debate

- EMH says:–
  - Prices reflect underlying value or the consensus value
  - Market prices may follow a random walk since unanticipated news by definition is random
  - The market risk premium must be positive at all times to persuade investors to hold the risky market instead of the riskless bank account
  - You cannot fool all the people all of the time
- EMH does not say:–
  - Prices are uncaused, they are caused by new news items
  - There is no upward trend in the market. In fact the market drifts up by R_m and the risk free asset increases at a rate $R_f < R_m$.
  - All shares have the same expected return, different firms can offer different rates of return
  - Investors should throw darts to select stocks
- Why are there still disbelievers?
  - There are optical illusions, mirages and *apparent* patterns in stock prices. Our brains have developed to detect patterns, however there are no "patterns" in random walk series!
  - The truth is less interesting than fiction (contrary to popular belief: the Truth is not out there!)
  - From *past* data, there is some weak evidence against efficiency (seasonality, insider trading); tests of the EMH are weak
- Market efficiency is the null hypothesis everyone would like to be able to reject

2.9 Martingales

- If the best guess (expectation) about tomorrow's value is *the same as* today's value, then the (unbiased) process is called a *martingale.*
- If the expectation about tomorrow is higher than today's, it is called a *submartingale.*
- If the expectation about tomorrow is lower than today's, it is called a *supermartingale.*
- Since interest rates and the market risk premium are positive, expected returns are positive and so the market process has to be adjusted for drift before it is a *martingale.*

Since financiers believe that many markets are efficient, martingales (once drift is accounted for) can be used to model future uncertain price movements. The random innovations that arrive day by day and move the market are the news items that affect market expectations.

2.10 Arithmetic Brownian motion

Robert Brown (19th century Botanist) observed smoke particles under a microscope and noticed that their movement (due to the bombardment by smaller gas molecules) was random. In particular over a sample time T the variance of their path length from start to finish σ^2 divided by T seemed to remain constant $\frac{\sigma^2}{T} = k$. Much work on the mathematics of Brownian Motions was done by Albert Einstein and Norbert Wiener at the beginning of this century[3].

Brownian (or Wiener sometimes W is used instead of Z) increments ΔZ_t have zero mean, "unit variance" and are serially uncorrelated (i.e. mutually independent or non anticipating), they accumulate in such a fashion that the distribution of

$$\begin{aligned} Z_T &= Z_0 + \sum_1^T \Delta Z_t \\ \Delta Z_t &\sim n\left(\text{mean} = 0, \text{sd} = \sqrt{\Delta T}\right) \\ \text{Cov}\left(\Delta Z_t, \Delta Z_{t'}\right) &= 0 \end{aligned}$$

[3]subsequently Finance has rediscovered earlier work on Brownian motions by Bachelier (1900) [2] [3].

is *always normal* with mean Z_0 and variance T (or standard deviation $\sqrt{T}$) *irrespective of the time horizon*. Thus the expectation of mean value of Z_T is Z_0 with a standard deviation of $\sqrt{T}$.

In continuous time where time increments dt can be considered arbitrarily small, Brownian increments are labelled dZ_t and the corresponding continuous time definition of Z_T is

$$Z_T = Z_0 + \int_0^T dZ_t$$

Alternatively the distribution of $Z_T - Z_0$ is normal with mean 0 and variance T.

$$\begin{aligned} Z_T - Z_0 &\sim n\left(0, \sigma^2 = T\right) \\ E\left[Z_T - Z_0\right] &= 0 \\ E\left[(Z_T - Z_0)^2\right] &= T \end{aligned}$$

See Figure 8 for a discrete example of 50 simulated arithmetic Brownian Motions and $\pm 1, \pm 2$ standard deviation confidence limits. At any instant, what proportion would you expect to find outside each of the confidence limits?

For a homoscedastic random walk, the expected variance of returns measured over different time horizons T will be proportional to T. Therefore the expected variance of returns over intervals T scaled by T will be a constant.

$$\begin{aligned} E\left[(Z_T - Z_0)^2\right] &= T \\ E\left[(Z_{T'} - Z_0)^2\right] &= T' \\ \frac{E\left[(Z_{T'} - Z_0)^2\right]}{T'} &= \frac{E\left[(Z_T - Z_0)^2\right]}{T} = 1 \end{aligned}$$

2.11 ABM Stock Prices?

We could represent stock prices as a cumulated continuous random walk (Arithmetic Brownian Motion) thus (setting $t_0 = Z_0 = 0$)

$$\begin{aligned} dS_t &= \mu dt + \sigma dZ_t \\ &\Updownarrow \\ S_t &= S_0 + \mu t + \sigma Z_t \end{aligned}$$

however the innovations may be sufficiently negative to make the cumulated process become negative! Thus ABMs are not often used to represent stock prices.

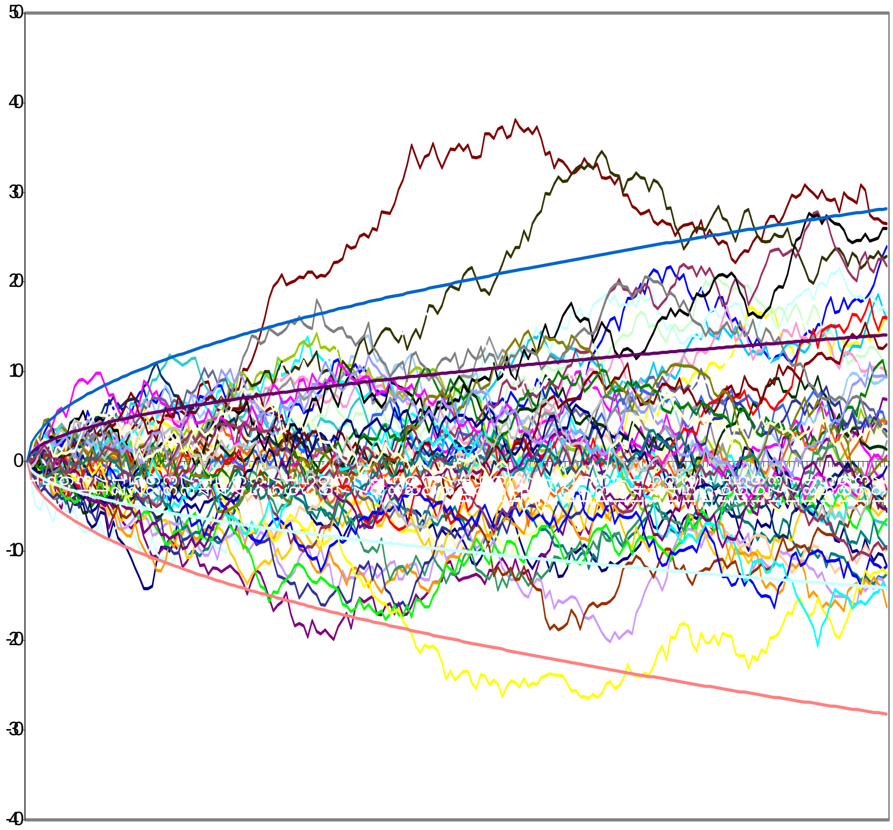


Figure 8: 50 zero mean unit variance simulated (200 obs) ABM's and $\pm 1, \pm 2$ s.d. confidence limits; value y over time x axes

2.12 Geometric Brownian Motion

An example of a geometric Brownian motion would be a heads tails game where the score started at one and is roughly *doubled* (multiplied by $e = 2.71828$) every time a head was tossed and roughly *halved* (divided by e) every time a tail was flipped. What would the expected score look like after T tosses?[4]

$$\begin{array}{cccl} & & & \text{Heads} \quad e^1 = 2.71828 \\ & \nearrow \frac{1}{2} & & \\ 1 & & & \\ & \searrow \frac{1}{2} & & \\ & & & \text{Tails} \quad e^{-1} = 0.36788 \end{array}$$

It can be described by

$$\text{Expected score } = E\left[e^{\#\text{Heads}-\#\text{Tails}}\right]$$

e.g. if 5 more heads than tails are flipped, the score is $e^5 \approx 148.4$ whereas if 5 more tails are flipped than heads the score is $e^{-5} \approx 0.0067$ The situation is not symmetric, the expected number of heads equals the expected number of tails but the function is non–linear so the expectation of the function is **not** the function of the expectation

$$\begin{aligned} E\left[\#\text{Heads} - \#\text{Tails}\right] &= 0 \\ e^{E[\#\text{Heads}-\#\text{Tails}]} &= e^0 = 1 \\ E\left[e^{\#\text{Heads}-\#\text{Tails}}\right] &\gg 1 \end{aligned}$$

What is the expected value? Even if you don't fancy trying this with a coin it is easy to do on a computer! It turns out that the *variance* of the arithmetic process that is exponentiated to generate the geometric process is key. For a small number of tosses the distribution of cumulative heads less tails follows a Binomial (see Formula 5 in later section) and for a large number of tosses the Binomial distribution converges to a normal so that the cumulative number of heads less tails is well described by a Normal Density

$$x_T = \#\text{Heads} - \#\text{Tails in } T \text{ tosses} \sim n\left(0, Tp(1-p)\right)$$

where $p' = 1 - p = \frac{1}{2}$ for an unbiased coin. For T =100 tosses the distribution is close to

$$n\left(0, \sigma^2 = 25\right) \text{ or } n\left(0, \sigma = 5\right)$$

[4] N.B. the expected **one period** payoff is $\frac{1}{2}(2.81828 + 0.36788) = 1.5431$

For a normally distributed variable $x_T \sim n\left(0, \sigma_T = 5\right)$ i.e.

$$\begin{aligned} E\left[x_T\right] &= 0 \\ E\left[x_T^2\right] &= \sigma_T^2 = 25 \end{aligned}$$

and a good statistics text book will tell you that for a normally distributed variable x_T

$$E\left[e^{x_T}\right] = e^{0.5\sigma_T^2}$$

so that the expected score from our game of 100 double or halves is actually a staggering quarter of a million!

$$e^{0.5*25} = e^{12.5} \approx 268,337$$

When many tosses are made, the expected per toss gain (this is like assuming that each toss generates a $n\left(0,1\right)$ outcome instead of a $[+1,-1]$ outcome) is

$$\begin{aligned} E\left[e^{x_1}\right] &= e^{0.5\sigma_1^2} = e^{\frac{1}{8}} = 1.1331 \\ 1.1331^{100} &= 268,337 \end{aligned}$$

By how much would we have to alter the probabilities (bias the coin) to make the expected outcome equal to one (i.e. to make the game a Martingale)?

$$\begin{array}{lll} & & \text{Heads} \quad e^1 = 2.71828 \\ & \nearrow p & \\ 1 & & \\ & \searrow 1-p & \\ & & \text{Tails} \quad e^{-1} = 0.36788 \end{array}$$

The mean #Heads less #Tails outcome is now $T\left(p - \left(1-p\right)\right) = T\left(2p-1\right)$ and the variance of this outcome is $Tp\left(1-p\right)$ and it is normally distributed around this mean. Therefore we can set the expectation of the exponentiated variable to one

$$\begin{aligned} 1 &= E\left[e^{T\left(2p-1+\frac{1}{2}p-\frac{1}{2}p^2\right)}\right] \\ 0 &= -\frac{1}{2}p^2 + \frac{5}{2}p - 1 \\ p &= \frac{5}{2} + \frac{1}{2}\sqrt{17} \approx 0.43845 \end{aligned}$$

i.e. if the payoff is based on the sum of many random tosses (otherwise the binomial will not converge to a normal) a 6% bias will do the trick and make the game fair and will ensure the expected payoff over many games (where the total number of heads or tails is normal) is one.

Here is our first warning that non–linear functions of random variables need special treatment in probability & calculus!

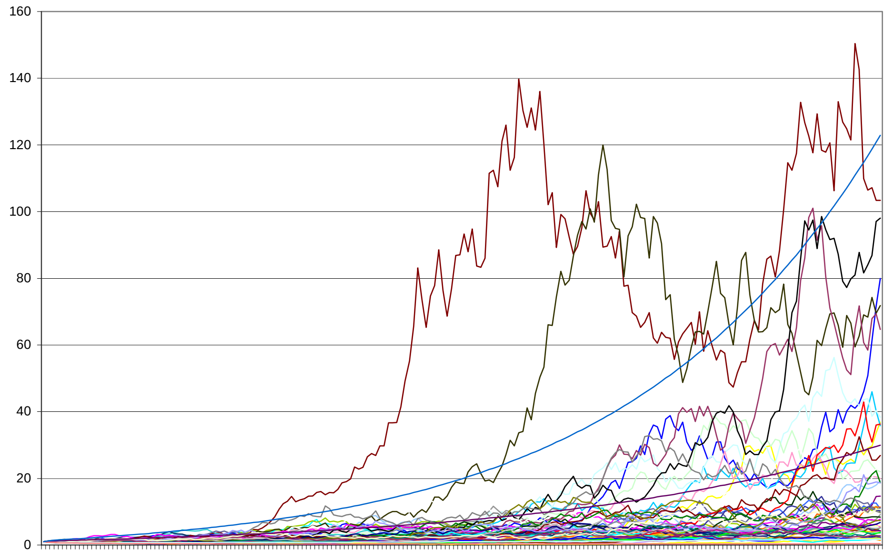


Figure 9: 50 zero mean unit variance simulated (200 obs) GBM's and $\pm 1, \pm 2$ s.d. confidence limits; value y over time x axes.

2.13 Continuous time GBM

If the arithmetic Brownian Motion variable is exponentiated, a geometric series is created (setting $t_0 = Z_0 = 0$).

$$\begin{aligned} e^{Z_T} &= e^{\int_0^T dZ_t} \\ \log e^{Z_T} &= \int_0^T dZ_t \end{aligned}$$

The advantage of representing series as GBM's is that when log returns are calculated, the linear Brownian motion is recovered. Figure 9 shows 50 simulated GBM's again with $\pm 1, \pm 2$ standard deviation confidence limits. For many of the simulations that return low or negative Z_T's, the plot is very near to the x axis and invisible, however the general random growth on growth and similarity to share price series is apparent.

Again we have the property that the expectation of a geometric series is not the geometric of the expectation

$$E\left[e^{Z_T}\right] = E\left[e^{\int_0^T dZ_t}\right] \neq e^{E\left[\int_0^T dZ_t\right]} = e^0 = 1$$

Again using our statistics result, for the normal variable x

$$E\left[e^x\right] = e^{0.5\sigma_T^2} = e^{0.5T\sigma^2}$$

Thus the geometric series in question is not a martingale, since $E\left[e^{Z_T}\right] > 1$, the arithmetic series returns will have to have their mean lowered in order to make the geometric series a Martingale. $Z_1 - Z_0$ has unit variance, $Z_T - Z_0$ has variance T,

$$\begin{aligned} E\left[e^{Z_T - 0.5T}\right] &= 1 \\ E\left[e^{Z'_T}\right] &= 1 \\ \text{where } dZ'_T &= -0.5dt + dZ_t \end{aligned}$$

where the new Brownian motion Z'_T is forced to have negative drift proportional to the time horizon

$$\begin{aligned} Z'_T &= Z_T - 0.5T = \int_0^T (dZ_t - 0.5dt) \\ dZ'_t &= dZ_t - 0.5dt \end{aligned}$$

since $Z_T = \int_0^T dZ_t$. This transformation of the mean return is called a change of measure in mathematical terms and is used in derivative pricing. (N.B. this is a similar transformation to the heads tails game).

2.14 GBM for stock prices

Better than ABM for stock prices, we can use GBM to prevent prices from becoming negative

$$\boxed{\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t}$$

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t}$$

Now returns are lognormally distributed or log returns are normally distributed. The $\frac{1}{2}\sigma^2$ correction to the drift is required because the expectation of an exponentiated normal variable depends on the variance so that when we take real world expectations[5]

$$E^P\left[S_t\right] = S_0 E^P\left[e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t}\right] = S_0 e^{\mu t}$$

This says that the current stock price is in equilibrium with its μ discounted expected payoff at any general horizon t.

$$\boxed{S_0 = E^P\left[S_t\right] e^{-\mu t}}$$

[5] If $x \sim n(\mu, \sigma)$ then $E\left[\exp(\theta x)\right] = \exp\left(\theta\mu + \frac{1}{2}\theta^2\sigma^2\right)$

3 Black Scholes

Definition of options vs. forwards

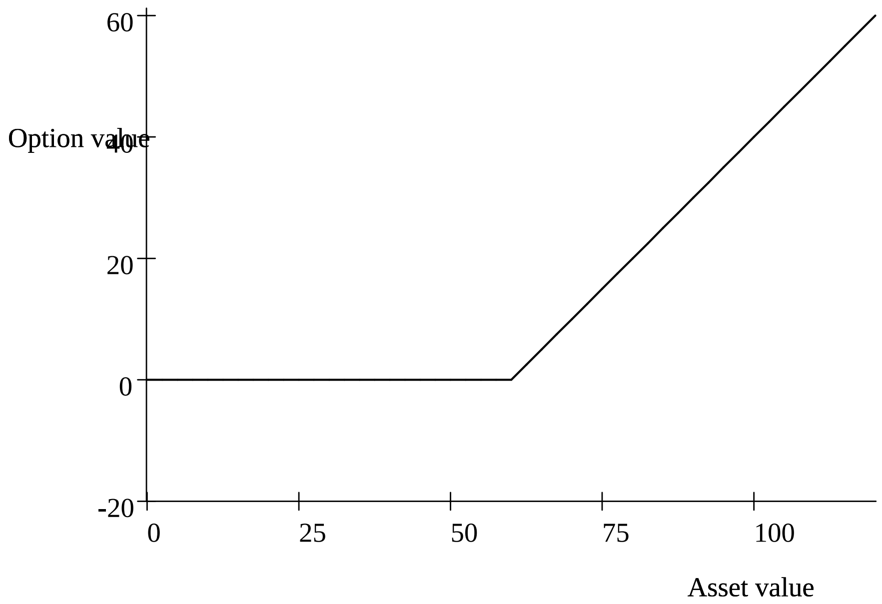
- Call Option: the right *but not* the obligation to buy (call) a stock (commodity etc.) from someone else at a future date
- Put Option: the right *but not* the obligation to sell (put) a stock (commodity etc.) to someone else at a future date
- Forward purchase: the right *and* the obligation to buy at a future date
- Forward sale: the right *and* the obligation to sell at a future date

Calls (puts) share in 100% of the gains/losses when the final stock price is above (below) the *exercise price* and none of the gains/losses when the final price is below (above) the *exercise price*. Being *long an option* means owning an option, and being *short an option* means having sold an option.

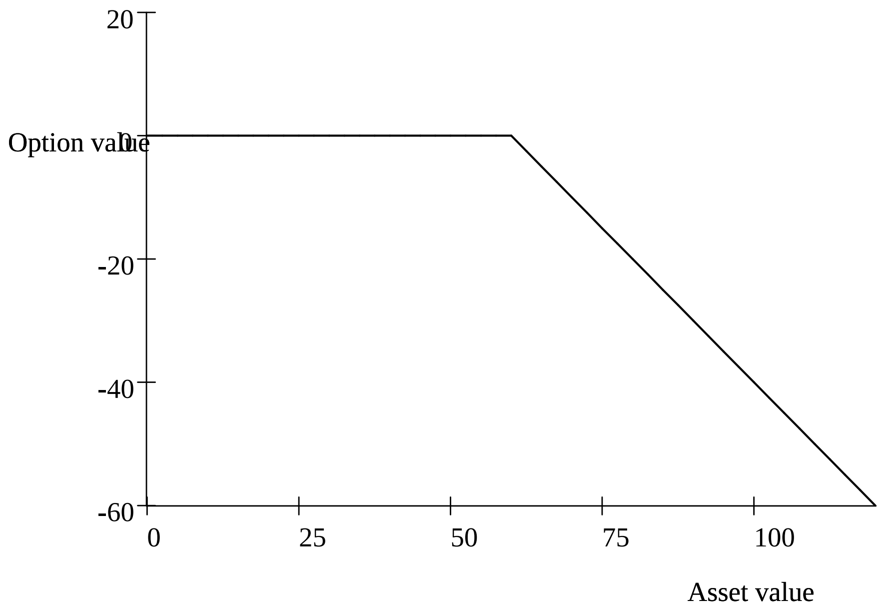
Some Options are exercisable at all times to maturity and these are known as American Options irrespective of their location of issuance or trading or underlying. Other, simpler to price options are known as European, and are exercisable at maturity only, again it is important to note that whether an option has European features may not depend on it's country of origin etc.

3.1 Payoffs at expiry

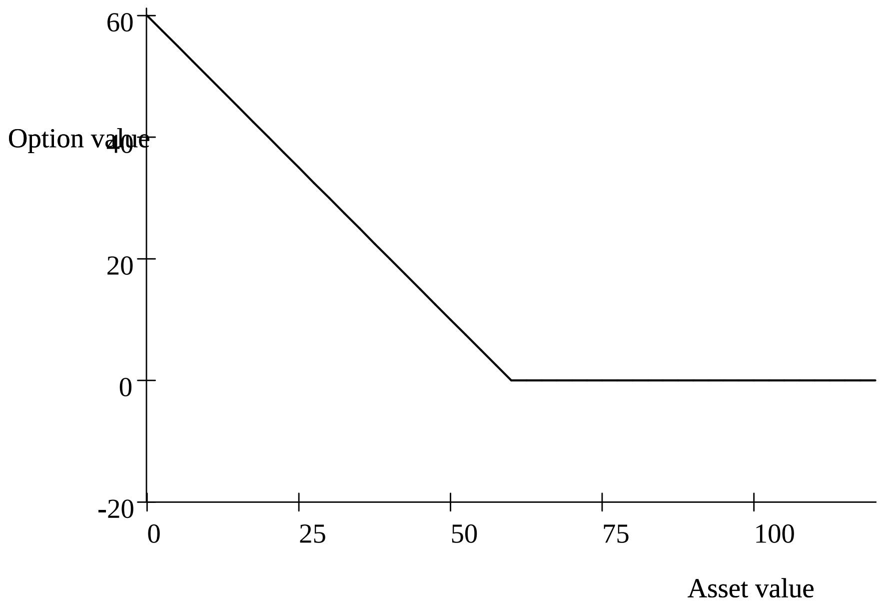
At expiry the option payoffs are graphed



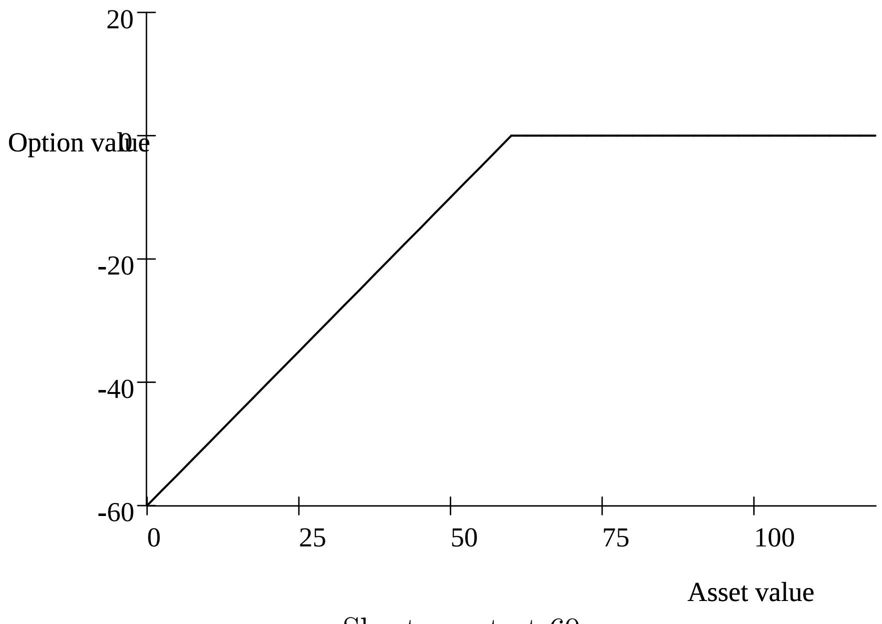
Long a call at 60



Short a call at 60



Long a put at 60



Short a put at 60

These values at expiry are often referred to as *intrinsic* values since they refer to the immediate underlying value of the option (N.B. this may not be immediately accessible through early exercise, i.e. may not be European).

Options are said to be *in the money* if they have positive intrinsic value and *out of the money* if their intrinsic value is zero (cannot be negative because an option once purchased cannot become a liability!)

3.2 Black Scholes Hedging

The seminal papers are Black and Scholes (1973) [5] and Merton (1973) [41]. The concept that won the 1997 Nobel Prize for Economics (for Myron Scholes along with Robert Merton, Fischer Black died in 1995) was that options could be priced assuming the option writer (seller) used a dynamic (changing over time) hedging strategy to defray his potential losses. If he had written a call that was currently out of the money (zero sensitivity to the stock price) then he had little to worry about or hedge but as the option came into the money and the sensitivity of that position reached 100% of the underlying, he had better adjust his hedge toward and up to 100% in the underlying (in some smooth and dynamic fashion) since deep in the money option price changes go 1:1 with underlying. Thus owning a fraction (*delta*) of the underlying (between 0 and 1) and adjusting this over time allow neither net profit or loss on the transaction (option sale plus hedge). One way to analyse a trading strategy where hedge ratios can change is to split time to maturity into chunks and form a tree.

3.3 Binomial Trees

N.B. C&W use q first and then p later, common terminology is now the other way round, real world probabilities (firstly) are here labelled p and (secondly) risk neutral probabilities are labelled q.

From a starting price at $t = 0$ of \$20 consider the two possibilities of increasing by +20%, ($u = 1.20$) or decreasing by −33% ($d = 0.67$) with general (to be determined) probabilities p and $1 - p$. The risk free rate will turn out to be 10%. The following conditions must hold

$$\begin{array}{ccccccc} u & > & 1+R_f & > & 1 & > & d \\ 1.2 & > & 1.1 & > & 1.0 & > & 0.67 \end{array}$$

Now

$$\begin{array}{cccll} t=0 & & & t=1 & \\ & & p & uS = 24.0 & \text{Call at 21} \\ & \nearrow & & & \text{has value 3} \\ S=20 & & & & \\ & \searrow & & & \text{Call at 21} \\ & & 1-p & dS = 13.4 & \text{has value 0} \end{array}$$

How would a one period call at \$21 be priced? First construct a risk-free hedge consisting of one share of stock, S and $m = \frac{1}{\Delta}$ shares of a call option written against the stock. If the call costs c then the payoffs will be as follows

$$
\begin{array}{cccc}
 & p & uS & -mc_u \\
 & \nearrow & & \\
S - mc & & & \\
 & \searrow & & \\
 & 1-p & dS & -mc_d
\end{array}
$$

By construct the end of period payoffs should be the same so that we are indifferent between the two (random) outcomes.

$$uS - mc_u = dS - mc_d = (13.40)$$

$$\boxed{m = \frac{S(u-d)}{c_u - c_d}} = 3.53$$

$$\frac{1}{m} = \Delta = \frac{c_u - c_d}{S\,(u-d)} = \frac{1}{3.53} = 0.283$$

Apart from thinking of 3.53 calls as being equivalent to 1 unit of stock, 1 call could be considered equal to $1/3.53 = 0.283$ units of stock. The option delta is the reciprocal of $m, \Delta = 1/m$ and is often called the hedge ratio since it is the number of units of stock needed to hedge one call option. A holding of one stock will balance 3.53 (one period) calls and $S - mc$ must appreciate to 13.40 at the riskless rate. The present value of this 13.40 can be thought of as the amount of borrowing B required to establish the position $S - mc$ in the first place, it can be repaid $B\,(1 + R_f)$ from either of the terminal values $uS - mc_u$ or $dS - mc_d$

$$
\begin{aligned}
S - mc &= B \\
uS - mc_u &= B\,(1 + R_f) \\
dS - mc_d &= B\,(1 + R_f) \\
B &= \frac{S\,(dc_u - uc_d)}{(1 + R_f)\,(c_u - c_d)} = \frac{13.40}{1.1} = 12.18
\end{aligned}
$$

Now the call price c can be established as a function of two pseudo probabilities $q, 1 - q$

$$
\begin{aligned}
(1 + R_f)\,(S - mc) &= uS - mc_u \\
c &= \frac{S\,[(1 + R_f) - u] + mc_u}{m\,(1 + R_f)} = \frac{qc_u + (1 - q)\,c_d}{1 + R_f}
\end{aligned}
$$

$$\text{where } \boxed{q = \frac{\left(1+R_f\right)-d}{u-d}}$$

$$\text{and } \boxed{1 - q = \frac{u-\left(1+R_f\right)}{u-d}}$$

the probabilities $p, 1-p$ do not appear because it is assumed that the stock itself must be already fairly priced

$$\begin{aligned} S &= \frac{E^P[S]}{1+R_f+\beta E[R_m-R_f]} \\ &= \frac{puS+(1-p)dS}{1+R_f+\beta E[R_m-R_f]} \\ p &= \frac{1+R_f+\beta E[R_m-R_f]-d}{u-d} \end{aligned}$$

i.e. there is a restriction on the weighted mean of u, d whereas there is no restriction on the dispersion of u, d (which turns out to be the volatility).

$$\boxed{c = \frac{qc_u+(1-q)c_d}{1+R_f}}$$

$$\begin{aligned} c &= \frac{\left(1-\frac{d}{1+R_f}\right)c_u + \left(\frac{u}{1+R_f}-1\right)c_d}{u-d} \\ &= \frac{\$3q+\$0(1-q)}{1.1} = \$2.2126 \end{aligned}$$

So the call option price c is a (discounted) expectation of the two possible outcome values c_u, c_d where the probabilities implied $q, 1-q$ are so called *risk neutral probabilities*[6] because they are equal to $p, 1-p$ if and only if the expected return on the stock itself is equal to R_f (as oppose to $R_f + \beta E[R_m - R_f]$ as the CAPM would predict).

Generally speaking

$$\begin{aligned} p &= \frac{1+R_f+\beta E[R_m-R_f]-d}{u-d} \\ q &= \frac{1+R_f-d}{u-d} \\ \frac{p}{q} &= \frac{1+R_f+\beta E[R_m-R_f]-d}{1+R_f-d} = 1+\frac{\beta E[R_m-R_f]}{1+R_f-d} \\ p-q &= \frac{\beta E[R_m-R_f]}{u-d} \end{aligned}$$

Alternatively, $q = p$ if $\beta E[R_m - R_f] = 0$ i.e. $\beta = 0$ or if $E[R_m - R_f]$! For this zero beta, calculation yields $p = q = 0.8113$, $1-p = 1-q = 0.1887$

[6] This rationalisation was provided by Cox & Ross (1976) [12].

and $c = \$2.2126$ so that the hedged portfolio has initial value $B = S - mc =$ \$12.18 and rate of return 10%. In fact the stock also has an expected rate of return of 10% (the risk free rate) $pu + (1-p)\,d = 1.1$; were the beta of the stock greater than zero, its expected rate of return would be greater than R_f and the option rate of return would also be greater than R_f and the stock return itself.

N.B. The option price does not depend on p (or equivalently the expected stock return $u + d$) since the stock price has already aggregated all diverse views about p (and $u + d$) in the current price! However, the option price does critically depend on $u - d$ the *dispersion* (or volatility) of future possible stock prices.

From a *series* of periods, trees can be made to encompass option pricing over any horizon. See example.

3.4 Moving to the Black Scholes formula

Considering a tree of T days length (i.e. with T branching points) we need to know the number of ways of arriving at the final points. This is like asking how likely we are to receive k heads if we toss a coin T times and is given by the Binomial distribution[7]

$$\text{Binomial Probability of } k \text{ from } T \;\; B\left(k|T,q\right) = \frac{T!}{(T-k)!k!} q^k \left(1-q\right)^{T-k} \quad (5)$$

The value of the call at expiry as a function of the final stock price level (the accumulated % changes from the current price) cannot be negative and so is given by a $\max\left(\cdot, 0\right)$ formula

$$c_{final} = \max\left(u^k d^{T-k} S - X, 0\right)$$

Summing the likelihoods multiplied by the payoffs and discounting gives the (certainty equivalent weighted) option price c

$$c = \frac{1}{\left(1+R_f\right)^T} \sum_{k=0}^{T} \frac{T!}{(T-k)!k!} q^k \left(1-q\right)^{T-k} \max\left(u^k d^{T-k} S - X, 0\right)$$

However for many of these final paths, the option will expire worthless ($u^k d^{T-k} S < X$) so many terms of the summation can be ignored (removing those items for which the 0 dominates in the max. statement), also the

[7]! means the factorial function $x! = \prod_{i=1}^{x} i$, e.g. $5! = 1.2.3.4.5 = 120$

two elements $u^k d^{T-k} S, X$ can be separated and defining a new probability q'

$$\begin{aligned} q' &= \frac{u}{(1+R_f)} q \\ 1-q' &= \frac{d}{(1+R_f)} (1-q) \end{aligned}$$

the call price can be written as the sum of two easier Binomial terms

$$c = S \sum_{k=a}^{T} \frac{T!}{(T-k)!k!} (q')^k (1-q')^{T-k} - \frac{X}{(1+R_f)^T} \sum_{k=a}^{T} \frac{T!}{(T-k)!k!} q^k (1-q)^{T-k}$$

where the Binomial terms are interpreted as probabilities of finishing above some threshold a, the smallest non–negative integer greater than a specific variable, if $a > T$, then $c = 0$

$$\begin{aligned} T &> a > 0 \\ & a \overset{int}{>} \frac{\ln\left(X/Sd^k\right)}{\ln(u/d)} \end{aligned}$$

Firstly however it is useful to move to the case where the number of time periods T is very large. We can use the continuous compounding result

$$\lim_{k\to\infty} \left(1 + \frac{j}{k/T}\right)^{k/T} = e^j$$

and by making the size of the steps in the tree very small as well

$$u = e^{\sigma\sqrt{T/k}} \qquad d = e^{-\sigma\sqrt{T/k}}$$

the Binomial probabilities reduce to cumulative areas under a normal curve

$$\begin{aligned} \lim_{T\to\infty} \frac{T!}{(T-k)!k!} (q')^k (1-q')^{T-k} &= n(q, q(1-q)) \\ \lim_{T\to\infty} \sum_{k>a} \frac{T!}{(T-k)!k!} (q')^k (1-q')^{T-k} &= \int_{-\infty}^{d} n(q, q(1-q))\, dq = N(d) \end{aligned}$$

and the price of a call option reduces to

$$\boxed{C = SN(d_1) - Xe^{-rT} N(d_2)}$$

where the normal areas are defined on two variables (concisely using $\pm$)

$$\begin{aligned} d_1 &= \frac{\ln S - \ln X + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln S - \ln X + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned}$$

$$\boxed{d_1, d_2 = \frac{\ln S - \ln X + rT \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}}$$

The hedge ratio we discussed earlier is given by

$$\boxed{\Delta = N\left(d_1\right)}$$

while the risk neutral probability of exercise is given by

$$\boxed{\text{RN prob. of exercise} = N\left(d_2\right)}$$

note that this is different to the real world probability of exercise (see later section also Shackleton and Wojakowski [58]).

3.5 Option Inputs

- Stock Price S : the traded value of the stock in question
- Exercise Price X: the critical price above (below) which calls (puts) will be exercised
- Time to Maturity (expiry) T: the fraction of years until the option must be exercised
- Risk Free Interest Rate r: the continuously compounded rate of interest (expressed annually) for the period to maturity of the option
- Volatility σ: the standard deviation (expressed as an annual quantity) of the % stock price movements
- (Dividend Yield δ: the % rate of dividend distribution)

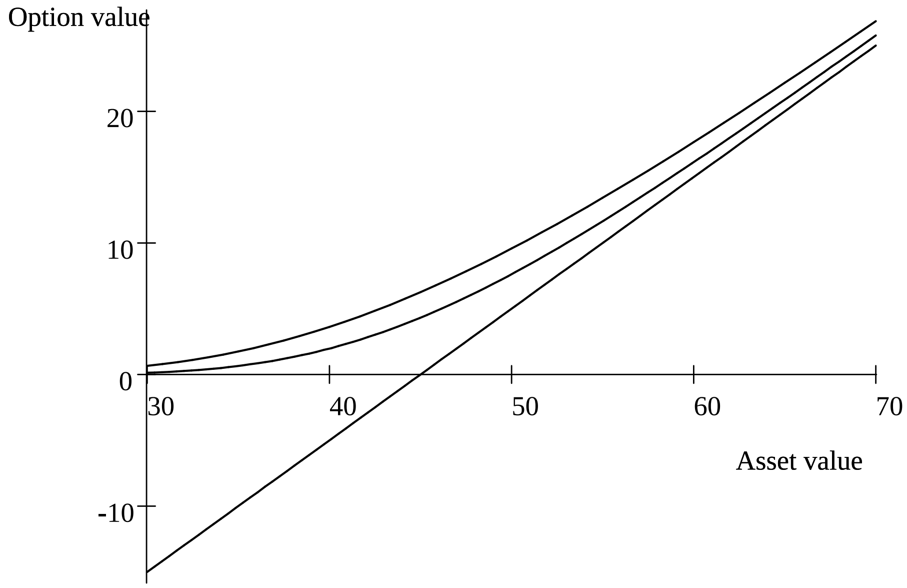
Stock Price	$S = 70$	
Exercise Price	$X = 45$	
Interest Rate	$r = 0.06$	
Volatility	$\sigma = \sqrt{0.2}$	$= 0.44721$
Time to Maturity	$T = 3/12$	$= 0.25$

derived inputs (ln means natural log - base e) Cumulative Normal Distribution

$$\begin{aligned} N\left(d\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} ds \\ &= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{d}{\sqrt{2}}\right)\right) \end{aligned}$$

3.6 Call Price

Using values from above, the call price can be plotted as a function of the stock price S.



Calls at 45 for different maturities T

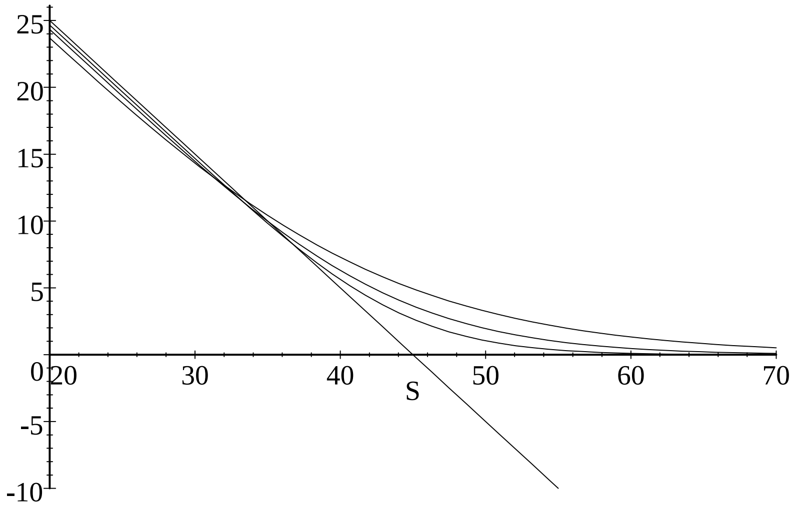
S	$C(S, X=45, T=0.25)$	$P(S, X=45, T=0.25)$	$S - Xe^{-rT}$
30	0.1325	14.462	$30 - 44.330 = -14.330$
40	1.9762	6.3038	$40 - 44.330 = -4.330$
50	7.6243	1.9517	$50 - 44.330 = 5.670$
60	16.136	0.46458	$60 - 44.33.0 = 15.670$
70	25.764	0.09327	$70 - 44.330 = 25.670$

3.7 Put Call Parity and Put Price

Call less Put (of the same exercise price) is equivalent to owning the stock less a borrowed amount since you either exercise the call (and own the stock) or someone will exercise their (put) option on you and sell it to you (so you will definitely own the stock either way if prices go up or down!).

$$
\begin{aligned}
C - P &= S - Xe^{-rT} \\
C &= SN(d_1) - Xe^{-rT}N(d_2) \\
P &= SN(d_1) - Xe^{-rT}N(d_2) - S + Xe^{-rT} \\
&= S(N(d_1) - 1) - Xe^{-rT}(1 - N(d_2))
\end{aligned}
$$

$$\boxed{P = Xe^{-rT}N\left(-d_2\right) - SN\left(-d_1\right)}$$



Puts of various maturity at 45

3.8 Put Call symmetry

There is also another potential relationship between puts and calls of different strikes, which holds only in a world where volatility is constant or symmetric. It is detailed in Carr and Chesney (1996) [10] but implies a symmetric graph of implieds against strike.

$$C\left(S, X, r, \delta, \sigma, T\right) = P\left(X, S, \delta, r, \sigma, T\right)$$

3.9 The Greeks (sorry not Aristotle or Plato!)

Remember Taylor's Expansion for the change in a function given some change x around the point a?

$$f\left(a\right) - f\left(a + x\right) = xf'\left(a\right) + \frac{x^2}{2}f''\left(a\right) + \ldots$$

Express Option Price sensitivities with respect to Stock Prices S

$$\begin{aligned}\text{first total derivative } f'\left(a\right) &= \left.\frac{df}{dS}\right|_a = \Delta \text{ (delta)} \\ \text{second total derivative } f''\left(a\right) &= \left.\frac{d^2f}{dS^2}\right|_a = \Gamma \text{ (gamma)}\end{aligned}$$

Changes in Call & Put Prices given change in share price are given as a series expansion (although we must use *partial* derivatives since many variables are involved)

$$dC, dP(dS) = \Delta dS + \frac{1}{2}(dS)^2 \Gamma + \dots$$

3.9.1 Delta Δ, Gamma Γ

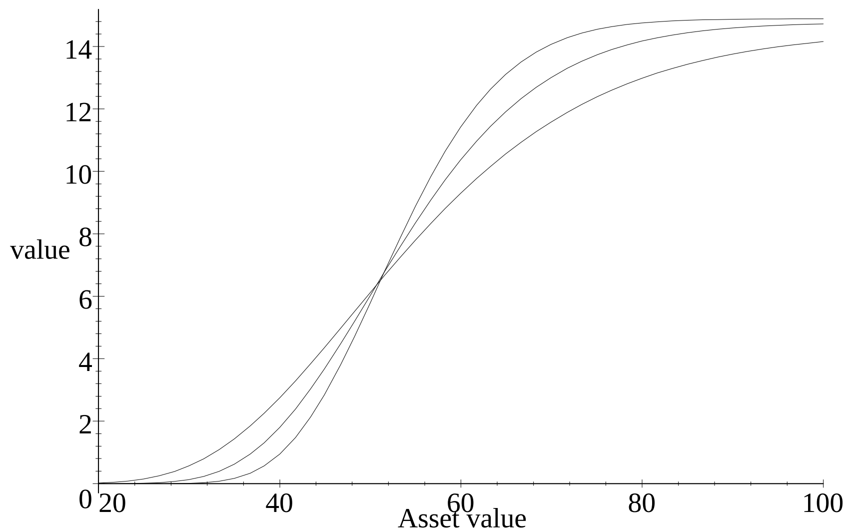
The delta of a call option is given by the first $N(d_1)$ term and therefore has a cumulative normal shape as a function of moneyness

$$\boxed{\Delta_{call} = \tfrac{\partial C}{\partial S} = N(d_1(S))}$$

Note that in the Black Scholes formula there are three dependencies on S, one explicit $SN(d_1)$ and two implicit through $d_{1,2}(S)$. When taking the partial w.r.t. S the first is easy and generates the $\Delta = N(d_1)$ term, the other two are trickier to calculate but generate terms of opposite magnitude that exactly cancel out[8].

$$\begin{aligned} \Delta_{call} &= \frac{\partial C}{\partial S} = N(d_1) > 0 \\ \Delta_{put} &= \frac{\partial P}{\partial S} = N(d_1) - 1 < 0 \\ \frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} &= \Delta_{call} - \Delta_{put} = 1 \text{ (put call parity)} \end{aligned}$$

[8]Note that this is not true of the binary call $BC = Xe^{-rT}N(d_2)$ where the partial w.r.t. S is not zero nor the partial w.r.t. X equal to $e^{-rT}N(d_2)$ because of the non cancellation of the derivative term w.r.t. $d_2(\ln S)$.



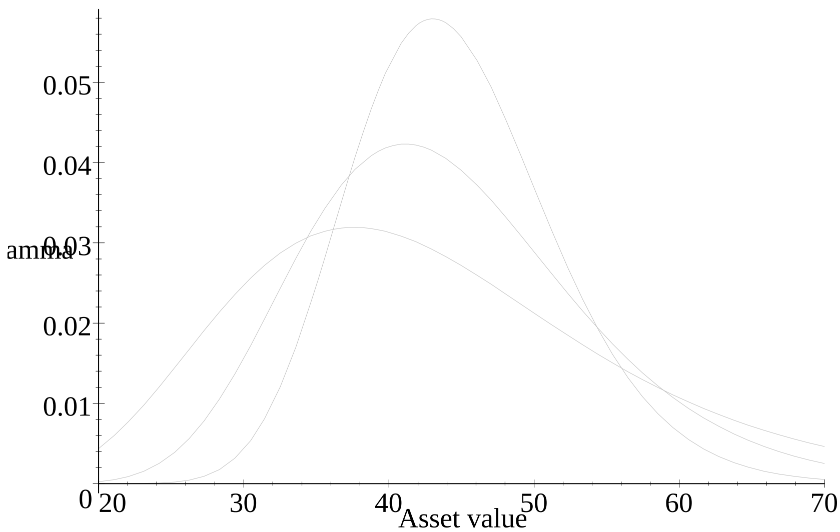
Delta $N(d_1)$ as a function of S

$T = 0.125$

Gamma is given by the partial of the cumulative density and is therefore the density itself

$$\Gamma_{call} = \frac{\partial^2 C}{\partial S^2} = \frac{n(d_1)}{S} = \frac{1}{S\sqrt{2\pi\sigma^2 T}} e^{-\frac{1}{2}d_1^2} = \frac{1}{S\sqrt{2\pi\sigma^2 T}} e^{-\frac{\left(\ln S - \ln X + rT + \frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}}$$

$$\begin{aligned} \frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} &= 1 \text{ (put call parity)} \\ \frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} &= 0 \\ \Gamma_{call} &= \Gamma_{put} \end{aligned}$$



Gamma as a function of S

3.9.2 Kappa κ

The partial differential of the call price w.r.t. to the exercise price X is called kappa and plays a role in the Black Scholes formula, it can be shown to be

$$\kappa = \frac{\partial C}{\partial X} = -e^{-rT} N(d_2)$$

so that the call option can be thought of as holding $\Delta = \frac{\partial C}{\partial S}\ (> 0)$ stocks by borrowing (shorting) $\kappa = \frac{\partial C}{\partial X}\ (< 0)$ bonds

$$C = S\frac{\partial C}{\partial S} + X\frac{\partial C}{\partial X} = \Delta S + \kappa X = SN(d_1) - Xe^{-rT}N(d_2)$$

The relationships with respect to the d parameter are as follows

$$\begin{aligned} \frac{\partial N(d)}{\partial d} &= n(d) \\ \frac{\partial n(d)}{\partial d} &= -n(d)\, d \\ Se^{-\delta T} n(d_1) &= Xe^{-rT} n(d_2)\,. \end{aligned}$$

3.9.3 Theta θ, Vega and rho ρ

Sensitivity to the other option pricing parameters (no second order terms considered here)[9] are again more difficult to evaluate than Δ, Γ, κ.

$$\begin{aligned} \text{Theta } \theta &= -\frac{\partial C}{\partial t} \\ \text{“Vega”} &= \frac{\partial C}{\partial \sigma} \\ \text{Rho } \rho &= \frac{\partial C}{\partial r} \end{aligned}$$

Note that the asset pricing equation itself (yet to be derived) utilises three partials, Δ, Γ and θ

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta) S \frac{\partial C}{\partial S} - rC &= 0 \\ -\theta(S,T) + \frac{1}{2}\Gamma(S,T)\sigma^2 S^2 + \Delta(S,T)(r-\delta) S - rC &= 0 \end{aligned}$$

and this helps shed some light on the nature of this equation. It says that the hedged reward for risk for holding an option rC is equal to the negative θ plus the (weighted) gamma plus the (weighted) delta of the option position.

3.10 Implied volatility

If options are no more than leveraged investments in the underlying and they can be replicated rather than purchased (this is precisely how they are priced) why is everyone not replicating instead of actually buying options? (If options are easily replicated are they not a redundant security?).

Well the answer lies in what (extra) they allow us to trade and the extra item that options allow us to observe and trade is *future expected volatility.* All other elements in the BS formula are known at the time of purchase but whether the option writer and hedger will profit or lose depends if the realised volatility he experiences over the life of the hedging programme falls short of or exceeds his estimate implied by his sale price. Thus option traders are really taking a view on future volatility and if you are long an option (all else equal) you will benefit (lose) if volatility increases (decreases).

Having bought or sold an option the particular price struck can be used to infer a volatility (*implied volatility*) which will likely differ from *historical*

[9] Note that as Wilmott (Derivatives, Wiley Press, 2000) points out, there are many “vegas” including Suzanne, Vincent and a car but alas no Greek letter!

Volatility	**Formula**	**Measurability**?
Historical or past	$\sigma^2_{-T,0} = \frac{1}{T}\sum_{i=-T}^{0}(r_i - \overline{r})^2$	measure at $T = 0$
Future, prospective	$\sigma^2_{0,T} = \frac{1}{T}\sum_{i=0}^{T}(r_i - \overline{r})^2$	forecast at $T = 0$
Option implied	$C_{BS}(S_0, X, r, T, \sigma_{imp}) = C_{market}$	imply at $T = 0$

Table 6: Historical, prospective and implied volatilities

volatility because the future market conditions could be more or less risky than the past. *Volatility is itself non constant, i.e. volatile!*

In fact if a trader buys an option but then proceeds to try to hedge it dynamically (using the Black Scholes formula for instance to eliminate the market level risk), he will benefit if market volatility increases. Conversely, if a trader sells an option and proceeds to hedge its market level risk, he will benefit if the market volatility decreases. Therefore options allow traders to trade in volatility itself, trying to guess if it will rise or fall in the future. If the former, participants try and buy volatility (through buying options and delta hedging), if the latter participants will sell options and delta hedge. If the market for options is efficient, the future implied may be a good estimate of future expected volatility. Not however that strictly speaking the Black Scholes formula does not allow for varying volatility, but this does not stop traders using it as a very handy tool.

The volatility level negotiated between buyer and seller should represent the best guess about future volatility else arbitrage opportunities may exist by out guessing the level of future volatility. Because of this many look at implieds as a forecast of future likely volatility. It is the extra information and opportunities to trade volatility that options afford that causes them to be so widely traded, but strangely if it is the non constancy of volatility that causes this, an option pricing model that specifically allows for changing volatility should be used (not BS). Such a model is again more complicated to implement than BS and so in practice BS (with fudge factors) is still applied! See xls with volatilities sampled over different time intervals.

$$\begin{aligned} \sigma_{implied} &\gtrless \sigma^2_{-T,0} \\ \sigma_{implied} &= \sigma^2_{0,T} \quad ? \end{aligned}$$

Solving for the option volatility in BS is a tricky task analytically (no formula is available) but on a spreadsheet it is easy, having set up the BS formula allow σ to depend on itself plus a change dependent on the magnitude the calculated option price differs from the traded price (so that it goes up if the calculated price is below the traded and vice versa), recalculate the spreadsheet again and again until the pricing error becomes negligible. Alternatively use something like Excel's GoalSeek function. Very often option traders buy and sell to each other simply by quoting and settling upon the implied volatility only later deriving the actual cash price!

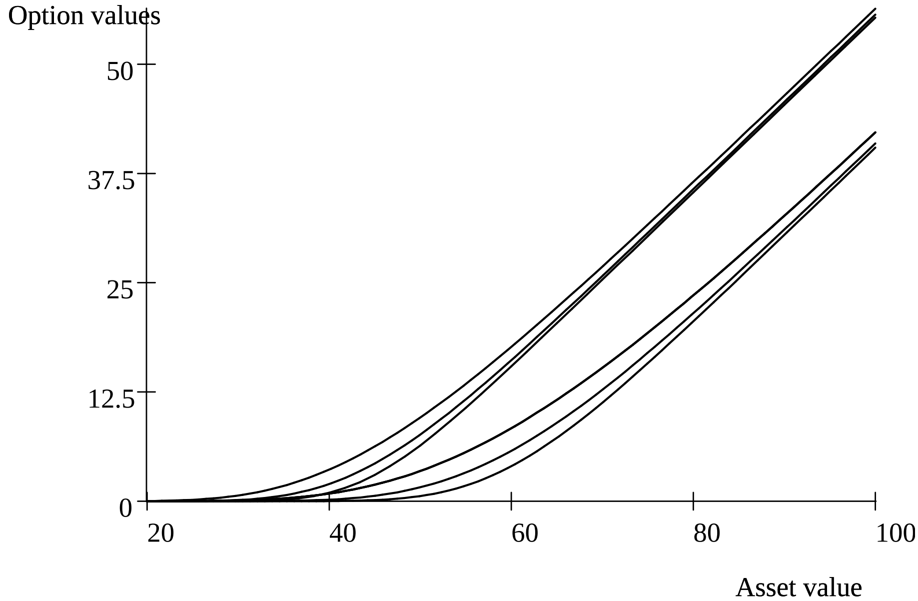
3.11 Formation of portfolios of options on the same underlying

Since a portfolio of options is worth the sum of its components[10], the sensitivity of the portfolio is the sum of the constituent sensitivities, e.g. consider buying a call at 45 and selling a call at 60 (this strategy of partly financing one option by selling another is called a bull spread)

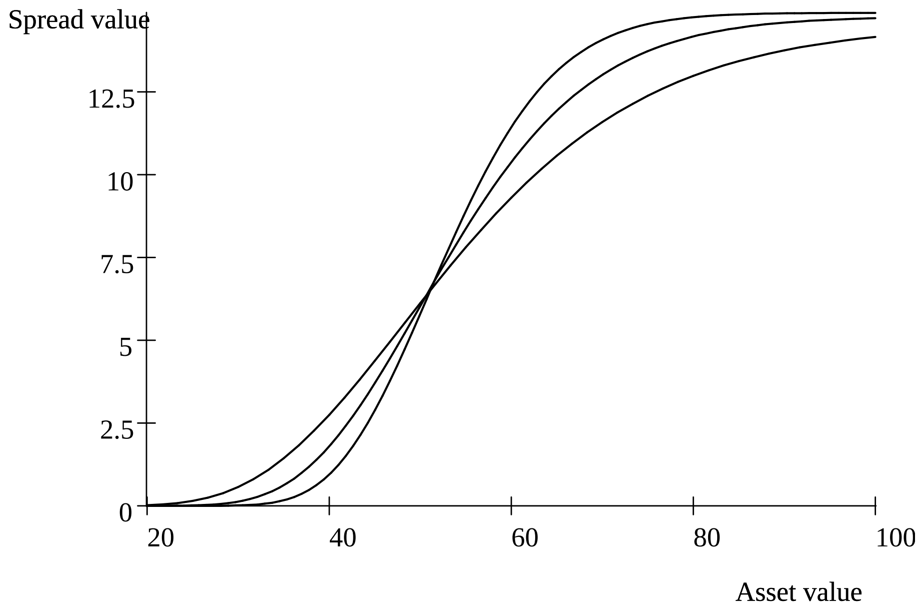
$$\begin{aligned} \text{spread} &= C_{X=45} - C_{X=60} \\ \frac{\partial \text{spread}}{\partial S} &= \frac{\partial C_{45}}{\partial S} - \frac{\partial C_{60}}{\partial S} \\ &= \Delta_{C45} - \Delta_{C60} \end{aligned}$$

(and so on for the other partial derivatives). This means that hedging of options can be done in aggregate, combining and netting their attributes before seeking external cover for the delta hedge.

[10] This is an MM result since there are no assumed frictions. It means that the NPV of a sum of options on a payoff is the same as the sum of NPVs of option on the same payoff because otherwise arbitrage would be possible. It does not say that the option on a portfolio of diverse assets (on different payoffs) is equal to the portfolio of options on diverse assets (on payoffs).



Calls at 45 and 60 for differing maturities



Bull spreads between 45 and 60

The graph shows the value of a (bull) spread between 45 and 60 for four maturities $(T = 0.5, 0.25, 0.125, 0.0 \text{ years})$. Assuming that the stock price does not go negative (under GBM it cannot), a *call at a strike price of zero* would *always* be exercised (the put would never be exercised) and the option (if it still can be called such) must be worth the stock price itself

$$\lim (X \to 0)\, C(S, X) = S \quad \forall T \qquad \text{i.e. } C(S, 0) = S$$

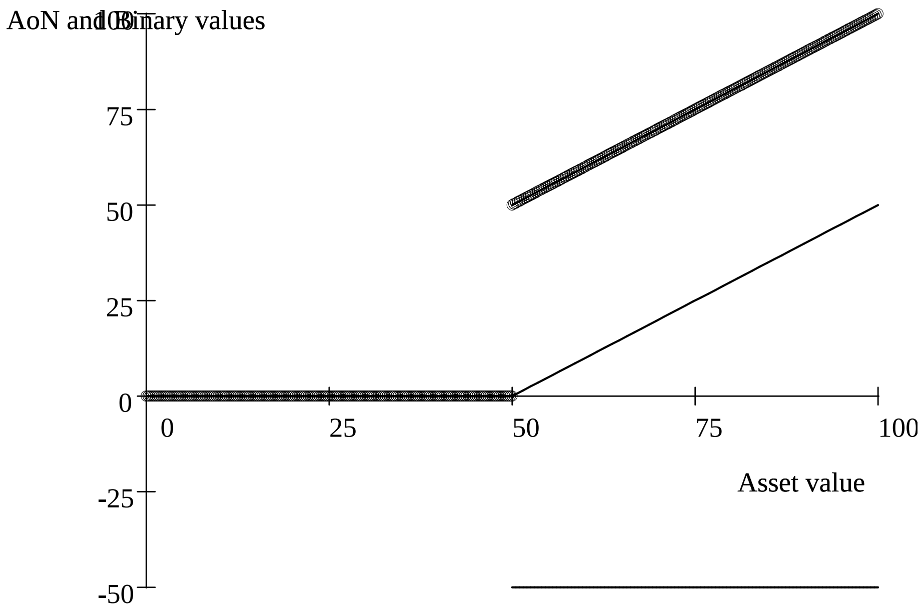


Figure 10: A long position in an Asset or Nothing call (ANC - dots) and a short position in a Binary Call (BC - dashes) make a Regular Call option payoff (solid line).

This allows a decomposition for the stock value

$$\begin{aligned} S &= C(S,0) \\ &= C(S,0) - C(S,45) + C(S,45) - C(S,60) + C(S,60) \\ &= \text{spread}(45,0) + \text{spread}(60,45) + C(S,60) \end{aligned}$$

This suggests that the stock value could be sold off in "segments" corresponding to spreads on its future unknown value. In this case the first spread claims up to the first 45 units of value but no more the second claimed the next 15 units of value (if the first is fully satisfied) and the third claims all value above 60 if the first two claimants are satisfied. The analogy with a hierarchy of liability claims will become apparent later.

3.12 Asset or nothing and binary options

The Black Scholes formula itself is actual a portfolio of two more primitive options on the same underlying, an *asset or nothing call* and a *binary call.*

$$C = \begin{bmatrix} \text{ANC} \\ SN(d_1) \end{bmatrix} - \begin{bmatrix} \text{BC} \\ Xe^{-rT}N(d_2) \end{bmatrix}$$

Trade		Asset, S	–Exercise Price, Xe^{-rT}
Long Call	$C =$	$ANC = SN(d_1)$	$-BC = -Xe^{-rT}N(d_2)$
+	+	+	+
Short Put	$-P =$	$ANP = SN(-d_1)$	$-BP = -Xe^{-rT}N(-d_2)$
=	=	=	=
Forward Purchase	$S - Xe^{-rT} =$	S	$-PVX = -Xe^{-rT}$

Table 7: Decompostion of put, call, binary and asset or nothing parities

The first component pays the asset S if $S_T > X$ but zero if $S_T < X$, its payoff is thus discontinuous at $S_T = X$. The second (short) component pays X if $S_T > X$ but zero if $S_T < X$, and its payoff is also discontinuous at $S_T = X$. However when the two components are combined the payoff becomes continuous at $S_T = X$ since it reduces to $\max(S_T - X, 0)$.

Note that option pricing is linear in the component payoffs only for payoffs on the same underlying[11].

$$\begin{aligned} C &= e^{-rT}E^Q\left[(S_T - X).1_{S_T>X}\right] \\ &= e^{-rT}E^Q\left[S_T.1_{S_T>X}\right] - Xe^{-rT}E^Q\left[1_{S_T>X}\right] \\ &= SN(d_1) - Xe^{-rT}N(d_2) \end{aligned}$$

The binary put and binary call sum to the discounted exercise price

$$\begin{aligned} \mathrm{BC} + \mathrm{BP} &= Xe^{-rT}N(d_2) + Xe^{-rT}N(-d_2) \\ &= Xe^{-rT}N(d_2) + Xe^{-rT}(1 - N(d_2)) = Xe^{-rT} = \mathrm{PVX} \end{aligned}$$

The regular put has a decomposition into a binary put (BP) and an asset or nothing put (ANP)

$$P = \begin{bmatrix} \mathrm{BP} \\ Xe^{-rT}N(-d_2) \end{bmatrix} - \begin{bmatrix} \mathrm{ANP} \\ SN(-d_1) \end{bmatrix}$$

The asset or nothing call and put have a summation parity condition

$$\mathrm{ANC} + \mathrm{ANP} = SN(d_1) + SN(-d_1) = S = \text{Asset}$$

Finally put all together, put call parity dictates that the binaries and asset or nothing options must satisfy the following conditions in Table 7.

3.13 Other underlying assets

- The pure Black Scholes is for non dividend paying assets (e.g. Microsoft Stock!)

[11] $1_{S_T>X}$ is a indicator function that is one if the subsrcipted funtion is satisifed and zero if not.

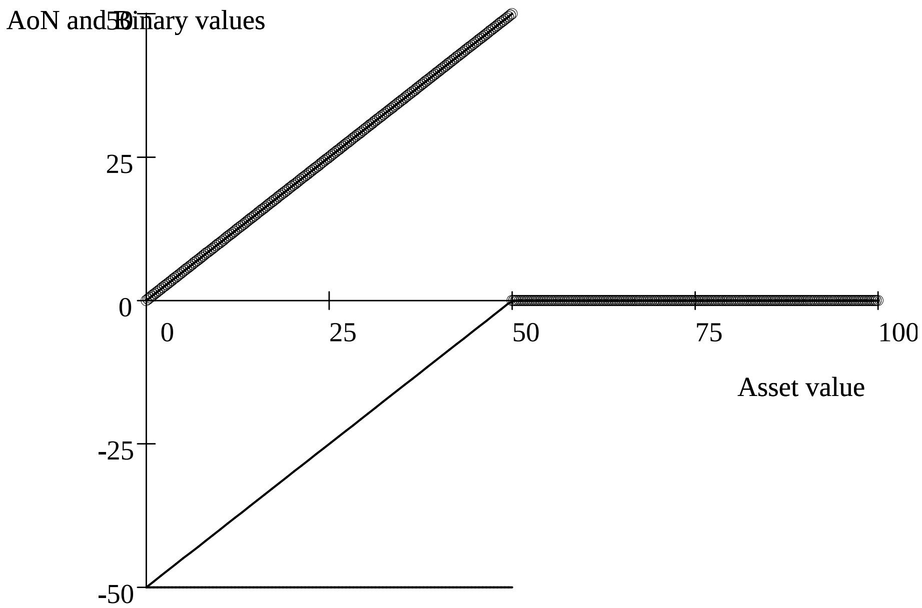


Figure 11: A long position in an Asset or Nothing Put (ANP - dots) and a short position in a Binary Put (BP - dashes) make a short regular Put option payoff (solid line).

- Most assets pay a dividend yield but the Black Scholes formula can be easily extended to value a European option on a dividend paying stock. For a dividend yield δ, it gains a term $Se^{-\delta T}$ to reflect the now diminished *forward price* of the stock and d_1, d_2 must be adjusted downward to reflect the lower drift of the underlying process (the growth rate g is now $\mu - \delta$). With dividends early exercise for American options is no longer ruled out.

$$\begin{aligned}
\frac{dS}{S} &= (\mu - \delta)\, dt + \sigma dZ \\
\text{Call} &= Se^{-\delta T} N(d_1) - Xe^{-rT} N(d_2) \\
\text{Put} &= Xe^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1) \\
d_1, d_2 &= \frac{\ln S - \ln X + (r - \delta \pm \sigma^2/2)\, T}{\sigma\sqrt{T}}
\end{aligned}$$

- Interest rates and bonds (rate process not Geometric Brownian Motion)
- Commodities (possibly mean reverting in the long run)
- Options on other processes such as inflation

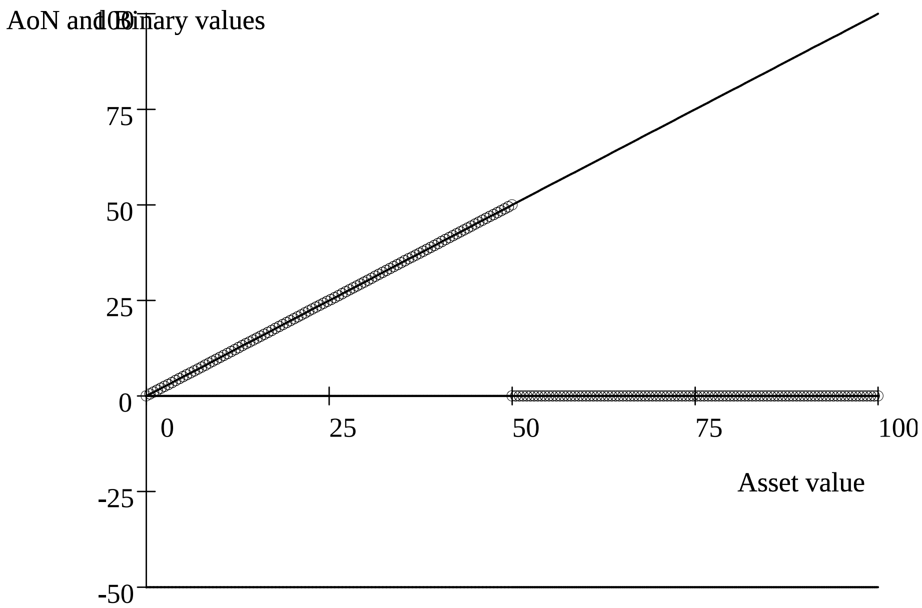


Figure 12: Long positions in Asset or Nothing Call and Put (ANC, ANP) and short positions in Binary Call and Put (BC, BP) make a long asset and short bond.

- Options to exchange on asset for another, e.g. one currency for another

3.14 Options on futures

For a futures spot relationship

$$\boxed{F_0^T = S_0 e^{(r-\delta)T}}$$

options on the spot and future are related (see Black (1976) [4])

$$\begin{aligned} C(S, X, r, \delta, \sigma, T) &= C\left(Se^{-\delta T}, X, r, 0, \sigma, T\right) \\ &= C(F, X, r, r, \sigma, T) \\ &= C\left(Fe^{-rT}, X, r, 0, \sigma, T\right) \end{aligned}$$

Thus for options on a futures contract (holding the future does not yield a dividend), the dividend yield is irrelevant. This is because the BS formula only cares about the remaining volatility, $\sigma^2 T$ and degree of moneyness variable

$$\begin{aligned}
\text{(Forward) Moneyness } M &= \frac{Se^{-\delta T}}{Xe^{-rT}} \\
\text{For a call; in the money } M &> 1 \\
\text{At the money } M &\approx 1 \\
\text{Out of the money } M &< 1
\end{aligned}$$

3.15 Garman & Kohlhagen, Grabbe

Garman and Kohlhagen (1983) [21] and Grabbe (1983) [23] used the foreign interest rate r^* as the effect dividend or opportunity cost of "stock" ownership. Thus the Black Scholes with dividend can be adjusted to yield the value of a call from the domestic currency (rate r) into the foreign (rate r^*) at a strike of X when the current exchange rate is S

$$\begin{aligned}
\frac{dS}{S} &= (r - r^*)\, dt + \sigma dZ \\
\text{Call} &= Se^{-r^*T} N(d_1) - Xe^{-rT} N(d_2) \\
\text{Put} &= Xe^{-rT} N(-d_2) - Se^{-r^*T} N(-d_1) \\
d_1, d_2 &= \frac{\ln S - \ln X + (r - r^* \pm \sigma^2/2)\, T}{\sigma\sqrt{T}}
\end{aligned}$$

3.16 Margrabe

Margrabe (1978) [36] showed that the option to exchange (call i.e. buy) asset X_1 for asset (by paying) X_2 is worth when the two assets have imperfect correlation $(\rho < 1)$. This intuitive formula reduces to a Black Scholes when the second asset is replaced by the money market account. It also reduces to the Black Scholes with dividends if the assets are also replaced by forward prices $X_1 e^{-\delta_1 T}$ etc. If the dividends are zero then an American Margrabe option would never be exercised early and would be worth its European counterpart.

$$\begin{aligned}
\frac{dX_{1,2}}{X_{1,2}} &= \mu_{1,2} dt + v_{1,2} dZ_{1,2} \\
dZ_1 dZ_2 &= \rho dt \\
\text{Call} &= X_1 N(d_1) - X_2 N(d_2) \\
d_1, d_2 &= \frac{\ln X_1 - \ln X_2 \pm 0.5 v^2 T}{v\sqrt{T}} \\
v^2 &= v_1^2 - 2\rho v_1 v_2 + v_2^2 > 0
\end{aligned}$$

In effect the Margrabe volatility v^2 depends on the volatility of a portfolio of one long unit in X_1 and one short unit in X_2, i.e. $v^2 = Var(X_1 - X_2) = v_1^2 - 2\rho v_1 v_2 + v_2^2$. If the amount of X_1, X_2 present in the call changes, then ν would change.

3.16.1 Margrabe application; relative calls

The Margrabe formula can be applied to calls (and puts) *relative to the market*. Suppose X_1 represents the price of a stock and X_2 the market. $X_1 N(d_1) - X_2 N(d_2)$ would represent the right to exchange X_2 for X_1, that is call the market value of the stock by paying the market value of the index (at that time). In order to calculate the Margrabe volatility ν^2 we need to know its idiosyncratic risk σ_{idio}, a function of the total stock risk σ_s, market risk σ_m and the beta of the stock $\beta = \rho_{sm}\frac{\sigma_s}{\sigma_m}$ where

$$\sigma_s^2 = \beta^2\sigma_m^2 + \sigma_{idio}^2$$

because the idiosyncratic risk is orthogonal to the market risk by definition. Now labelling the Margrabe volatility of the option ν, the market risk $\nu_2 = \sigma_m$ and the stock volatility $\nu_1 = \sigma_s$

$$\begin{aligned} \nu^2 &= |\sigma_s - \sigma_m|^2 = \sigma_s^2 - 2\sigma_s\sigma_m\beta\frac{\sigma_m}{\sigma_s} + \sigma_m^2 \\ &= \sigma_{idio}^2 + \beta^2\sigma_m^2 - 2\sigma_s\sigma_m\beta\frac{\sigma_m}{\sigma_s} + \sigma_m^2 \\ &= \sigma_{idio}^2 + \sigma_m^2(\beta - 1)^2 \end{aligned}$$

The effect of using the market portfolio as numeraire is to reduce the risk of the Margrabe call by reducing the market risk by up to one unit (until $\beta = 1$ after which total risk starts to increase again). The idiosyncratic risk, being orthogonal to the market risk is unaffected by the beta of the stock. For a given level of idiosyncratic risk, the Margrabe volatility is minimised for a stock beta of one.

Note that for a stock of zero beta (all risk is idiosyncratic) the Margabe volatility is larger than that of a unit beta stock, this is because in this case the market hedge actually increases the risk. Only if the beta is one is the Margrabe volatility minimised and equal to the idiosyncratic risk. Also note that this idiosyncratic risk affects all option probabilities $N(d_{1,2..})$ and therefore the leverage even if it does not affect the underlying rate of return.

The drift (instantaneous expected rate of return) on this contract is

$$\begin{aligned} E[R_s - R_m] &= R_f + \beta E[R_m - R_f] - E[R_m] \\ &= (\beta - 1) E[R_m - R_f] \end{aligned}$$

thus if the original stock beta is greater than one, the contract is "underhedged" and still contains market risk but if beta is less than one the contract is "overhedged" and will act like a put contract overall with a negative rate of return..

The other key input along with the idiosyncratic risk and beta is the difference in dividend yield $\delta_2 - \delta_1$ because it determines the cost of carry of the hedge position (which is not determined by r the risk free rate because the delta neutral hedge would not contain a risk free position). Note that the actual market and stock returns $\mu_m = E[R_m]$ and $\mu_s = E[R_s]$ do not appear in the pricing formula but would affect the true (E^P) probabilities of exercise if not the risk neutral ones.

$$\begin{aligned} \frac{dX_{1,2}}{X_{1,2}} &= \left(\mu_{1,2} - \delta_{1,2}\right) dt + v_{1,2} dZ_{1,2} \\ C &= X_1 e^{-\delta_1 T} N(d_1) - X_2 e^{-\delta_2 T} N(d_2) \\ d_{1,2} &= \frac{\ln X_1 - \ln X_2 + (\delta_2 - \delta_1 \pm \nu^2) T}{\nu T^{0.5}} \end{aligned}$$

For example such an option might be used to incentivise staff while controlling for total market performance. If the current stock price of the firm is $S_{now} = \$100$ and the equity holders wish to reward managers for any *differential* performance above the market adjusted value of $S\&P_{now} = \$5000$ then establish the call as yielding the stock price if $\frac{1}{50}$ of the current index level is paid $X_1 = 100$, $X_2 = \frac{1}{50}5000 = 100$ and using the dividend yield of the stock and market for δ_1, δ_2 will allow the option on excess relative performance to be evaluated.

3.16.2 Margrabe application; β relative calls

Finally the right to acquire X_1 by foregoing βX_2 may also be priced, now however the Margrabe volatility is different (less for all beta)

$$\begin{aligned} \nu^2 &= |\sigma_s - \beta\sigma_m|^2 \\ &= \sigma_s^2 - 2\sigma_s\sigma_m\beta^2\frac{\sigma_m}{\sigma_s} + \beta^2\sigma_m^2 = \sigma_{idio}^2 \end{aligned}$$

Here, because the market risk of the two positions is the same (for all beta), it always cancels out and the only risk element left is the idiosyncratic risk σ_{idio}^2. Thus the Margrabe option to call a stock X_1 by paying its beta time the market index value βX_2 ($\beta \neq 0$) is a pure call on the idiosyncratic risk

$$\begin{aligned} C &= X_1 e^{-\delta_1 T} N(d_1) - \beta X_2 e^{-\delta_2 T} N(d_2) \\ d_{1,2} &= \frac{\ln X_1 - \ln \beta X_2 + (\delta_2 - \delta_1 \pm \sigma_{idio}^2) T}{\sigma_{idio} T^{0.5}} \end{aligned}$$

Here rewarding the manager with an option on $X_1 - \beta X_2$ takes out the market risk and leaves just the idiosyncratic risk within the option. The drift (instantaneous expected rate of return) on this contract is different to the previous case

$$\begin{aligned} E\left[R_s - \beta R_m\right] &= R_f + \beta E\left[R_m - R_f\right] - \beta E\left[R_m\right] \\ &= R_f\left(1 - \beta\right) \end{aligned}$$

In the previous case $X_1 - X_2$ left some market risk present unless the special case of $\beta = 1$ pertained, here all systematic risk is eliminated and the return contains no market premium. If $\beta > 1$ the portfolio is "borrowed" and will decline over time while if $\beta < 1$ it will increase over time. For $\beta = 1$ it is neither an asset or a liability and has zero drift.

3.16.3 Implied correlation

Unlike the market price of regular call options (see next section for implied volatility) Margrabe options allow inference of implied volatility. This is the value of ρ derived from ν that solves Margrabe theoretical price equal to the market price. Just as standard options allow inference of implied volatility, empirical Margrabe prices allow inference of implied correlation which solves

$$C_{Magrabe}\left(X_1, X_2, T, \sigma_1, \sigma_2, \rho_{implied}\right) = C_{Market}.$$

3.17 Other strategies

Cylinders, butterflies, calendar spreads, straddles, strangles (see Hull). The list grows.

3.18 Other option types

- European: exercisable at the *expiry date only*
- American: exercisable at *any time up to and including the expiry date* (N.B. cannot be less valuable than a European option and might be more valuable). Perpetual Merton options are included in this category.
- Bermudan: exercisable at *one of many specific times up to and including the expiry date*
- Asian: average rate options pay the excess of the average price over the period above the (prespecified) strike price

- Russian: perpetual American option which at any time chosen by the holder, pays out the maximum price realised to date
- Barrier: optionality is triggered/cancelled once a critical threshold is reached
- Lookback: options whose payout depends on the maximum (or minimum) over the option life
- Compound: options on options
- Chooser: the right to choose either a put or a call in the future
- other Exotics: ladder, range forward, exchange, rainbow, cliquet – the list goes on expanding! Many of these are much more difficult to price.

3.19 Risk neutral pricing

We have already seen how to derive the BS formula is to use the Risk Neutral Density (RND) but it can be done straight away in continuous time. Geometric Brownian motion implies that the future stock price is log normally distributed around some mean, under the true probability the mean depends on the average return of the stock μ, however under the RND this is not the case. These two distributions are labelled P, Q and are called the objective and risk neutral distributions respectively. Both are lognormal, both have variance $\sigma^2 T$ but they have different means. For the probability distribution of some future log price $\ln S_T$ conditional on the current price $\ln S_0$, the distributions are

$$\begin{aligned} P &: \phi^P (\ln S_T | \ln S_0) \sim n \left(\ln S_0 + \left(\mu - \delta - \frac{1}{2}\sigma^2 \right) T, \sigma^2 T \right) \\ Q &: \phi^Q (\ln S_T | \ln S_0) \sim n \left(\ln S_0 + \left(r - \delta - \frac{1}{2}\sigma^2 \right) T, \sigma^2 T \right) \end{aligned}$$

$$\begin{aligned} \frac{dS}{S} &= (\mu - \delta)\, dt + \sigma dZ^P \\ \frac{dS}{S} &= (r - \delta)\, dt + \sigma dZ^Q \\ dZ^Q &= dZ^P + \frac{\mu - r}{\sigma} dt \end{aligned}$$

The Black Scholes formula can be shown (Cox & Ross (1976) [12]) to be the discounted expectation of the payoff if positive *under the RND*[12]

$$\begin{aligned}
\text{BS} &= e^{-rT}E^{Q}\left[(S_T-X)^{+}\right]=e^{-rT}\int_{-\infty}^{\infty}\phi^{Q}\left(\ln S_T|\ln S_0\right)(S_T-X)^{+}\,d\ln S_T \\
&= e^{-rT}\int_{\ln X}^{\infty}\phi^{Q}\left(\ln S_T|\ln S_0\right)(S_T-X)\,d\ln S_T \\
&= e^{-rT}\int_{\ln X}^{\infty}\phi^{Q}\left(\ln S_T|\ln S_0\right)S_T d\ln S_T-e^{-rT}\int_{\ln X}^{\infty}\phi^{Q}\left(\ln S_T|\ln S_0\right)X d\ln S_T \\
&= Se^{-\delta T}N\left(d_1\right)-Xe^{-rT}N\left(d_2\right)
\end{aligned}$$

where

$$\begin{aligned}
d_{1,2} &= \frac{\ln S_0-\ln X+\left(r-\delta\pm\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\
N\left(d_{1,2}\right) &= \frac{1}{\sqrt{2\pi\sigma^2T}}\int_{-\infty}^{d_{1,2}}e^{-\frac{1}{2}\frac{\left(\ln S_T-\ln S_0+\left(r-\delta-\frac{1}{2}\sigma^2\right)T\right)^2}{\sigma^2T}}d\ln S_T
\end{aligned}$$

3.20 Boness "pricing"

What if the objective density was taken, what "price" would this yield. This is something that Boness (1962) [6] did a *decade* before Black & Scholes. Galai (1978) [20] shows how the call formula of Boness (1964) [7] can be reconciled with that of Black Scholes (1973) [5]. Taking expectations under the objective P distribution yields an expected payoff under the objective measure

$$\begin{aligned}
E^{P}\left[C\left(T\right)\right] &= E^{P}\left[(S_T-X)^{+}\right]=\int_{-\infty}^{\infty}\phi^{P}\left(\ln S_T|\ln S_0\right)(S_T-X)^{+}\,d\ln S_T \\
&= \int_{\ln X}^{\infty}\phi^{P}\left(\ln S_T|\ln S_0\right)(S_T-X)\,d\ln S_T \quad \text{Boness expected call payoff} \\
&= \int_{\ln X}^{\infty}\phi^{P}\left(\ln S_T|\ln S_0\right)S_T d\ln S_T-\int_{\ln X}^{\infty}\phi^{P}\left(\ln S_T|\ln S_0\right)X d\ln S_T
\end{aligned}$$

$$\boxed{E^{P}\left[C\left(T\right)\right]=Se^{(\mu-\delta)T}N\left(d_3\right)-XN\left(d_4\right)}$$

$$\boxed{d_{3,4}=\frac{\ln S-\ln X+\left(\mu-\delta\pm\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}$$

[12] This again shows how BS is the difference between two digital options, one that pays the stock price S if $S>X$ and one that pays X if $S>X$. They are valued as $Se^{-\delta T}N\left(d_1\right)$ and $Xe^{-rT}N\left(d_2\right)$ respectively.

Here the real world probability of exercise is given by

$$\boxed{\text{RW prob. of call exercise} = N(d_4).}$$

Boness then discounted this expected (under the objective density) payoff $C(T)$ by $e^{-\mu T}$ (i.e. he incorrectly put it in the same risk–return class as the stock itself which has a total yield of μ) to arrive at a "price"

$$\text{Boness "Price"} = Se^{-\delta T}N(d_3) - e^{-\mu T}XN(d_4) \neq \text{Black Scholes}$$

This is very close to the BS formula, indeed it reverts to it if $\mu = r$, however Boness had no argument available at the time to justify valuing the option as if its μ was r; the contribution of Black Scholes was precisely the fact that under their hedging strategy, the hedge portfolio should earn the riskless rate only. Boness "prices" are not arbitrage free because they attach the wrong rate of return to the option.

Knowing the actual expected payoff, the correct rate of return can be inferred using the Black Scholes price. If we label ν the expected continuous return on the call from now until T then

$$\begin{aligned} e^{\nu T} &= \frac{\text{Expected payoff under objective measure } P}{\text{Black Scholes price (discounted expectation under RND } Q)} \\ \nu &= \frac{1}{T}\ln\frac{Se^{(\mu-\delta)T}N(d_3) - XN(d_4)}{Se^{-\delta T}N(d_1) - Xe^{-rT}N(d_2)} \end{aligned}$$

See Shackleton & Wojakowski (2001) [58] from Rubinstein (1984) [55] who covered this form of calculation in discrete time, showing it to be equivalent to use of the BS formula with $Se^{(\mu-\delta)T}$ substituted for S.

3.21 Parity in expected call and put payoffs

Expected call and put values can be taken and a form of put call parity can be formed under P expectations (put call parity)

$$\boxed{C_T - P_T = S_T - X}$$

$$E_0^P[C_T - P_T] = E_0^P[S_T - X]$$

$$Se^{(\mu-\delta)T}N(d_3) - XN(d_4) - \left[XN(-d_4) - Se^{(\mu-\delta)T}N(-d_3)\right] = S_0e^{(\mu-\delta)T} - X$$

which is the expected payoff for a forward purchase (time T at price X) since the stock will most likely appreciate at a rate $\mu - \delta$.

This analysis allows us to calculate the rate of return to a covered call writing strategy, showing that the net premium received does not come without a cost, the rate of return is adjusted accordingly.

3.22 Local option returns and betas

Local option returns[13] are derived in this section. The BS formula satisfies the following partial differential equation

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+(r-\delta)\,S\frac{\partial C}{\partial S}-rC=0$$

As an alternative to the C&W derivation [14], we can use it to evaluate the instantaneous expected rate of return (drift) on the call price C. Ito's Lemma tells us that we need not only the first differential wrt time and stock price but the second wrt to stock price (think back to our non–linear claim)

$$\begin{aligned}
E^P[dS] &= (\mu-\delta)\,Sdt \qquad (dS)^2=\sigma^2 S^2 dt\\
E^P[dC] &= E^P\left[\frac{\partial C}{\partial t}dt+\frac{\partial C}{\partial S}dS+\frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2\right]\\
&= dt\left(\frac{\partial C}{\partial t}+\frac{\partial C}{\partial S}(\mu-\delta)\,S+\frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)\\
\frac{E^P[dC]}{dt} &= \frac{\partial C}{\partial S}(\mu-r)\,S
\end{aligned}$$

[13] Local means over the very next short instant of time dt.

[14] Over a short horizon, dt we can write the local % return on a call $R_C dt=\frac{dC}{C}$ as a function of the local rate of return on the stock $R_S dt=\frac{dS}{S}dt$

$$\begin{aligned}
\frac{dC}{C} &= \frac{\partial C}{\partial S}\frac{dS}{C}+\frac{\partial C}{\partial t}dt\\
&= \frac{\partial C}{\partial S}\frac{S}{C}\frac{dS}{S}+\frac{1}{R_f}\frac{\partial C}{\partial t}R_f dt
\end{aligned}$$

Thus the local rate of return on a call is linked to that of the stock

$$R_C=\frac{\partial C}{\partial S}\frac{S}{C}R_S+\frac{1}{R_f}\frac{\partial C}{\partial t}R_f$$

where the hedge ratio $\frac{\partial C}{\partial S}$, stock to price ratio $\frac{S}{C}$ and time partial $\frac{\partial C}{\partial t}$ are locally (but not globally) fixed. Using our CAPM definitions of beta and the fact that the $R_f dt$ part of the return is anticipated and only the $\frac{dS}{S}$ contributes to the market covariation

$$\beta_C=\frac{Cov\,(R_C,R_m)}{Var\,(R_m)}$$

and substituting for R_c the option beta becomes

$$\beta_C=\frac{\partial C}{\partial S}\frac{S}{C}\frac{Cov\,(R_S,R_m)}{Var\,(R_m)}=\frac{\partial C}{\partial S}\frac{S}{C}\beta_S$$

(N.B. the risk free term does not contribute to the covariation calculation).

by making use of the Black Scholes asset pricing equation.

$$\begin{aligned}
\mu &= r + \beta_S \text{Market Risk Premium} \\
\upsilon &= r + \beta_C \text{Market Risk Premium} \\
\frac{\mu - r}{\beta_S} &= \text{MRP} = \frac{\upsilon - r}{\beta_C}
\end{aligned}$$

Now the relative rate of return or beta β_C (drift) on the call price can be inferred as a function of the stocks beta by equating risk premia

$$\begin{aligned}
\beta_C \text{MRP} &= \frac{\partial C}{\partial S}(\mu - r)\frac{S}{C} = \frac{\partial C}{\partial S}\frac{S}{C}\beta_S \text{MRP} \\
\beta_C &= \frac{\partial C}{\partial S}\frac{S}{C}\beta_S
\end{aligned}$$

The BS hedge ratio $\frac{\partial C}{\partial S}$ is given by $e^{-\delta T}N(d_1)$ (between 0 and 1) and so the option beta is derived from a levered stock beta

$$\boxed{\beta_C = e^{-\delta T}N(d_1)\tfrac{S}{C}\beta_S}$$

If $e^{-\delta T}N(d_1)\frac{S}{C} > 1$ (can be shown easily since $C < Se^{-\delta T}N(d_1)$) options have higher betas than the underlying and therefore higher expected rates of return (in a CAPM setting) because they represent a *levered investment in the underlying*. Even though they have a global non-linear payoff, because they have a local linear payoff they still conform to the CAPM and obey regular (local) CAPM risk pricing! It is just that their future beta is dynamic (changing over time). The instantaneous drift is given $(\upsilon, \mu, r) = (R_C, R_S, R_f)$, $R_m - R_f$ =MRP

$$\begin{aligned}
E[R_S - R_f]\,dt &= \beta_S E[R_m - R_f]\,dt \\
E[R_C - R_f]\,dt &= \beta_C E[R_m - R_f]\,dt
\end{aligned}$$

then imply

$$\begin{aligned}
R_C &= R_f + e^{-\delta T}N(d_1)\frac{S}{C}(R_S - R_f) \\
&= e^{-\delta T}N(d_1)\frac{S}{C}R_S - \left(e^{-\delta T}N(d_1)\frac{S}{C} - 1\right)R_f \\
&= \frac{Se^{-\delta T}N(d_1)}{C}R_S - \frac{Xe^{-rT}N(d_2)}{C}R_f
\end{aligned}$$

i.e. a weighted average return of the return R_S on a long stock position $Se^{-\delta T}N(d_1)$ funded by a borrowing cost r on a short bond position $Xe^{-rT}N(d_2)$. Note that the weights add up to one.

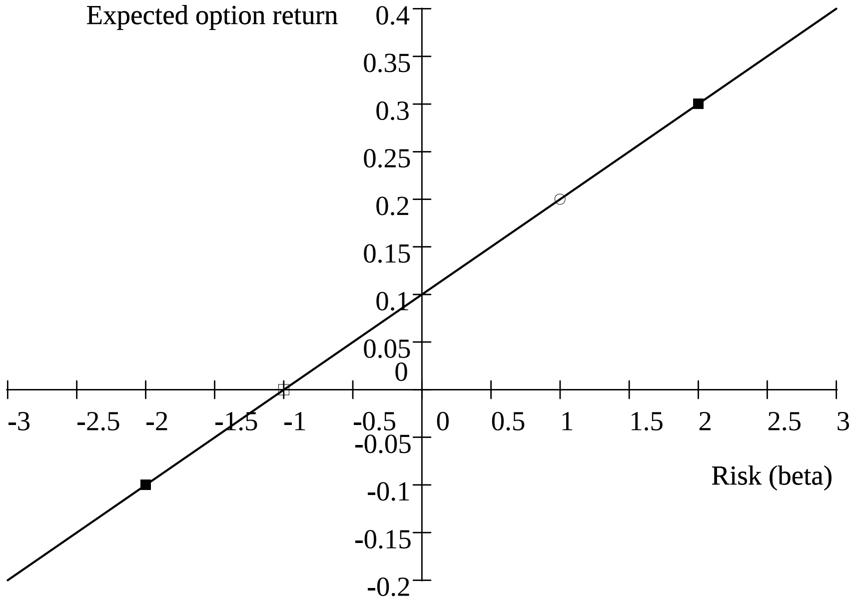


Figure 13: SML returns from long (circle), short (diamond) stock holdings and call & put options (crosses) as a function of beta ($R_f = 10\%$, $R_S = 20\%$).

3.23 Options and the security market line

Thus call options at a minimum have a local beta and rate of return equal to the underlying stock and this occurs as $X \to 0$ (in the money) when exercise is assured and the option behaves as if it were stock. Conversely, as $X \to \infty$, exercise becomes increasingly unlikely and both the option beta and rate of return increase pushing the (out of the money) up the Security Market Line away from the position of the stock. Figure 13 shows the SML for $R_f = 10\%$ and a $\beta = 1$ stock with $R_S = 20\%$ and two sample options one call and one put.

O'Brien and Shackleton (2005) [50] show an empirical version of this figure for FTSE100 index options.

4 Company liabilities as options

In this section we assume that the asset value of the firm A follows the GBM.

$$\boxed{\frac{dA}{A} = \mu dt + \sigma dZ}$$

4.1 Risky debt and equity

This section is derived from the work of Merton 1974 [42]. If the asset return is very certain to exceed the interest expense, then the debt, considered to be riskless should yield the risk free rate (otherwise investors would not hold the debt in preference to a government security) and the equity assumes all the risk.

We know that convertibles contain an option and therefore have a non-linear payoff but what about straight debt and equity if the future value of the firm is uncertain and could fall below the promised debt value? Well then the debt becomes *risky* since its repayment is no longer certain and its yield must compensate investors accordingly. N.B. so far we have just extrapolated a constant R_f for the cost of debt fudging the issue of what will happen to make it rise to R_a as the firm becomes fully debt financed. If payoffs at expiry have some structure, then we could use option pricing to determine the relative yield or cost of debt and equity.

Consider a one period firm (say a project with life one year) with unknown future *firm value* A_T[15] that has issued debt with a face value of X, what is the firm worth and what are the components of firm value (debt and equity) worth if the debt holders have a claim that pays at the most X but might pay less than X if A is less than X? This looks like an option and the value of the debt and equity at expiry (payoffs at the end of the period) are simply

$$\boxed{\text{Debt value } D_T = \min(X, A_T)}$$

$$\boxed{\text{Equity value } E_T = \max(0, A_T - X)}$$

$$\text{Asset value } = A_T$$

This is to say that because of the priority rule a loan to a limited liability company is worth at most the face value and could be worth less if the value

[15] Note that now the firm value A (the dividend yield δ is assumed sero here) is the unknown state variable that follows a zero dividend yield GBM instead of the stock price of the firm S.

$$\frac{dA}{A} = \mu dt + \sigma dZ$$

We will think of the equity E or stock price S as an option on and a function of A.

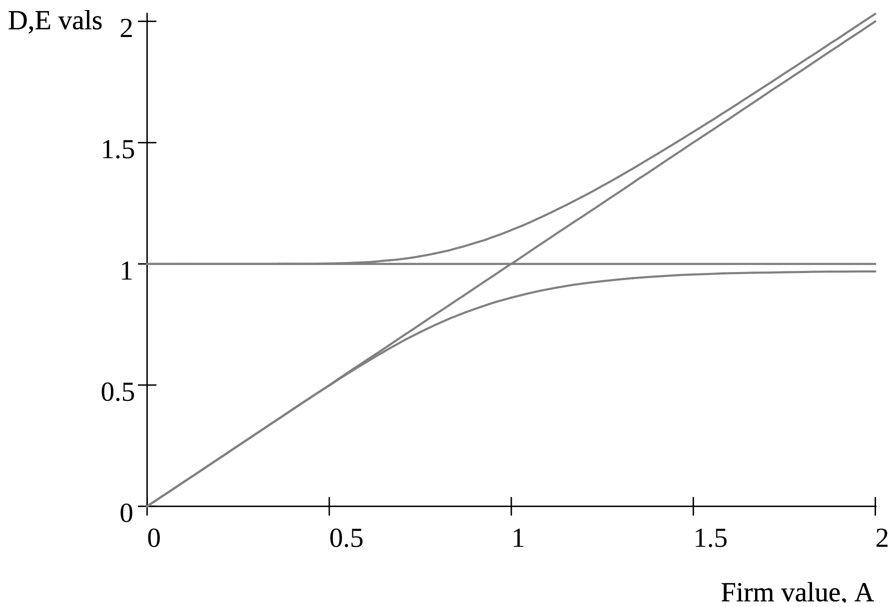
falls below the face value and that the equity is effectively a call on the asset of the firm at a strike price of the face value of the debt!

Thus in some sense it is the *debtholders who actually own the firm* but they have written a call to the equity holders with a strike at the face value of the debt. The debt value is

$$D = A - E$$

even for dynamic options positions and under the circumstances that the call becomes worthless ($E \to 0$), the debt holders indeed become the economic beneficiaries of all the (now diminished) assets $D \to A$.

Thus before realisation of the risky asset return, both debt and equity values will be sensitive to the asset return and this risk will be shared fractionally by both parties. Who bears the greater risk will determined by the proximity of the asset return to the covenanted return to debt holders, if well covered, debt is secure and equity is the residual (risky) claim, if not covered at all (zero or negative asset return) debt holders (or other parties!) will bear all the risk and equity holders enjoying limited liability will bear *no further risk*, their investment already having fallen to zero value.



Contingent Debt and Equity values as a function of firm value

There is another way of thinking about the transfer of an option from debtholder to equity holders, just as we considered equity to be a call at the face value of debt, and debt was equal to asset ownership less this call we can use put call parity $C - P = A - Xe^{-R_f T}$ to determine another relationship.

	Firm value A		Debt value D		Equity value E
	$\parallel$		$\parallel$		$\parallel$
(i)	A	$=$	$\boxed{A - C}$	$+$	$\boxed{C}$
	$\parallel$		$\parallel$		$\parallel$
(ii)	A	$=$	$\boxed{Xe^{-R_f T} - P}$	$+$	$\boxed{A - Xe^{-R_f T} + P}$

Table 8: Dual representations of option components of firm value

Here in Figure (8) it can be seen that as well as asset ownership less call on assets, debt is like a (zero coupon) risk free bond and a short position in a put. Thus it can be seen that risky debt is like risk free debt but with the added risk that the assets will be put back to the debtholders by the shareholders in those circumstances when the firm value has fallen below the face value of the debt and similarly equity is like a levered holding of the firm $A - Xe^{-R_f T}$ with downside protection provided by a put option from the debtholders. Either way we look at it the debtholders are short an option and the equity holders are long the corresponding option.

Other approaches to debt equity modelling are possible including infinite and rolling fixed horizon debt (Leland [29] 1994 and Leland and Toft [31] 1996) as well as using maturity profit caps and floors (Shackleton and Wojakowski [60] 2007). These use real options approaches.

4.2 Dynamic WACC

If the return on assets is given by a CAPM and asset beta β_a (the following relationships are strictly in expectational terms)

$$R_a - R_f = \beta_a (R_m - R_f)$$

then we should be able to relate the equity beta and now since debt is risky, the debt beta to this using option pricing theory

$$\begin{aligned} R_e - R_f &= \beta_e (R_m - R_f) \\ R_d - R_f &= \beta_d (R_m - R_f) \end{aligned}$$

Since equity is a call on A at X its beta is

$$\begin{aligned} E\beta_e &= AN(d_1)\beta_a \\ E(R_e - R_f) &= AN(d_1)(R_a - R_f) \end{aligned}$$

where $N(d_1)$ is the hedge ratio of the embedded (default) option and since debt is equal to A less E then its beta is

$$\begin{aligned} D\beta_d &= A(1 - N(d_1))\beta_a \\ D(R_d - R_f) &= A(1 - N(d_1))(R_a - R_f) \end{aligned}$$

and unsurprisingly we can return to a familiar WACC

$$\begin{aligned} AR_a &= DR_d + ER_e \\ A\beta_a &= D\beta_d + E\beta_e \end{aligned}$$

but this is more subtle than the Modigliani Miller WACC and capital structure irrelevance because implied within it are *dynamic not static* values and betas! This means that even if the asset beta remains the same, if the firm value changes the components of firm value will also change and the component capital costs will also have to change to leave the asset beta the same. This means that under this model (and this model only) we can actually describe the locus of the debt/equity returns. It will follow a different path under another model.

MM said that the value of the firm must be the independent of the financing because investors could replicate/hedge the static leverage, this WACC says that the firm value must be independent of the financing because investors could replicate/hedge this leverage dynamically as they would hedge options. Some long position in equity balanced by a short position in risky debt will be a perfect local hedge and so must return the risk free rate.

4.3 Debt returns and yields

Please note that the asset drift rate $\mu = R_a$. The risk free bond yields R_f while the risky bond is cheaper by the amount of the put option and yields $R_d \geq R_f$. We can describe three quantities, i) the expected drift on the bond, ii) the expected continuous return to maturity and iii) the yield to maturity assuming no default (Merton (1974) [42] calls this a risk premium to the risk free rate despite the fact that it does not contain the risk premium on the underlying asset.)

The YTM and the total expected return R_d are linked by the default probability and payout in default, in fact the YTM always offers a premium to R_d because it assumes that default does not occur, the magnitude of this premium is dependent on the default probability and the expected value of the assets in the default state (see Table 9). $N(-d_4)$ gives the probability that default occurs and $N(d_4)$ that default does not occur. $Ae^{\mu T}N(-d_3)$

	Risk Free	**Risky** $(P>0)$
Price	$\boxed{D = Xe^{-R_f T}}$	$\boxed{D = Xe^{-R_f T} - P}$
Expected instant. yield	$R_d = R_f$	$R_d = R_f + \beta_d (R_m - R_f)$ $= R_f + \frac{A}{D}(1 - N(d_1))(R_a - R_f)$ $\beta_d = \frac{A(1-N(d_1))\beta_a}{D} \qquad 0 \le \beta_d < \beta_a$
Expected return to mat.	$R_d = R_f$	$e^{R_d T} = \frac{\text{Expected under } P}{\text{BS expected under } Q} =$ $\frac{X - E^P\left[(X-A_T)^+\right]}{X - Xe^{-rT}N(-d_2) + AN(-d_1)} = \frac{1-N(-d_4)+Ae^{\mu T}N(-d_3)/X}{1-e^{-rT}N(-d_2)+AN(-d_1)/X}$
Promised mat. yield (no def.)	$\text{YTM}_d = R_f$	$\text{YTM}_d = \frac{1}{T}\ln\left(\frac{X}{Xe^{-R_f T} - P}\right) > R_d$

Table 9: Riskless and risky debt returns and yields

	Unlevered	**Levered**
Price	$\boxed{E = A - Xe^{-R_f T}}$	$\boxed{E = A - Xe^{-R_f T} + P = C}$
Expected instantaneous yield	$R_e = R_a$	$R_e = R_f + \frac{A}{E}N(d_1)(R_a - R_f)$ $= R_f + \frac{A}{E}N(d_1)(R_a - R_f)$ $\beta_e = \beta_a \frac{A}{E} N(d_1) \qquad \beta_a < \beta_e$
Expected return to mat.	$R_e = R_a = \mu$	$e^{R_e T} = \frac{\text{Expected under } P}{\text{BS expectation under} Q} =$ $\frac{E^P\left[(A_T - X)^+\right]}{AN(d_1) - Xe^{-rT}N(d_2)} = \frac{Ae^{\mu T}N(d_3) - XN(d_4)}{AN(d_1) - Xe^{-rT}N(d_2)}$
Yield to mat. (no def.)	$\text{YTM}_e = R_a$	?

Table 10: Unlevered and levered equity returns and yields

gives the expected asset value if default occurs.

$$
\begin{aligned}
Xe^{-YTM*T} &= Xe^{-R_f T} - P = D = e^{-R_d T}\left(X - E^P\left[(X - A_T)^+\right]\right) \\
&= e^{-R_d T}\left(X - XN(-d_4) + Ae^{\mu T}N(-d_3)\right) \\
\text{YTM}_d &= R_d + \frac{1}{T}\ln\frac{X}{X - XN(-d_4) + Ae^{\mu T}N(-d_3)} > R_d
\end{aligned}
$$

In discrete terms for a par $(X = 100)$ zero coupon bond that pays $\min(100, A_T)$

$$
\boxed{\frac{100}{(1+\text{YTM}_d)^T} = \frac{100}{\left(1+R_f\right)^T} - P(A_T, 100) = \frac{E^P[\min(100, A_T)]}{(1+R_d)^T}}
$$

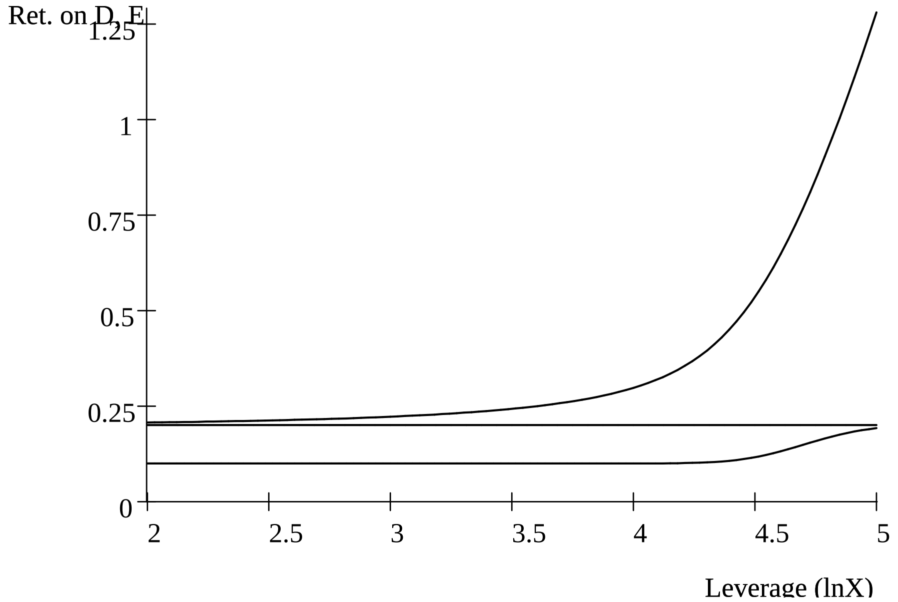


Figure 14: The WACC can remain constant at 20% even though both component costs of Capital (R_E and R_D) rise with increasing leverage ($x = \ln X$).

4.4 Equity returns

We can repeat similar calculations for the equity component Table (10). Again in terms of the one period, zero coupon world:-

$$\frac{E^P\left[\max\left(A_T - 100, 0\right)\right]}{\left(1 + R_e\right)^T} = C = A - \frac{100}{\left(1 + R_f\right)^T} + P$$

Debt and equity costs and their combination all lie on the security market line, with a weighted average at R_a.

$$\begin{aligned} T &= 1 \\ \sigma &= 0.2 \\ r &= 0.1 \\ A &= 100 \\ R_A &= 0.2 \end{aligned}$$

4.5 Specific risk

Finally finance theory admits a role for specific risk in pricing company assets! σ the volatility of firm assets is effectively the total risk of the firm and will generally be a function of market risk (through a beta) and some residual *specific risk*. The higher σ, the higher the specific risk, but all changes in specific risk do is to reallocate wealth between the stock and bond holders, it does not affect firm value and firm rate of return. Rises (decreases) in σ benefit stock (bondholders). See Section 8.2.5 for Debt covenants and the asset substitution problem.

4.6 Time to maturity of debt and option

Of course few firms actually anticipate being wound up at some fixed point in time, most going concerns expect to continue operation beyond the maturity of their debt. If firm value will allow, they will refinance their debt and continue. In reality therefore the bankruptcy option is infinite maturity irrespective of the explicit maturity of the debt and therefore is American not European in the nature of its exercise.

Hsia (1991)[16] used a particular assumption to use such an option based model for real world firms. (Without tax) he assumed that the asset of the firm evolve as a GBM without income distribution so that finite zero coupon claims could be priced in the way that Merton used as illustrated in this section.

$$\begin{aligned}
\frac{dA}{A} &= \mu dt + \sigma dZ \\
A &= D + E \\
E &= AN(d_1) - Xe^{-rT}N(d_2) = C \\
D &= Xe^{-rT} - P = Xe^{-iT}
\end{aligned}$$

where i is the YTM of the zero. The *duration* of a bond is the time weighted repayment or the interest rate sensitivity, for a zero coupon it is just the maturity time T, generally it is given by

$$\text{Duration } = -\frac{1}{D}\frac{\partial D}{\partial i} = T$$

Hence if we could estimate the duration of a real firm's debt, we may be able to use it as a basis for the maturity of an equivalent amount of zero debt. If

[16] Estimating a firm's cost of capital: An option pricing approach, Jounral of Business Finance and Accounting ,18(2), Jan 1991.

the firm current pays a \$ amount (pa) I in perpetual interest payments then the value of that debt on a perpetual basis at a YTM of i is D

$$\begin{aligned} D &= \frac{I}{i} \\ \text{Duration} &= \frac{1}{i} = T = \frac{D}{I} = \frac{1}{\text{current yield}} \end{aligned}$$

so the current yield on the debt can be used to fix the effective maturity of the (perpetual) debt. Now the strike X price needs to be fixed

$$\begin{aligned} D &= Xe^{-iT} \\ T &= \frac{1}{i} \\ X &= De \approx 2.72D \end{aligned}$$

Now use $A = D + E$ and $X = De$ to find

$$\begin{aligned} E &= (E + D) N(d_1) - De^{1-rT} N(d_2) \\ E(1 - N(d_1)) &= D\left(N(d_1) - e^{1-rT} N(d_2)\right) \\ T &= \frac{D}{I} \text{ the payback period of the debt} \end{aligned}$$

$$d_{1,2} = \frac{\ln\left(\frac{D+E}{eD}\right) + \left(r \pm \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

If we know the market value of equity E, and approximate the market value debt D with book value debt and approximate T with the debt payback period $\frac{D}{I}$ we can solve for the remaining variable σ. This allows $N(d_1), N(d_2)$ to be established and instantaneous costs of capital to be evaluated.

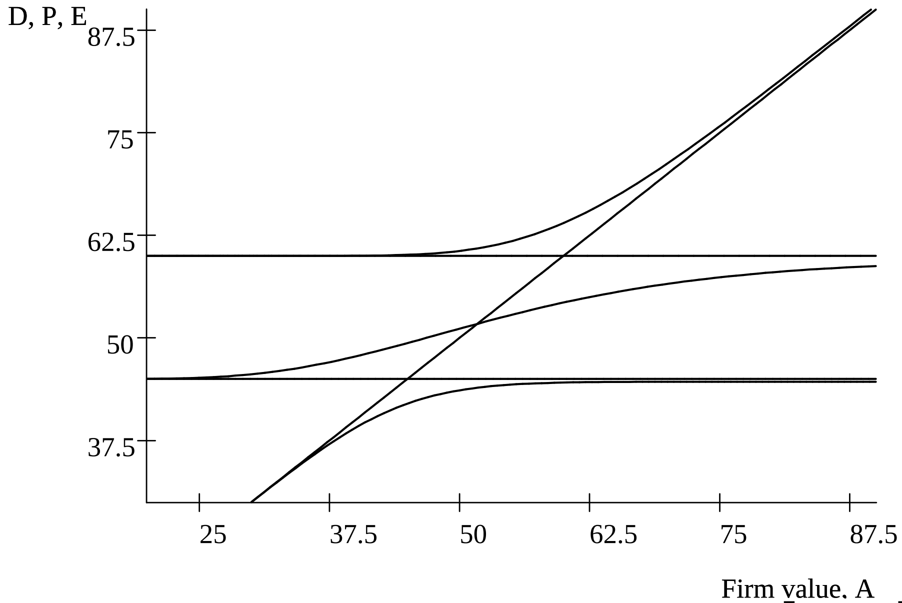
$$\begin{aligned} R_d &= R_f + \beta_d (R_m - R_f) \\ R_e &= R_f + \beta_e (R_m - R_f) \\ \beta_e &= \beta_a \frac{A}{E} N(d_1) \text{ etc} \end{aligned}$$

See example and reading.

4.7 Multiple claimants

Finally this analysis can be extended to multiple claimants ($A = D + P + E$), using the Bull Spread as the difference between two calls at different strike prices we could add a third tier of capital and produce a value diagram.

There is still no internal optimum for financing, any mix of claims is as good as any other. It is not until third party interests and latent holdings with contingent (and non negotiable) cashflows come along and break up the party that our theory of dynamic firm value yields an optimal debt equity mix.



Multiple hierarchical liability claims on firm asset value $\begin{bmatrix} & D \\ A & P \\ & E \end{bmatrix}$

Note that this finite debt horizon model is not the only way that debt default can be modelled. Infinite horizon, boundary crossing models (structural) and stochastic decay (reduced form) models abound. Some continuous versions are treated in the real options section, discrete versions are also possible.

5 Options on company liabilities

In this section we first assume that the equity value of the firm E follows the GBM, then later that A follows the GBM!

$$\boxed{\frac{dE}{E} = \mu dt + \sigma dZ}$$

5.1 Equity Calls

Investment Banks often offer options on particular stocks in order to satisfy some clientele; they write calls (or puts even) against a firms equity at a certain strike price and (net) hedge the position in a BS manner, using the BS price to determine a fair value and adding a profit margin. This does not increase or decrease the total amount of stock in circulation; as the stock price rises and the call come into the money, the Bank has to purchase more stock from the market in order to remain hedged (delta neutral). This could have nasty consequences if the underlying amount of the option issue were large compared to the total number of shares in issue but if the number is small there should be no problem. BS tells us that the fair price of the Equity Warrants or calls is given by

Stock Price	S	186.5p
Exercise Price of Call	X	200.0p
Interest Rate (annualised 3 month)	r	0.06
Volatility (which one?)	σ	$\sqrt{0.2} = 0.44721$
Time to Maturity of Call	T	$3/12 = 0.25$

$$C = SN(d_1) - Xe^{-rT}N(d_2) = 12.32\text{p}$$

where the numbers have been chosen to represent a three month call on Waste Management International (WMI chosen because it is a company that is currently paying no dividends).

If the company is paying dividends there is a problem with the BS formula (since it assumes that no dividends are paid). The problem is that the future value of the share and the dividend policy are not independent. Obviously as dividends are paid, the value of the share *ex-dividend* decreases, all else equal if WMI paid a dividend of 10p then we would expect its price to fall from 186.5 to 176.5p. N.B. this does not say that the value of the firm is dependent on the dividend policy, total value is preserved (186.5p) it is just that its distribution is altered between retaining it in the firm and distributing it to the share holders.

Even thought we know current dividends, we cannot perfectly anticipate future dividends and so part of the share price uncertainty is due to the dividends receivable before the call expires (the remaining uncertainty is driven by the dividends beyond the call maturity). Thus the firm could depress its share price below *any* exercise price by the payment of extreme dividends! Because of this, call options are often *dividend protected* where the effective exercise price can be adjusted to allow for any dividends giving the net effect of buying a call on a no dividend share. If the share pays out 10% of its value as a cash dividend the exercise is (roughly) decreased by 10% ($S' = S/1.1$) so that the call value only suffers itself by 10%, the solution is to decrease the exercise price by 10% ($X' = X/1.1$) and increase the number of calls held by 10% ($\#' = \# * 1.1$). $d_{1,2}$ will still remain the same and the total value associated with the calls will remain the same

$$\begin{aligned} d_{1,2} &= \frac{\ln(S'/X') + (r \pm 0.5 * \sigma^2) T}{\sigma T^{\frac{1}{2}}} \\ \text{Value} &= \#' \left(S' N(d_1) - X' e^{-rT} N(d_2) \right) \\ &= \# \left(S N(d_1) - X e^{-rT} N(d_2) \right) \end{aligned}$$

If calls are not *dividend protected* then we can still value them using BS by assuming we know the dividend policy, for example if WMI pursued a constant payout ratio, or that dividends grow at the same rate g as share prices S

$$\begin{aligned} \delta &= \frac{D_t}{S_t} \\ E[S_t] &= S_0 (1+g)^t \text{ or } S_0 e^{gt} \\ E[D_t] &= D_0 (1+g)^t \text{ or } D_0 e^{gt} \end{aligned}$$

($r_e > g$) then the total PV of the share is the PV of the share at expiry T plus PV of the dividends before expiry, for example if expiry is between the payment of $D_2 \& D_3$

$$\begin{aligned} S_0 &= \frac{D_0(1+g)}{1+r_e} + \frac{D_0(1+g)^2}{(1+r_e)^2} + \frac{D_0(1+g)^3}{(1+r_e)^3} + ... + \frac{D_0(1+g)^\infty}{(1+r_e)^\infty} \\ &= \frac{D_1}{r_e - g} \\ \text{share price now} &= \text{PV(dividends until 2)} + \text{PV(dividends after 2)} \end{aligned}$$

Thus the dividend yield is defined as

$$\boxed{\delta = \frac{D_t}{S_t} = r_e - g}$$

and firms with high payout ratios will have lower growth than firms with low payout ratios unless growth is funded by debt. In continuous time the current start and forward start perpetuities can be used to value the annuity from 0 to T

$$\begin{aligned} S_0 &= \int_0^\infty e^{-r_e t} D_0 e^{gt} dt = \frac{D_0}{r_e - g} \\ e^{-r_e T} E[S_T] &= e^{-r_e T} \int_T^\infty e^{-r_e t} D_0 e^{gt} dt = \frac{e^{-r_e T} E[D_T]}{r_e - g} \\ &= \frac{D_0 e^{(g - r_e)T}}{r_e - g} = S_0 e^{-\delta T} \text{ since } r_e = g + \delta \\ S_0 - e^{-r_e T} E[S_T] &= S_0 \left(1 - e^{-\delta T}\right) \end{aligned}$$

so the only adjustment necessary is to lower the (full dividend) price from S_0 to $S_0 e^{-\delta T}$ where δ is the dividend yield and then input this reduced amount in the BS formula. For example, if WMI paid a dividend yield of 1% p.a. of its then value then the share price for BS use would be reduced to

$$\begin{aligned} C &= S e^{-\delta T} N(d_1) - X e^{-rT} N(d_2) = 12.11\text{p} \\ d_1, d_2 &= \frac{\ln S - \ln X + (r - \delta \pm \sigma^2/2) T}{\sigma \sqrt{T}} \end{aligned}$$

This can be thought of as replacing S in the BS with $S e^{-\delta T}$

$$\begin{aligned} S_0 e^{-\delta T} &= 186.5 e^{-.01 * 0.25} = 186.03 \\ C &= 12.11 \end{aligned}$$

giving a slightly lower call value. If short term dividends ran ahead of the 1% the share and call prices would fall further, if all future dividends ran ahead of the 1% projected rate of growth, two effects would compete, first prices would fall due to the higher distributions but this would be more than compensated for by an increase due to higher value from the long term dividends.

N.B. the longer the time horizon the further the call price falls below the value if dividends are non–zero, it is therefore no longer the case that the infinite horizon option price is the stock value itself, it is actually zero.

5.2 Stock splits & scrip issues

Sometimes firms simply double the number of shares in issue, all this does is half the share price and leaves the firms market value and book value unaltered. Cash flows and book balances remain the same. In the US this happens because investors have come to accept that shares trade in a \$1–\$100 range.

Equity calls that actually create new shares do exist, indeed firms sometimes want to sell new equity (raising equity capital) and do so by issuing calls on their shares rather than just by selling shares now. They could either sell new shares or calls on new shares to (new) third party investors or offer them to existing shareholders first. In the US and UK existing share holders have a right of *pre-emption* over new investors to subscribe to new capital and so rights for new shares are given to existing holders to exercise as they will or sell in the market for their option value. This right ensures that minority holders cannot have their interest further diluted by the issuance of more equity to the majority, they must be offered their fair quota and be given the opportunity to maintain the same fractional ownership of the firm.

Suppose WMI had a rights issue of 1:1 at a price of zero (usually the subscription price is greater than zero but less than the current price!), what will be the effect on the MV and BV balance sheets? Well since no new capital is raised the BV & MVs will remain the same and only the number of shares will change, WMI has about 375M in issue (MV of approx. £700M @ £1.865p) so the new figure will be 750M and the share price will halve (because firm value $= Sn = \frac{S}{2}.2n = \pounds 700M$ must remain unchanged). What has this achieved? This is called a *scrip issue* or stock split since no new funds were raised. All it does is reduce the value of the shares by the same fraction as the increase in their number. A two for one scrip issue would reduce the share price by a factor of three.

Since scrip issues and stock splits reduce share prices, equity calls are often *scrip protected* against this effect. If the share price halves and the number in issue doubles, the calls can be value protected by halving the exercise price and doubling the number in issue.

5.3 Rights Issues

What about a non-zero rights price? Well if WMI had a rights issue of 1:10 (one new share for every 10 old held) at a price of £1.00 then 37.5M new share if purchased would raise £37.5M and the post rights balance sheet (MV & BV) would increase by £37.5M, the number of shares would also increase however to 412.5M and so we would expect dilution to a new share price of

$$
\begin{aligned}
A &= E = Sn = 1.865 * 375\text{M} \\
1.00n' &= 1.00 * 37.5\text{M} \\
n + n' &= 375 + 37.5 = 412.5\text{M} \\
S'(n + n') &= 699.38 + 37.5 = \pounds 736.88\text{M} = A' = E' \\
S' &= \frac{736.88}{412.5} = \pounds 1.7864
\end{aligned}
$$

Why are shareholders prepared to accept a decline in the value of their shares? Because they own 10% more shares now (net of the £1 paid) and the total value of their holdings is preserved

$$\boxed{\text{10 shares @ }\pounds 1.865 + \pounds 1 \text{ on a new share} = \text{11shares @ }\pounds 1.7864}$$

$$\pounds 18.65 + \pounds 1 = \pounds 19.65$$

They can either subscribe to the new share or sell the right, in the former case their fractional holding will not be diluted and in the latter it will. If they do not want the one new share they can sell it in the market for $\pounds 1.7864$ having subscribed the $\pounds 1$ or alternatively they can sell the rights, either way they will be diluted (own a lower fraction of the firm). The 10 combined rights must have value $\pounds 1.7864 - \pounds 1 = 78.64$p since they allow purchase of a share worth $\pounds 1.7864$ for $\pounds 1$. One right is worth 7.864p and on the ex–rights day the cum–rights share price will fall by 7.864p to the ex-right price of $\pounds 1.7864$ since

$$\begin{aligned}
\text{1 share @ }\pounds 1.865 &= \frac{\text{11shares @ }\pounds 1.7864 - \pounds 1 \text{ on a new share}}{10} \\
&= \text{1share @ }\pounds 1.7864 + \frac{\text{1share @ }\pounds 1.7864 - \pounds 1 \text{ new share}}{10} \\
\pounds 1.865 &= \pounds 1.7864 + \frac{\pounds 0.7864}{10} \text{ or } \pounds 1.7864 + \pounds 0.07864
\end{aligned}$$

There may not seem to be an option involved here but the firm cannot force shareholders to convert at any price, for example nobody would exercise rights on WMI to buy at $\pounds 10$! Actually shareholders have the right but not the obligation to exercise their pre-emption rights, i.e. an option. Typically subscription prices (£1 in the case above) are set below the current price so the *option to exercise you right* is in the money and most likely conversion will occur especially since the Rights exercise time is usually kept short, normally about three months.

For a given amount of investment proceeds, the firm can choose the degree of dilution, i.e. number of share on issue and can therefore either give a cheap rights option to its share holders (at the money) or an expensive one (in the money). The cheaper the option value the less likely that it will be exercised and the less likely that new money will be raised.

For any rights subscription price there is a small chance that the ex-rights price will fall below the exercise price and the option will fall out of the money and not be called. If this happened the firm would not get its new capital and so to prevent this firms often go to Investment Banks for *underwriting of new (rights) issues.* Here the firm is guaranteed to get the new funds because it purchases a matching put to offset the call given to its shareholders. This put is not free and (subject to the dividend problem) can be priced via BS *so long as the expected dilution is accounted for.*

5.4 Equity Warrants & Dilution

In reality the shares the share price does not actually fall by the exact amount of the theoretical rights price but falls by the amount of the option to purchase the share at the rights price. This option can then be used to infer the volatility of the share in question. The amount of the share price fall (equal to right value) can be estimated using the Black–Scholes model. We must however consider the impact of *dilution.*

When the firm issues (i.e. *sells*) *calls on new shares* they are labelled *equity warrants* and are very similar to rights issues (they both involve an option to purchase shares at some future date, rights are *given* to existing investors in exchange for immediate dilution, warrants are typically sold maybe to new investors and dilution cannot be ignored either).

If an Investment bank writes calls on a firms equity and backs it by owning a fractional delta hedge then no new shares have been created or will be created (it must buy the fractional deltas hedge shares from market participants), but if the *firm itself* sells warrants and pockets the cash (just as if it gives the rights to existing investors), then there is a chance that new shares will be issued upon the exercise of the warrants in question and this will dilute firm value over more equity holders. Thus the further factor that must be accounted for on exercise in option pricing, is the fact the number of shares increases and therefore the *expected* share price will be lower due to anticipated *dilution.*

q	:	the full dilution factor $\frac{n'}{n}$
$qN(d_1)$	:	the risk neutral expected dilution factor $\frac{N(d_1)\,n'}{n}$
S	:	current share price $= A/n$
X	:	exercise price on shares
T	:	time to maturity
r	:	annualised interest rate to maturity
σ	:	annualised standard deviation of stock returns
(δ	:	annualised dividend yield)

Suppose $n' = nq$ shares are callable by the warrant holders against a base of n. Copeland & Weston (p473 uses V but you could think of A) shows how we can still revert to the BS formula. If the firm value is A then the share price before warrant issue and anticipated dilution is

$$S = \frac{A}{n}$$

and if the exercise of the nq warrants generates proceeds of nqX (i.e. number of shares times the subscription/exercise price) then the value of the firm post exercise (including the cash generated) is $A + nqX$ and the new share price is a weighted average of the old and the investment proceeds

$$S' = \frac{A + nqX}{n + nq} = \frac{S + qX}{1 + q} = \frac{S}{1 + q} + \frac{qX}{1 + q}$$

Options pay off the maximum of the new diluted share price only $S' - X, 0$ i.e. the max. of 0 and $S' - X$

$$\begin{aligned} W &= e^{-rT} E^Q \left[\max\left(S' - X, 0\right)\right] \\ S' - X &= \frac{S + qX}{1 + q} - X = \frac{S - X}{1 + q} \\ C &= e^{-rT} E^Q \left[\max\left(S - X, 0\right)\right] \end{aligned}$$

so that the warrants will be converted as if they were naked calls or naked warrants but that the payoff to the warrant holders is $\frac{1}{1+q}$ times that of the naked call holder due to the dilution that they will encounter

$$\boxed{W = \frac{1}{1+q} C}$$

where W represents the value of warrants that dilute and C represents the value of calls that do not. Thus the way to value warrants is to first calculate call prices using the BS inputs (maybe need to adjust for dividends) and then apply a dilution factor $\frac{1}{1+q}$.

Note that this analysis assumes that the share price S has not yet impounded the dilution effect and therefore started to trade like S'. Once irrevocably announced, the share S will start to trade as if diluted S' but if the new issue were subsequently rescinded, it would revert to S. This leaves the tricky problem when estimating parameters, should one assume that the current share price represents S (no issue envisaged) or S' (new contingent issue envisaged)? In reality the share price is always adjusting to new potential issues and it may well reflect an anticipated issue even before the managers consider or even announce such an issue! That is to say that the number of shares n used in market valuation is not really deterministic but it too actually anticipates all future potential stock issues.

	Time T	$T = 0$
"Stock" Price $S =$	186.500p	186.500p
Call $C =$	$\begin{array}{c} SN(d_1) - \\ Xe^{-rT}N(d_2) \end{array} = 88.008\text{p}$	86.500p
10 Rights, 1 Warrant $W =$	$\frac{1}{1+q}C = \frac{C}{1.1} = 80.007\text{p}$	78.636p
1 Right $R =$	$qW = \frac{W}{10} = 8.001\text{p}$	7.864p
Ex. Rights Price $S - R =$	178.499p	178.636p
RN Prob. of takeup $N(d_2)$	99.695%	100%

Table 11: Optional rights valuation with no cash

5.5 Rights as Warrants that raise no initial cash

Assuming that no dividends are to be paid before rights exercise, the BS formula applied to the full share price will give the fraction of share value attributable to the warrant. This is because before exercise the value of the firm has not changed and the warrants were in effect given away. A different input price must be used if the warrants are sold and cash is taken in at the time.

Warrants sold for no initial cash		
Stock Price	S	186.5p
Exercise Price of Call	X	100.0p
Interest Rate (annualised 3 month)	r	0.06
Volatility	σ	$\sqrt{0.2} = 0.44721$
Time to Maturity of Call	T	$3/12 = 0.25$

In effect we are repeating the calculation from above except for *non–zero time to maturity* where the option not to subscribe is valuable as can be seen by varying the time to maturity T in the option calculation, as T goes to zero, the call price reverts to its payoff value used in the section on Rights Issues.

The risk neutral probability associated with rights take up is $N(d_2)$ (or for the true probability of takeup $N(d_4)$, which depends on R_e).

Negative issues of shares (repurchases) are also possible, indeed some firms consider them very tax efficient means of redistributing cash.

5.6 Warrants that raise new initial cash

When warrants are sold and new cash is raised immediately, the value of the firm increases the moment the warrants are paid for. This makes the calculation more complex (Hull p254). We need to know the volatility of the

	time T	$T = 0$
"Stock" Price $S =$	$186.500 + qW = 195.300$p	195.150p
Call $C =$	$SN(d_1) - Xe^{-rT}N(d_2) = 96.798$p	95.150p
10 Rights, 1 Warrant $W =$	$\frac{1}{1+q}C = \frac{C}{1.1} = 87.998$p	86.500p
1 Right $R =$	$qW = \frac{W}{10} = 8.800$p	8.650p
Ex. Rights Price $S - R =$	186.500p	186.500p
RN Prob. of takeup	99.840%	100%

Table 12: Optional rights valuation with cash

assets post subscription, however we could assume that the warrant funds are immediately invested in similar assets to the existing ones. Rather than use the stock price of 186.5p we need to allow for both the dilution and the increase in stock price associated with the (fair) proceeds from the Warrant issue. In fact solving for the fair price of the warrants is tricky, since the new share price is replaced by one that incorporates the warrant proceeds themselves and it is no longer the case that the share price will suffer on ex–right stripping because the rights are *sold* separately

$$S' \to S + qW$$

W effectively solves a tricky equation with no closed form solution

$$\boxed{W = \frac{1}{1+q}C(S + qW, X, r, \sigma, T)}$$

Warrants sold for initial cash		
Stock Price	S	$186.5 + qW$p
Exercise Price of Call	X	100.0p
Interest Rate (annualised 3 month)	r	0.06
Volatility	σ	$\sqrt{0.2} = 0.44721$
Time to Maturity of Call	T	$3/12 = 0.25$

For Rights issues and Warrants with short lives, the assumption of zero dividend yield is OK but for longer term warrants (say 10 years or more) that may be embedded in other instruments, the dividend yield cannot be ignored.

5.7 Convertible Debt

Convertible bonds are purely bonds that can be exchanged before redemption (maturity) for some new equity in the firm, for example

Face Value of debt	\$50M
Par Value	\$1000
Number of Bonds	50,000
Maturity	25 years
Effective Option Mat.[17]	5 years
"Regular" Coupon	17%
Convertible Coupon	10%
Face value converts to	35.71 shares
i.e. Exercise Price of	\$28
Current share price	\$25
Dilution factor $1 + q$	1.05

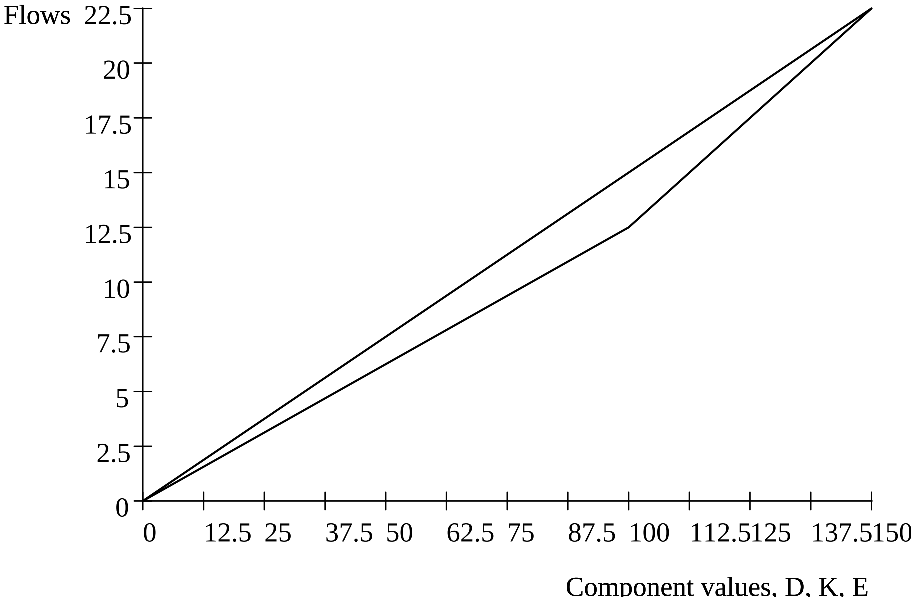
The firm can enjoy a coupon advantage of 7% if it is prepared to give away the embedded option on equity. What is the cost of capital and the effect on WACC?

It is similar to issuing a third tranche of capital like the preferred equity we considered earlier. Instead of preferred equity, suppose the firm funded 50 units more of existing assets by issuing 50 units of a hybrid instrument called a convertible bond, which partly mirrored debt and partly equity. How is this incorporated into the WACC? Defining π_k, R_k, K as the expected flow, return and market value to convertible holders (respectively.), the cashflow identity from which WACC can be calculated becomes

$$\begin{aligned}
\pi_a &= \pi_d + \pi_k + \pi_e \\
R_a A &= R_d D + R_k K + R_e E \\
R_a &= \frac{D}{A} R_d + \frac{K}{A} R_k + \frac{E}{A} R_e \\
\text{and } A &= D + K + E
\end{aligned}$$

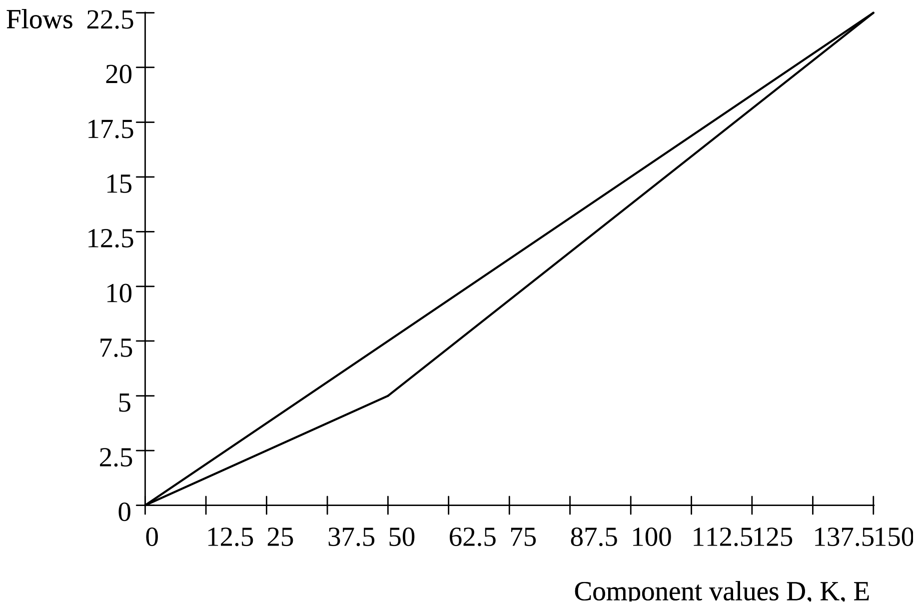
and we are back to the Figure with the Preferred claim being replaced by the Convertible claim. Under some circumstances the Convertible will behave as debt and $R_k \to R_d$ (at worst it will yield the same as debt)

[17] Convertible bonds often have call features which prevent the option holder from holding the option to maturity.



Liability returns (slope) for D, K, E with convertible yielding the debt return

and in some circumstances it will behave like equity and $R_k \rightarrow R_e$ (its rate of return cannot exceed that of equity's or the equity holders will act to force conversion into equity)



Liability returns (slope) for D, K, E with convertible yielding the equity return

Thus a convertible security is a compound of debt and equity warrants ($K = B + W$) and its rate of return is a (linear again) combination of that of debt and equity through an equity warrant

$$\boxed{R_k = \tfrac{B}{K} R_d + \tfrac{W}{K} R_w}$$

where the weights correspond to the fraction of the Convertible attributable to debt and equity ($B + W = K$) and the closeness of R_k to R_e is determined by the proximity to conversion (or exercise of the option) so that the WACC is given by

$$\begin{aligned} R_a &= \frac{D}{A} R_d + \frac{K}{A}\left(\frac{B}{K} R_d + \frac{W}{K} R_w\right) + \frac{E}{A} R_e \\ &= \frac{D+B}{A} R_d + \frac{W}{A} R_w + \frac{E}{A} R_e \\ \text{and } A &= D + B + W + E \end{aligned}$$

We also know that a warrant and the rate of return on a warrant is linked to that of the underlying through

$$\beta_w = qN(d_1)\frac{E}{W}\beta_e$$

where $N(d_1)$ is the option delta so that

$$R_a = \frac{D+B}{A} R_d + \frac{E + EqN(d_1)}{A} R_e$$

i.e. issuance of K units of a Convertible bond is equivalent to issuing B debt and the effective new amount of equity is

$$\begin{aligned} EqN(d_1) &= \frac{n'}{n} EN(d_1) \\ &= n'SN(d_1) \end{aligned}$$

(since $E = nS$) i.e. n' units with an effectiveness of $N(d_1)$ at price S!. Option pricing theory will allow us to determine $B, W, N(d_1)$ and therefore the proportions of debt and (implicit) equity present in a convertible at the time of issue. Subsequent to issue, this fraction will change *dynamically* and the fraction will change smoothly from its starting value to its final value of either $K = B$ if the bond is not converted but is retired at maturity or $K = qE = n'S$ if the bond is converted to equity. The market value of the convertible will of course also wander away from its starting value and will either end up at the pure debt level or at the value of the converted equity content again either assuming no conversion or conversion.

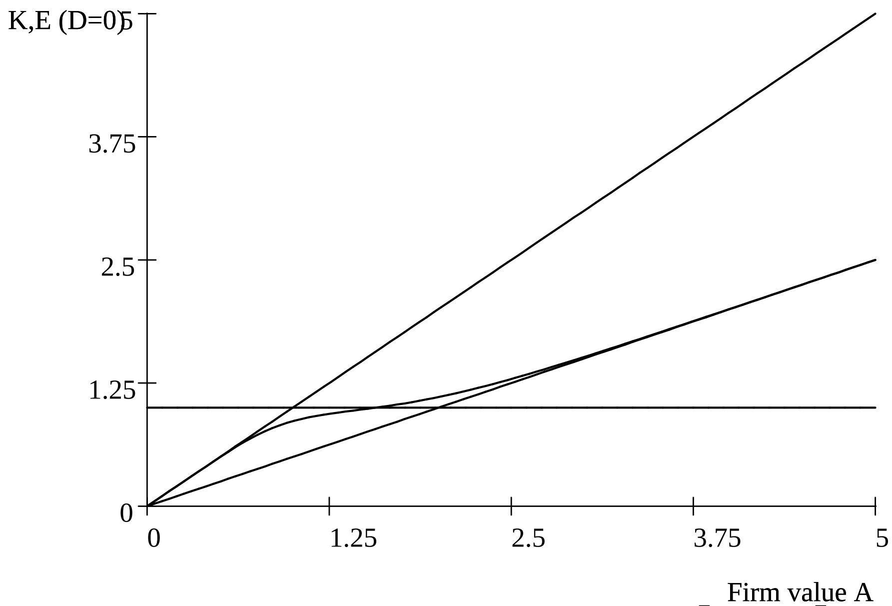
If you need further motivation as to why convertible bond financing is no better than straight debt, consider a special mortgage with a low interest rate, the catch is that on mortgage repayment the lender has an option on part of your house price increases. The interest rate discount could be offset by the expected option loss you suffer on exercise!

5.8 Convertible Bond Example

Firm Value (incl. all proceeds)	A	200.00
Firm Value (incl.) less PV of divs	$Ae^{-\delta T}$	180.97
Face Value of Zero Coupon Bond	X	100.00
Interest rate	r	10%
Dividend Yield	δ	2%
Asset Volatility	σ	20%
Eff. Option Maturity in Years	T	5
Call at X	$C(A, X)$	120.42
Put at X	$P(A, X)$	0.11
Straight Bond Value $D =$	$Xe^{-rT} - P = A - C$	60.54
Straight Equity Value $E =$	C	120.42
Conversion Price of Convertible	X'	200.00
Call at X'	$C(A, X')$	66.28
Dilution Factor $q = \frac{X'-X}{X} = 1$	$\frac{1}{1+q} = \frac{X}{X'}$	$\frac{1}{2}$
Warrant	$W = \frac{1}{1+q}C(A, X')$	33.14
Convertible Value K	$B + W$	93.69
Equity Value E'	$E' = E - W$	87.28
	$Ae^{-\delta T}$	180.97

Convertible bonds cannot be valued without a dividend yield assumption since even current non–dividend paying stock will probably pay dividends over a long horizon.

Typically the exercise option is *American* not *European* and BS will give an answer that is too low. However the uncertainty over (bondholder) exercise is often offset by equity holder call options on the bond to force conversion at $R_k = R_e$ (i.e. the point just before bondholder start to get a better return than equity holders). This correspond to a point where the dividend yield on the underlying share is just equal to the coupon on the face value of the bond (10%).



Value of convertible before expiry of option $\begin{bmatrix} & K \\ A & \\ & E \end{bmatrix}$

6 Bankruptcy costs, tax & optimal capital structure

6.1 Tax

If corporate taxes are levied on profits after interest payments only, money can be spirited out of the company to debtholders before tax and all capital holders would wish their capital claims to be labelled as "debt" to benefit from this tax exempt status. Thus all firms would wish to be 100% debt financed and there would be no role for equity at all! What factors could motivate a role for equity and lead to an internal solution (instead of corner solution) for optimal capital structure?

6.2 Third party and external claims in bankruptcy

We need a theory that can encompass cash flow costs that are contingent on the capital structure and potential insolvency. It is not until debt and equity holders are threatened with losses to third parties that they can agree that there is a best capital structure. We know that taxes drive a wedge between debt and equity holders since tax must be paid before dividends can be distributed. Traditional analysis shows that if there is a tax advantage to debt then it would be value maximising to 100% debt finance[18] the firm and have no equity at all! What expected cashflows can persuade a firm to accept anything other than a 100% level of gearing?

The answer is bankruptcy costs that the firm must pay to some third party (such as legal fees to lawyers) in the event of technical or absolute bankruptcy. Alternatively these third party costs can be thought of as loss of revenue as customers cease to trade with the firm as it becomes increasingly likely that liquidation will occur.

6.3 "Stages" of bankruptcy

Firms do not often suddenly go bankrupt, normally there are a number of stages than run beforehand, some of which are listed here.

1. Normal healthy profitable firm, paying full interest and dividends, with equity and investment grade debt outstanding.

2. Profit warning. Profits may be forecast to fall or becomes negative (losses).

[18] Modigliani Miller [43][46]

3. Downgrading. Firm's debt credit rating is downgraded by rating agencies (Moody's, Standard & Poors – AAA to D etc.) indicating increased riskiness of debt. Debt values fall and yields increase following a presumed fall in equity price. Bond's start to trade like junk bonds.

4. Dividend reduction/suspension. Share price may or may not fall on news depending on whether the event was expected or not.

5. Financial Distress. Some early indication that the firm may be not be able to meet all its contractual future obligations when they become due.

6. Interest suspension. Firm may or may not applying for suspension of interest payment or debt moratorium, bank's may withdraw or suspend short term credit facilities preventing the firm from financing other interest payments. Bonds trading well below their face value.

7. Creditor protection. Firm may apply for formal protection from all its creditors (US Chapter 11). This places certain restrictions on its activities, it is effectively under control of a Court (Judge). This process can run for a long time (firm's in the US have lived under Chapter 11 for years, some emerging and some sinking).

8. Petition for Receivership. Appeal to the court by the creditors for a receiver to be appointed.

9. Receivership. If the petition is successful, a receiver is appointed (in place of the company Directors) to sell the assets of the firm on behalf of the creditors. Occasionally a buyer may be found for the whole firm, otherwise priority rules of liquidation are followed with differing degree of strictness in different regimes, but generally the amount returned to shareholders if the process has got this far is very small (most often zero).

10. Winding up petition. Alternatively creditors may petition a Court for the firm to be wound up (liquidated).

11. Liquidation. The fees of the Liquidation firm and government taxes due are generally paid first before any other creditors, secured creditors next, trade creditors, next and unsecured creditors last. Firm ceases to exist.

It is important to note that it is not 100% sure that 2 will follow 1 or 3 will follow 2. At each stage the company may reverse its fortunes and start to unwind the sequence of event's that were leading it to formal bankruptcy. It is also possible for the events to happen is slightly different sequence, e.g. for 3 to happen before 2. Items can occur simultaneously as well.

Public companies are hardly ever liquidated, rather they become subject to takeover well beforehand as their assets become increasingly cheap. Thus much of M&A activity is driven by poor corporate performance and failure.

On rare occasions, firm's go from the initial state 1 straight to an advanced state of bankruptcy such as 6 or 7. Polly Peck plc was an example of this but fraud on the part of the Director's is normally associated (it is illegal to trade knowing that your firm does not have the means to meet its obligations). Otherwise it is generally unlikely for a large firm to be hit by such extreme negative changes in value, rather more often a sequence of events unfolds over a period, to which the firm may or may not have the ability to react in time.

Too often bankruptcy is associated with failure, but it should be recognised that eventually *all* lines of business will be either greatly reduced or disappear (where are the great Canal Companies of the 18th Century?). *Firms* on the other hand have the ability to outlive dying businesses by divesting them before death and moving into new businesses, sometimes it is cheaper for firms to reorganise and divest themselves, sometimes cheaper to actually go bankrupt (this may represent a valuable option the option to terminate negative expenditure).

It is probably better to think of actual bankruptcy as a firm recycling its durable and still valuable assets (land, scrap, labour etc.) to other productive and more economic activities than as the result of collective failure, sometimes this gets done with the firm, sometimes with the whole firm.

6.4 Bankruptcy and Capital Structure Example

6.4.1 The firm's assets

A firm has invested in a risky project. It is constructing a plant on a "turnkey" basis, and when it is completed it will be sold at the then current market price which is subject to uncertainty.

It is common knowledge that the current value of the firms assets gross of tax and other costs is A, but that the future sale value is uncertain. All parties agree that A will diffuse from its current level according to a geometric Brownian motion and future prices will therefore be lognormally distributed with variance $\sigma^2 T$ (the Black Scholes conditions). There are no interim asset or liability cash flows.

Payoff to Claimant	**Value A exceeds X**	**X exceeds A**
Equity	$(1-\tau)(A-X)$	0
Tax	$\tau(A-X)$	0
Debt	X	$X-(1+\alpha)(X-A)$
Third Parties	0	$\alpha(X-A)$
Sum	A	A

Table 13: Payoffs to various parties in and out of bankruptcy

6.4.2 Project loan

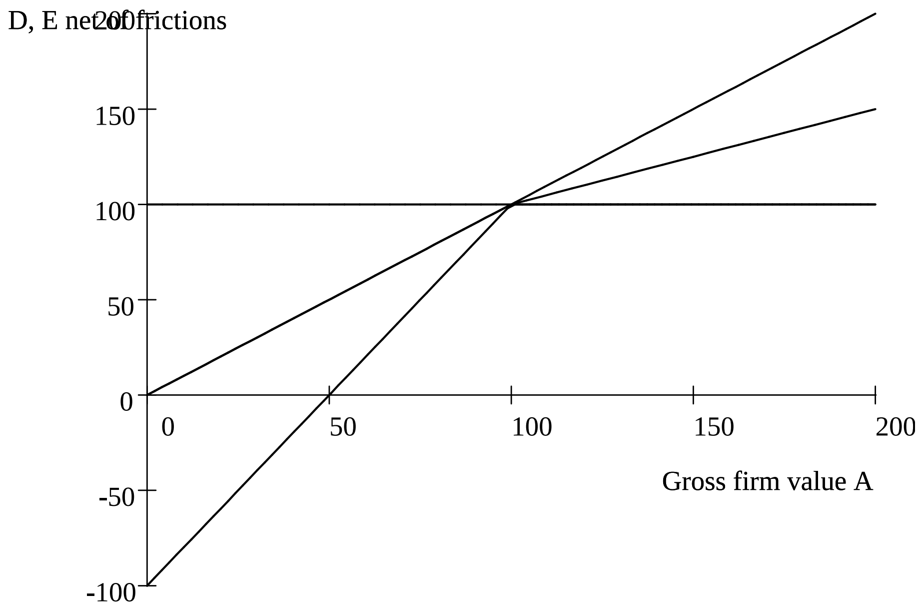
The bankers have lent money to this firm. The promised repayment is X (bullet repayment i.e. inclusive of interest and principal) at time T.

6.4.3 Project Equity

The remainder of the cash initially required for investment was contributed by the equity holders.

6.4.4 Bankruptcy code

The future value of the firm will be shared amongst competing claimants in a non–linear fashion. If the firm value exceeds the promised debt repayment amount, the firm is not in default, the debt claim is paid in full without incurring any third party expenses and the residual is shared between the equity holders and the government who get a tax slice on the profits. If however the debt claim cannot be met in full, the equity holders and government get nil and third party expenses proportional to the degree of shortfall must be paid before any distribution can be made to the debt holders. Figure **??** graphs the payoffs at time T to the various parties (E Equity Holders, T Government Tax, D Debtholders and TP Third Parties). Summarising the payoffs in Table form or graphically

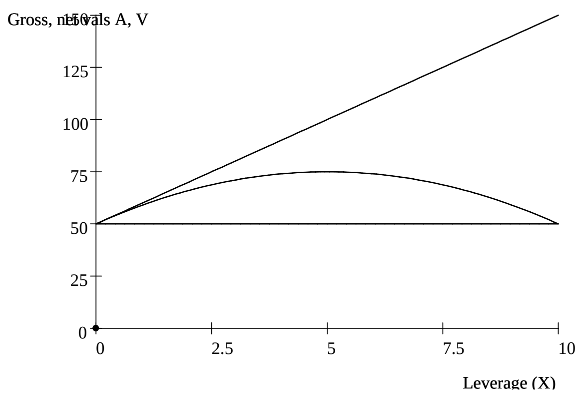


Payoffs at expiry to contingent claim holders in and out of bankruptcy

Remember, this is only a stylized bankruptcy code but it is relatively easy to analyse. It does however conform to the traditional analysis of firm value starting from an all equity financed firm, considering both the tax advantages and financial distress disadvantages of debt

$$
\begin{aligned}
\text{Firm Value } (V_L) &= \text{Value if all equity financed } (V_U) \\
&+ \text{Tax Benenfits of Debt} \\
&- \text{Expected Financial Costs of Debt}
\end{aligned}
$$

Again the optimal capital structure is when marginal benefits of tax shield equate to marginal costs of distress.



MM (limited) gains to leverage

6.4.5 Bankers

In the absence of bankruptcy costs, the market value of the loan could be established using a certain bond less put formula $Xe^{-rT} - P$ (or equivalently assets less call, $A - C$).

However, if the firm value A at time T is less than X then substantial bankruptcy costs will have to be borne by the bankers and under these conditions the equity holders will not share the losses.

The bankers know from experience that the more the asset value shortfall $(X - A)$ at time T, the greater the costs and specifically that for every \$ the asset value A falls below X at time T they will incur another \$ of expenses in the form of loss of asset value due to forced sale, negotiation and legal costs.

Thus their *net* debt claim (contractual less costs of liquidation) is actually $Xe^{-rT} - (1 + \alpha) P$ where $\alpha = 1$ represents \$1 for \$1 and the debt investment is worth less than would be the case if there were no bankruptcy costs. Thus it is possible for the debt to have negative value, in the extreme if A were very close to zero at T (an unlikely event), the debt would have negative worth $-\alpha X$.

It is important to note that the equity holders may be able to *estimate* the costs the bankers will incur if bankruptcy occurs, but they cannot know it precisely since this information is proprietary to the bank and its cost position. Conversely, the bank is only able to *estimate* information that is specific to the equity holders.

6.4.6 Equity holders

If corporate and equity holder taxes were zero, the equity holders value could be established as a call on the assets A at a strike price of X. However, if the firm value exceeds the promised repayment to debtholders the firm is deemed to be in profit and corporation tax will be levied at a rate τ. Thus Equity is a fractional call worth $(1 - \tau) C$ and the government tax call is worth τC.

6.4.7 Caveats

It is important to note that although this example and assignment captures several real life features of firm capital structure and valuation, it is not fully realistic. Firstly, the exact nature of the bankruptcy costs have been stylized in order to make them easy to value (Black Scholes Put). Secondly, firms typically have long maturity debt and asset life, with interim interest payment and asset cash flows becoming due before maturity. Bankruptcy therefore is more likely triggered by a flow condition rather than a stock condition at maturity, i.e. a firm is declared bankrupt when it can no longer

	(Expected Present) Value	
Value to claimant	**using Calls**	**or using Puts**
Equity $E =$	$\boxed{(1-\tau)\,C}$	$(1-\tau)\left(A - Xe^{-rT} + P\right)$
Tax $T =$	$\boxed{\tau C}$	$\tau\left(A - Xe^{-rT} + P\right)$
Debt $D =$	$A - C - \alpha\left(C - A + Xe^{-rT}\right)$	$\boxed{Xe^{-rT} - (1+\alpha)\,P}$
Third Party $TP =$	$\alpha\left(C - A + Xe^{-rT}\right)$	$\boxed{\alpha P}$
Sum A	A	A

Table 14: Payoff values

pay its interest payments well before the debt in question is ever due for repayment.

6.4.8 Valuation

Under the BS framework established, the payoffs can be valued using standard option theory in terms of Calls $C(A, X, r, \sigma, T)$ or Puts $P(A, X, r, \sigma, T)$ if the Put Call parity theorem is used

$$\boxed{C - P = A - Xe^{-rT}}$$

By way of example

$$A$$

$$X = 100$$

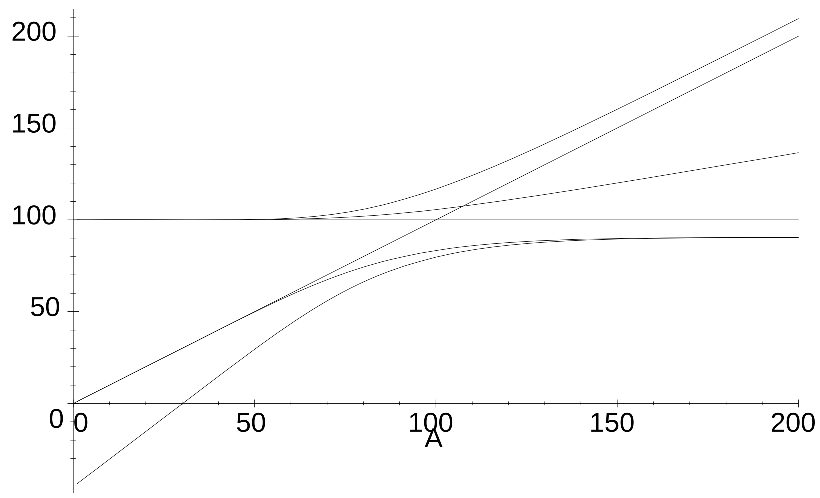
$$r = 0.10$$

$$\sigma = 0.30$$

$$T = 1$$

$$\tau = \frac{1}{3}$$

$$\alpha = \frac{1}{2}$$



Components of firm value as a function of firm value A

6.4.9 Debt value

As can be seen debt values can be negative reflecting the fact that the debt holders are committed to paying the bankruptcy costs in the case of severe firm value falls. However as the firm value A rises, D quickly rises above 0 and then approaches the contacted value as the firm value rises enough to make the likelihood of default small.

6.4.10 Third Party Bankruptcy Costs

These are positive as $A \to 0$ and fall rapidly (they are a Put) as A rises. They are virtually zero if A is above X.

6.4.11 Equity value

Being a call on A in excess of X, E is initially zero and starts to rise at values of A close to X and then asymptotes toward a fraction of $A - X$.

6.4.12 Tax value

The remaining fraction below $A - X$ is of course the tax, tax and equity claim are strictly proportional in the ratio $\tau : 1 - \tau$. All components of firm value if added together sum to $A = E + D + T + TP$.

6.4.13 Optimum choice of contracted debt

Now how does firm value vary as a function of leverage? Well our intuition should be that the firm will try to avoid too many payments to other parties, i.e. that it is costly to both equity and debt holders to have lawyers involved (their claim erodes both the debt and equity values) but that if the firm remains too secure (low gearing) it will be giving too much away to the government in tax.

Now the shareholders and debtholders are free to exchange monies and payments now in return for changing future contracted payments, this means they are able to control X in order to maximise their *joint* claim. Thus they can choose X so as to maximise the value of $D+E$ or alternatively minimise $T+TP$ [19]

$$A = 100$$

$$r = 0.1$$

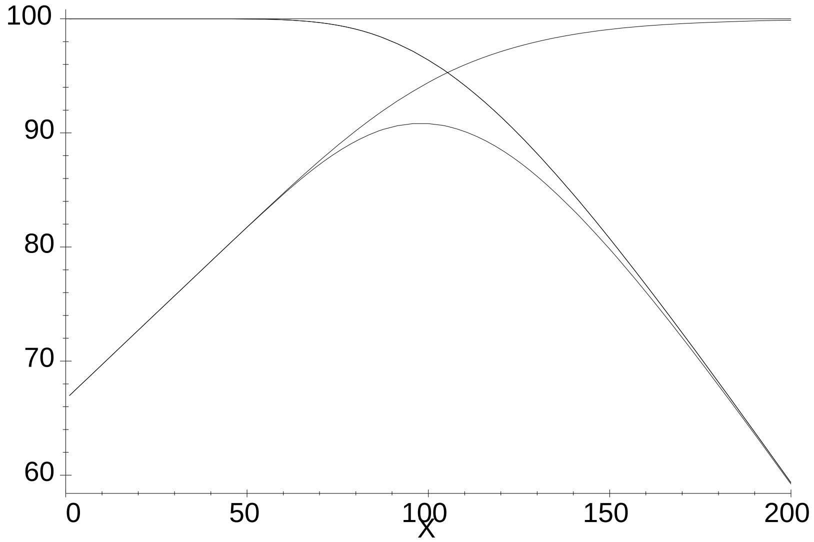
$$\tau = \frac{1}{3}$$

$$\alpha = \frac{1}{2}$$

$$\sigma = 0.3$$

$$T = 1$$

[19]
$$N(d) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{d}{\sqrt{2}}\right)\right)$$



$D+E$ for $A=100$ net of Tax & Third Parties as a function of debt payment X.

This problem admits a solution for the maximum value labelling $V = D+E$ alternatively $V = A - TP - T$, either way X is determined through

$$\boxed{\text{Net Value } V^* = \max_X V = \max_X (D+E)}$$

$$\boxed{\text{Frictions} = \min_X (TP+T)}$$

Now we know the form for D, E from above and also, Black Scholes gives us

$$\begin{aligned} \frac{\partial C}{\partial X} &= -N(d_2)\, e^{-rT} \\ \frac{\partial P}{\partial X} &= N(-d_2)\, e^{-rT} = (1 - N(d_2))\, e^{-rT} \end{aligned}$$

so that the maximisation problem is solved by equating the first differential of V w.r.t. X equal to zero

$$\begin{aligned} 0 &= \frac{\partial V}{\partial X} \\ &= \frac{\partial T}{\partial X} + \frac{\partial TP}{\partial X} \\ &= \tau \frac{\partial C}{\partial X} + \alpha \frac{\partial P}{\partial X} \\ &= (-\tau N(d_2) + \alpha (1 - N(d_2)))\, e^{-rT} \end{aligned}$$

$$\boxed{\text{Optimal risk neutral prob. of survival: } N(d_2) = \tfrac{\alpha}{\alpha+\tau}}$$

$$\boxed{\text{Optimal risk neutral prob. of default: } N(-d_2) = \tfrac{\tau}{\alpha+\tau}}$$

and therefore from the definition of d_2, optimal debt face value X^* can be recovered

$$\begin{aligned} d_2 &= \frac{\ln A - \ln X^* + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = N^{-1}\left(\frac{\alpha}{\alpha+\tau}\right) \\ X^* &= Ae^{\left(r-\frac{1}{2}\sigma^2\right)T - \sigma\sqrt{T}N^{-1}\left(\frac{\alpha}{\alpha+\tau}\right)} \end{aligned}$$

$$\boxed{X^* e^{-rT} = Ae^{-\frac{1}{2}\sigma^2 T - \sigma\sqrt{T}N^{-1}\left(\frac{\alpha}{\alpha+\tau}\right)}}$$

This point corresponds to the optimum in the Figure above. At this debt level the call is worth

$$\begin{aligned} C &= AN(d_1) - Xe^{-rT}N(d_2) \\ C^* &= AN(d_1^*) - \frac{\alpha}{\alpha+\tau}X^* e^{-rT} \\ \frac{C^*}{A} &= N\left(N^{-1}\left(\frac{\alpha}{\alpha+\tau}\right) + \sigma\sqrt{T}\right) - \frac{\alpha}{\alpha+\tau}e^{-\frac{1}{2}\sigma^2 T - \sigma\sqrt{T}N^{-1}\left(\frac{\alpha}{\alpha+\tau}\right)} \end{aligned}$$

where[20]

$$\boxed{d_1^* = N^{-1}\left(\tfrac{\alpha}{\alpha+\tau}\right) + \sigma\sqrt{T}}$$

6.5 Maximised Value

Substituting the optimal face value debt into the value equation call and put components yields

$$V = E + D = (1-\tau)C + Xe^{-rT} - (1+\alpha)P$$

or using calls alone

$$\begin{aligned} V &= (1-\tau)C + A - C - \alpha\left(C - A + Xe^{-rT}\right) \\ &= (1+\alpha)A - (\alpha+\tau)C - \alpha Xe^{-rT} \end{aligned}$$

now substitute for the call value using the optimal level X^*

$$\begin{aligned} V^* &= (1+\alpha)A - (\alpha+\tau)\left(AN(d_1^*) - \frac{\alpha}{\alpha+\tau}X^* e^{-rT}\right) - \alpha X^* e^{-rT} \\ &= A\left(1 + \alpha - (\alpha+\tau)N(d_1^*)\right) \\ &= \boxed{A\left(1 + \alpha - (\alpha+\tau)N\left(N^{-1}\left(\tfrac{\alpha}{\alpha+\tau}\right) + \sigma\sqrt{T}\right)\right)} \end{aligned}$$

[20] $\boxed{N^{-1}(z) \neq \frac{1}{N(z)}, \text{ its the inverse cumulative normal function, NORMSINV() in Excel.}}$

This can be shown to be always less than A and also it can be shown to be decreasing in σ, α and τ. This means that the maximum firm value is decreasing with the total risk of the firm which motivates a role for hedging and is also decreasing in bankruptcy costs and tax.

Taking a first order approximation to the cumulative normal function around one point a

$$\begin{aligned} N(a+b) &\approx N(a) + bn(a) \\ a &= N^{-1}\left(\frac{\alpha}{\alpha+\tau}\right) \\ b &= \sigma\sqrt{T} \end{aligned}$$

yields

$$\begin{aligned} \frac{V^*}{A} &\approx 1+\alpha-(\alpha+\tau)\left(\frac{\alpha}{\alpha+\tau}+\sigma\sqrt{T}n\left(N^{-1}\left(\frac{\alpha}{\alpha+\tau}\right)\right)\right) \\ &= \boxed{1-(\alpha+\tau)\,\sigma\sqrt{T}n\left(N^{-1}\left(\tfrac{\alpha}{\alpha+\tau}\right)\right)} \end{aligned}$$

Thus the relative decrease in value of maximised V wrt A can be seen to be approximately proportional to $(\alpha+\tau)\,\sigma\sqrt{T}$.

6.6 Optimal WACC

From the definition of WACC

$$\boxed{VR_V = ER_E + DR_D}$$

and from the definitions of R_E and R_D

$$\begin{aligned} R_E &= r + \frac{A}{E}N(d_1)(R_A - r) \\ R_D &= r + \frac{A}{D}(1-N(d_1))(R_A - r) \end{aligned}$$

we can derive

$$\begin{aligned} VR_V &= r(D+E) + A(R_A - r) \\ &= AR_A - r(A-D-E) \end{aligned}$$

which when differentiated on both sides wrt X the face value of debt yields, and noting that $V = D + E$ yields,

$$\begin{aligned} R_V\frac{\partial V}{\partial X} + V\frac{\partial R_V}{\partial X} &= \frac{\partial\left[r(D+E)\right]}{\partial X} + \frac{\partial\left[A(R_A - r)\right]}{\partial X} \\ V\frac{\partial R_V}{\partial X} &= (r - R_V)\frac{\partial V}{\partial X} \end{aligned}$$

$$\boxed{(V \neq 0) \qquad \frac{\partial V}{\partial X} = 0 \Rightarrow \frac{\partial R_V}{\partial X} = 0}$$

from which we conclude that WACC minimisation is equivalent to Value maximisation. Temptation in walking close to the edge of a cliff to enjoy the view is tempered by the cost and probability of falling off the cliff!

In practice we do not see companies substantially adjusting their capital structure very often, partly because it is costly to do so (there are equity and debt issuance costs), these in theory could be incorporated into another (more complex) model. However, changes in dividend policy, scrip issues and special dividends are one way that firms can continually adjust their capital structures. N.B. many other structures could be used for bankruptcy costs (see second structure).

6.7 Maximum debt capacity

Note that the value of X that maximises firm value, is not the same as the one that maximises debt value alone. Using the put shape bankruptcy costs

$$\begin{aligned} D &= Xe^{-rT} - (1+\alpha)\,P \\ \frac{\partial D}{\partial X} &= e^{-rT} - (1+\alpha)\,e^{-rT} N(-d_2) = 0 \end{aligned}$$

$$\boxed{\text{Optimal risk neutral prob. of default: } N(-d_2) = \tfrac{1}{1+\alpha}}$$

$$\boxed{d_2 = \frac{\ln A - \ln X^{\max} + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = N^{-1}\left(\tfrac{1}{1+\alpha}\right)}$$

Thus, although not optimal, the maximum debt capacity of the firm is given by

$$\boxed{X^{\max} = Ae^{\left(r-\frac{1}{2}\sigma^2\right)T - \sigma\sqrt{T}N^{-1}\left(\frac{1}{1+\alpha}\right)}}$$

Note that this is not optimal because even introducing a small amount of equity will increase firm value, that is to say the bankers get richer faster than the rate at which equity holders contribute cash or that they could be better off paying the equity holders to increase their firm equity holding.

6.8 Summary

The form of bankruptcy costs chosen here are very stylized and almost certainly applicable in no real firm. All the intuition gained however is transportable to a general setting where real down and upside frictions (bankruptcy costs and taxes) are convex. The advantage of the stylized setting is

	$\alpha = 0$	$\alpha > 0$
$\tau = 0$	MM 1 $D\&E$ are equivalent	all firms 100% E D redundant
$\tau > 0$	MM 2 all firms 100% D E redundant	optimal capital structure $D\&E$ both have a role

Table 15: Roles of Debt and Equity

that comparative statics are available and firm conclusions can be drawn as to relevant sensitivities.

If there is no tax or bankruptcy costs, debt and equity are essentially equivalent, if there is tax alone, debt would drive equity out of the market since it is tax preferred. If tax *and* bankruptcy costs exist there is a role for both debt and equity, since if bankruptcy costs were present without tax, equity would drive debt out of the market (see Table 15)

Thus in some sense, debt financing exists to shelter tax payments and equity financing exists to buffer bankruptcy costs and most firms cannot do without both form of financing. What fraction of both forms of financing they chose will depend on the amounts of tax payments and bankruptcy costs that their assets generate.

In any debt and equity financing situation, think about the costs of under– (tax etc.) and over– (distress) leverage. Think about the relative magnitude and importance of each as a guide to how they should be traded off against each other.

6.9 Pecking order theory of capital structure

If friction exist and there is an optimal capital structure then firms will try and keep close to that optimal capital structure. If however it is costly to return to your optimal capital structure, firms will not continually do so. They will exert the cheapest capital structure control mechanisms first and resort to the most expensive last. Heuristically firms seem to order sources of financing thus:–

1. Retained earnings, i.e. available liquid assets

2. Straight Debt Financing

3. Lease Financing

4. Convertible Debt Financing

5. Preferred Equity Financing
6. Ordinary Equity Financing

Thus firms may chose their dividend policy (as well as their borrowing and equity issuance policy) to optimally control their capital structure.

6.10 New financial instruments

As firms become increasingly aware of the issues involved in firm maximisation, new types of financial instruments will evolve to aid the process such as gold loans for gold producing firms etc. This brings this text onto the topic of Risk Management.

7 Hedging

7.1 Forwards and futures

- Spot market: deal today for immediate (maybe tomorrow) settlement (no credit risk beyond settlement)
- Forward market: deal today for delayed (maybe months ahead) settlement (credit risk until settlement). Zero dividend, stochastic capital gain at expiry.
- Futures: deal today for delayed (maybe months ahead) settlement but make good or receive losses each and every day until settlement (no credit risk since all debts squared at the end of each and every day!). Stochastic dividend, zero capital gain at expiry.

The sum of all cashflows is the same for futures and forward contract with the same terms!

7.2 Bilateral v. clearing house markets

Most spot and forward markets (FX etc.) are self organising and policing so that things are simple but counterparty credit risk is present. Futures markets are run through a clearing house to make sure that losses are made good and therefore counterparty credit is not a risk.

7.3 Exchanges (clearing houses)

- US: Chicago Board of Trade, Chicago Mercantile Exchange, NYMEX, PBOT etc.
- UK: LIFFE, LME, IPE etc.
- Other: MATIF, DTB, TOPIX, SIMEX etc.

7.4 Contract "underlying"

- Stock index futures: S&P500, FTSE100, Nikkei225, CAC, DAX etc.
- Interest rate futures: TBond, TBill, Eurodollar, Gilt, Eurosterling etc.
- Currencies: All major currencies against the U.S.Dollar and cross currencies on local exchanges

- Energy: Oil(s), Petrol, Gas, Coal, other fuels
- Metals: Gold, Silver, Copper, Platinum
- Agricultural: Wheat, Corn, Oats, Soybeans, FCOJ, (Eggs!),Cattle, Hogs, PorkBellies!

7.5 Return to speculators and hedgers

If these (and the other) markets are efficient, everything should be fairly priced (no expected abnormal risk adjusted profit or loss) and the NPV (expected future discounted profit) of any transaction should be zero! Speculators should not make excessive risk adjusted returns after costs or there would be incentives for more speculators to enter. Hedgers will also face zero NPV transactions and will neither expect to win or lose from hedging. Hedging should neither create nor destroy value, it will just transfer risk from one party to another. If there are returns to risk (CAPM), then the transfer of risk will also involve the transfer of expected return but in a risk adjusted sense parties should be indifferent between hedging and not. Hedgers are thus simply shifting their position of the SML! For example, for a future on a commodity, the CAPM gives

$$E^P[R_i] = R_f + \beta_i E^P[R_m - R_f]$$

but the expected return can be expressed through an expected future spot price $E^P[100_{iT}]$ and a current spot price S_{i0}

$$E^P[R_i] = \frac{E^P[S_{iT}] - S_{i0}}{S_{i0}}$$

so that

$$\begin{aligned}
\frac{E^P[S_{iT}] - S_{i0}}{S_{i0}} &= R_f + \beta_i E^P[R_m - R_f] \\
E^P[S_{iT}] &= S_{i0}\left(1 + R_f + \beta_i E^P[R_m - R_f]\right) \\
S_{i0} &= \frac{E^P[S_{iT}]}{1 + R_f + \beta_i E^P[R_m - R_f]}
\end{aligned}$$

i.e. current market prices must be expected future prices discounted at a risky rate of return $R_f + \beta_i E^P[R_m - R_f]$. Alternatively

$$\begin{aligned}
S_{i0}(1 + R_f) &= E^P[S_{iT}] - S_{i0}\beta_i E^P[R_m - R_f] \\
S_{i0} &= \frac{E^P[S_{iT}] - S_{i0}\beta_i E^P[R_m - R_f]}{1 + R_f} = \frac{E^Q[S_{iT}]}{1 + R_f}
\end{aligned}$$

This indicates that under uncertainty, we can formulate spot prices as discounted expected future values using the risk free rate if we adjust the expected value for the systematic risk (subtract $S_{i0}\beta_i E^P[R_m - R_f]$). This approach where uncertainty is included in the NPV through the expected cashflows rather than the discount rate is called the *certainty equivalent* and is important for futures prices because futures are deferred spot purchases

$$\begin{aligned} {}_0F_{iT} &= (1 + R_f) S_{i0} \\ &= E^P[S_{iT}] - S_{i0}\beta_i E^P[R_m - R_f] \end{aligned}$$

Thus futures prices give us the certainty equivalent of expected spot prices and if the commodity in question has low or zero market β then this will be equal to the expected price but if β is non zero, there will be a risk premium $S_{i0}\beta_i E^P[R_m - R_f]$.

7.6 Risk neutral expectations

Thus we can define risk neutral expectations

$$E^P[S_{iT}] - S_{i0}\beta_i E^P[R_m - R_f] = E^Q[S_{iT}] = \quad {}_0F_{iT}$$

and we think of futures as representing the risk neutral expectation of the uncertain amount. (This could also be represented in continuous time). The future price ${}_0F_{iT}$ is also often called the *certainty equivalent* amount.

7.7 Futures on the index

If the commodity in question is the market index itself then $\beta_i = 1$ and assuming dividends on physical holdings over the period of $D_{i0,T}$

$$\begin{aligned} \beta_i S_{i0} E^P[R_m - R_f] &= E^P[S_{iT} + D_{i0,T} - S_{i0}] - R_f S_{i0} \\ {}_0F_{iT} &= E^P[S_{iT}] - E^P[S_{iT} + D_{i0,T} - S_{i0}] + R_f S_{i0} \\ &= \left(1 + R_f - \frac{D_{i0,T}}{S_{i0}}\right) S_{i0} \end{aligned}$$

so that the futures price of the index is a compounded version of the spot, but using a reduced rate due to futures not benefiting from physical dividends.

7.8 Managerial and shareholder attitudes to hedging

Even if we left the CAPM aside and said that firm managers and/or shareholders care about specific risks and hedging is desirable, the shareholders may not wish them to hedge individually since it may be easier (or more cost effective) for them to hedge *in aggregate* (through the formation of portfolios and then hedge only the net). If firms do care about specific risks, this could be a reason to form holding companies and conglomerates.

Consider an importer and exporter who stand to win and lose respectively if the home currency appreciates. Consequently they both face an uncontrollable risk (FX) but their shareholders could be indifferent to this risk if they held shares in both firms! (since one would appreciate if the other fell). Obviously this might not help the two managers of the two firms, one of whom will "do well" and one will "do badly", and if risk averse they may still desire hedging to eliminate this risk but it should be the shareholders who determine overall managerial policy of the firm including hedging. Furthermore since these risks are genuinely uncontrollable, it should be credible for the "losing" manager to say that it was not his fault and not credible for the "winning" manager to claim the credit. It would be better for the shareholder to base managerial compensation on the outcome of risks *over which the managers had some control* (removing uncontrollable FX risk, e.g. *volume* of business controlling for changing foreign price, *timeliness* of customer payment, *domestic cost* control etc.) to potentially prevent managers from trying to manage risk that the shareholders either wish to retain or could manage better themselves.

7.9 Currency risk

Conglomerates and shareholders are not prevented from holding a portfolio of international firms and so may have the ability to diversify away currency risk. Furthermore since currencies may revert in the long run toward a fundamental level (Purchasing Power Parity), there *may be no risk in the long term* (if you can survive that long!).

This course is not specifically about International Financial Management, but this table deals with the Parity conditions between spot & forward FX rates and interest & inflation rates. By way of summary see Table 16

The consequence for finance is that foreign discount rates must be used for foreign flows but that due to the parity laws holding (in expectation) this hedged NPV will be the same as the unhedged NPV! Consequently if a firm makes a bond issues in a foreign currency it need not necessarily hedge it as the investors (particularly if they have international exposure already) may be indifferent between the unhedged and hedged firm. Even if investors are not indifferent, they can hedge the resulting exposures of all the firms in aggregate rather than individually.

Hedging can be through a series of forward foreign exchange contracts or

Forward/spot differential $\frac{{}_0F_1}{{}_0S_0}$	FPT expectations & speculation $\longleftrightarrow$	**Expected change in spot** $\frac{E[{}_1S_1]}{{}_0S_0}$
IRPT covered interest rate parity $\updownarrow$	$\searrow\nwarrow$ RRR real rate of return $\nearrow\swarrow$ UIP uncovered interest rate parity	$\updownarrow$ PPPT market law of one (factor) price
$\frac{1+\$r}{1+£r}$ **Interest rate differential**	$\longleftrightarrow$ IFR Fisher equal expected real returns	$\frac{1+E[\$i_1]}{1+E[£i_1]}$ **Expected inflation differential**

Table 16: Interest rate, currency and inflation parities

via a currency swap (p830), but since the currency swap will be priced in a competitive market, it too (at the margin) will be $NPV = 0$ for both buyer and seller.

$$\begin{aligned}
\text{NPV}\left(\begin{array}{l}\text{foreign discounted,}\\ \text{spot exchanged}\end{array}\right) &= \text{spot rate} \times \frac{£\text{ flows}}{1 + £\text{ rate}} \\
\text{NPV}\left(\begin{array}{l}\text{foreign forward hedged,}\\ \text{domestic discounted}\end{array}\right) &= \frac{£\text{ flows} \times \text{forward rate}}{1 + \$\text{ rate}} \\
\text{NPV}\left(\begin{array}{l}\text{foreign unhedged,}\\ \text{domestic discounted}\end{array}\right) &= \frac{£\text{ flows} \times \begin{array}{c}\text{expected future}\\ \text{spot rate}\end{array}}{1 + \$\text{ rate}}
\end{aligned}$$

All three NPVs are expected to be equal. Even if the currency swap remains off the book value balance sheet, because it affects the cashflows of the firm it will reside on the market value balance sheet and will transform the exposure of the market value of liabilities from one currency to another.

This has strong consequences for international capital budgeting, namely that projections based on proprietary forecasts are highly questionable and subject to arbitrage.. The only “forecasts” that are safe to use are those embedded in market prices, foreign flows can either be foreign discounted and spot exchanged, forward hedged and domestic discounted or unhedged and domestic discounted but to project translation at any other rate is dangerous.

7.10 Transaction costs

In practice for hedgers the hedge may be $NPV < 0$ due to transactions costs (bid/ask spread commission etc. this may be how market makers earn their money) so that some extra benefit must be earned from hedging for it to be $NPV \geq 0$.

7.11 Interest rate hedges

Firms and investors might also care about variable rates of interest on long term floating rate debt or on short term loan that is rolled over. How does speculation and hedging impact market expectations here?

Remember that fixed coupon bond prices should equal the sum of the discounted coupons, so that the sum of the purchase price and the discounted elements should be zero

$$0 = -P_{fixed} + \sum_{t=1}^{T} \frac{E\left[C_t\right]}{\left(1+r_d\right)^t} + \frac{E\left[100.1_{ND}\right]}{\left(1+r_d\right)^T}$$

There will be some coupon that will make the bond worth exactly 100 (what is it?). Similarly for a floating rate bond that pays LIBOR[21] every six months

$$0 = -P_{float} + \sum_{t=1}^{T} \frac{E\left[LIBOR_t\right]}{\left(1+r_d\right)^t} + \frac{E\left[100.1_{ND}\right]}{\left(1+r_d\right)^T}$$

Again this bond could be worth exactly 100 (if the next LIBOR setting had just been made). An interest rate swap simply transforms the first series of cash flows into the second (or vice versa) for example for 4 years the cashflows on a swap of 10% fixed (s.a.) v. LIBOR would be

$t = 0.5$ years	-5%	$+LIBOR_{0.0-0.5}$
$t = 1$ year	-5%	$+LIBOR_{0.5-1.0}$
$t = 1.5$ years	-5%	$+LIBOR_{1.0-1.5}$
$t = 2.0$ years	-5%	$+LIBOR_{1.5-2.0}$
$t = 2.5$ years	-5%	$+LIBOR_{2.0-2.5}$
$t = 3.0$ years	-5%	$+LIBOR_{2.5-3.0}$
$t = 3.5$ years	-5%	$+LIBOR_{3.0-3.5}$
$t = 4.0$ years	-5%	$+LIBOR_{3.5-4.0}$

The swap needs no upfront payment to arrange and indeed as constructed is $NPV = 0$ since for both the bonds at 100 we can ascertain

$$\sum_{t=1}^{T} \frac{E\left[LIBOR_t\right]}{\left(1+r_d\right)^t} = \sum_{t=1}^{T} \frac{R\left[C_t\right]}{\left(1+r_d\right)^t}$$

[21] London InterBank Offered Rate is a market rate of interest determined between the major banks in the Euromarket for deposits (from one of many currencies) for a fixed but short term e.g. 6 months.

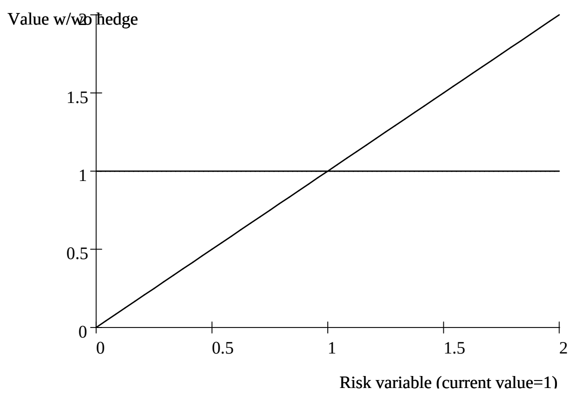
Thus (for each maturity T) swap rates imply expected values for LIBOR (until time T), whether or not the firm or the shareholder fixes or floats the interest expense or not will make no difference to firm value (NPV) since swaps are $NPV = 0$, it will however change the risk and therefore the position on the SML of the firm/investors. Remember, if the expectations implied in the swaps formula did not hold, arbitrage opportunities would be available in swaps trading or alternatively, the market for fixed and floating rate debt would not be in equilibrium. These fixed/floating rate hedges could also have been accomplished with interest rate futures.

Whether or not these hedge instruments are put on the book value balance sheet or not, because their cashflows will be felt they are on the market value balance sheet anyway. Thus taking out an interest rate swap (or buying interest rate futures) will simply adjust the fixed floating debt structure of the firm.

7.12 Commodity exposure

Inputs and outputs, e.g. oil, gold. Why invest in a gold mine rather than in some gold itself? Gold futures, gold swaps and finally other denomination debt e.g. gold loans.

7.13 Conclusion on hedging



Hedge/No Hedge in an MM World.

- In an MM world there is no benefit to hedging. Due to arbitrage, hedging is at best a zero NPV activity. All we can do is change our sensitivity (slope) to the market return, not our current wealth level.
- Furthermore hedging simply moves your location on the SML

- However if there are frictions (costly bankruptcy, tax etc.) hedging can add value because it can reduce these expected frictions
- Just as these frictions create an optimal capital structure they also admit an optimal hedge portfolio.
- These frictions are borne at the firm level so the investor cannot replicate this hedging and firm hedging can create firm value.
- However, if firm frictions are present, hedging may add value by reducing the variability of the future capital structure and therefore the PV frictions.

Situation/Project	Principals	Agents
A Joint Stock Company	Shareholders	Managers
A Mutual Firm[22]	Members	Managers
A Partnership	Partners	
A Building Project	Occupier	Builder
A Farm	Farm Owner	Farm Tenant
A Married Couple/Household	Husband & Wife	
A Football team	Fans	Manager & Players
The Nation	The People	The Government
The Environment	Everybody	Everybody
This Course	AccFin Dept.	Lecturer
An AccFin Degree	University	AccFin Dept.
A University Degree	Students??	University

Table 17: Examples of Principal Agent Relationships

8 Agency theory

8.1 The separation of ownership and control

Agency theory is the branch of economics that deals generally with the relationship between the owner or *principal* and his manager or *agent* although it can also apply to non–owner situations as well (such as government). The problem occurs when the owner or owners are not able to manage the project themselves and must employ managers or agents on their behalf to do the job. The term project must be taken in a broader sense when looking at non–company situations. Examples of principal agent problems include those in Table 17

This topic will be treated in three sections, Incentives, Information & Control.

8.2 Incentives

8.2.1 Diversity of interests

Firms have many stakeholders, they all have different incentives to work toward a collective and individual goal.

- Investors: Equity & Debt Holders (aims are not always aligned)
- Labour force: Workers & Managers
- Trade counterparties: Customers & Suppliers

- Other companies: Competitors & Allies
- Others: Government Tax & Lawyers Legal Fees
- Everyone on Earth: Environmental Interest

For example, firms as a whole have an incentive to consume environmental resources at the expense of everyone on Earth. Governments have incentives to make promises they might not keep. Workers have incentives to shirk[23] and to covertly apply effort elsewhere or steal company resources (see Dilbert on office supplies), even managers have incentives to consume company resources as perquisites ("perks" such as company cars, jets etc.) if they own less than 100% of the firm (See Jensen & Meckling (1976) [26]).

Recognising that each actor is subject to differing incentives in different situations and at different times is an important step in explaining the behaviour of individuals within and outside a firm environment as well as individually and collectively. Most of the time shareholders employ managers to represent their interests but it would be foolish of shareholders not to recognise that managers like themselves are only human and being subject to temptation will inevitably engage in some selfish behaviour. To expect anything else is foolish, and so this form of selfishness must be anticipated in the principal agent relationship beforehand.

8.2.2 Contracting

In essence Agency Theory describes the economics of this relationship and how the principal and agent can contract (agree beforehand) to behave under a range of circumstances or eventualities. The conclusion is that because parties may have an incentive to cheat on any agreement, measures must be put in place beforehand that prevent or limit the incentives of one party to cheat on the other. The result of these measures is that the very best outcome, where everyone works hard and no–body cheats despite the lack of any penalty measures, is not achievable! Penalties and ex–ante losses are necessary beforehand if everyone is to behave properly. Thus valuable time and effort must be spent on the design, monitoring and enforcement of contracts that, if all goes as planned and contract conditions are met, will be redundant! That is not however to say that they could be done away with, the consequence of non implementation or enforcement of a contract will always be a change in incentives that may or may not lead to a change in behaviour.

[23] not work as hard as their wage would require

Thus the so called "first best" outcome, where the agents always do what the principal wants and where no contracting or monitoring costs are incurred is almost always unachievable and the resultant "second best" outcome, where the desired result is achieved with the attendant contracting costs, is always more costly or has a lower result than that which is theoretically possible (but practically impossible)! If the agent cooperates and fulfills the contract without penalty then after the event it is always tempting to say that the contract was not needed but this is of course not true. Contracts are a necessary evil of principal agent relationships and if one party has to resort to the penalty clauses in a contract then it could be considered that the contract has failed, in that it has not achieved the desire outcome of compliance without needing to resort to the contract details.

8.2.3 Agency costs

This loss of value between the first and second best (or other yet lower) outcomes is often referred to as the Agency Cost of the relationship. Although explicit costs are often involved the agency cost also involves the potentially shared lower outcomes that agency generates. Although difficult to measure in $ terms (very often an unobserved opportunity cost is present e.g. a managers reservation or alternative wage), these Agency costs no less real than other costs such as transaction costs and are (some of) the main drivers of incentives and behaviour and can explain a wide range of Corporate Finance activity.

Examples of agency costs include the result that unchecked managers will chose projects that they prefer over the projects with the highest NPV. *Costly* specific counter–incentives are required to motivate the manager to undertake the best (from the shareholders perspective) project and prevent the manager from taking the project that has the earliest cashflow or the least risk, both of which might be of benefit to the manager in his wage review.

8.2.4 The free cash flow problem (cost of underleverage)

Another example of agency cost is the cost of free cash flow described by Jensen (1986) [25]. Here in order to avoid the temptation to spend the cashflow that a company generates, managers have to borrow money and pledge the cash flow of the firm to banker in order to pre–commit not to waste company resources. Thus corporate debt can prove useful in avoiding some of the principal agent costs of delegated management.

8.2.5 Asset substitution and bond covenants

Since an option has been transferred from bondholder to stockholder and we know option prices are sensitive to several pricing factors, how do debt and equity prices react to the same factors. Call options become more valuable as interest rates rise so unanticipated rises in interest rates will cause equity to rise and debt to fall in price (ceterus paribus) but other effects may well swamp this. Call options become more valuable and so bondholders will not want to extend the maturity of the bond (and increase the value of the stockholders call at their expense) without gaining some reward. Also the bondholders will want to prevent the shareholders asset stripping - taking all the money and running which like payment of super dividends would decrease the stock price and rob them of value.

Finally control over σ the volatility of firm value is key, once debt is pledged and assets are in place, the shareholders always have an incentive to increase the riskiness of the firm since they will then enjoy a higher call value on assets. Because bondholders are short and stockholders long an option, a fundamental asymmetry exists that the latter could use to exploit the former. How do the bondholders protect themselves? In the terms of the loan they specify many things, amongst others what fraction of earnings can be paid out as dividends, what rank debt the firm can issue above and below the existing debt and what the purpose if the firm and use of funds is to be. These all give the bondholders some legal protection against non-performance of stockholders on their promises! Even given this protection, these so called *Agency Problems* exist and are of interest to financiers who study delegated management.

8.2.6 The underinvestment problem (cost of overleverage)

For a firm that is in default on its debt, the equity holders have little incentive to recapitalise the firm since the majority of all the early funds they put into the firm will benefit the debtholders and not themselves. This is known as the debt overhang problem and is another agency cost of financial distress, in that it leads to bad an perverse decision making at times of financial distress (see Section on Agency). Myers (1977) [47] and Myers & Majluf (1984) [49] describe the overhang problem.

8.2.7 The overinvestment problem (cost of overleverage)

Indeed equity holders might even want to chose projects with negative NPV if they increased the riskiness of the firm, this is because under certain conditions they have limited downside and will not have to bear large losses but might gain from large gains.

Asset substitution, under and over investment are all agency costs of the *debt equity separation*, i.e. the fact that the equity holders may control some of the wealth attributable to the debtholders.

For a much more complete description of the interaction between valuation and agency theory, see Froot, Scharfstein and Stein (1993) [19].

8.2.8 Some drivers of Agency Costs

Some of the variables that might determine the degree of conflict between the *shareholder and the manager* are:–

- Concentration of shareholders
- Holding by the manager
- The Managers alternative wage
- Amount of debt issued by the firm (although this then generates another agency problem)

and between the shareholders and the debtholders

- How specific the asset are
- The degree of asset volatility
- The firm's hedging policy

8.2.9 The free rider problem

Why can shareholders not perfectly control their managers? As a group, their ownership is generally too diffuse to easily force managers to follow any set of strict instructions they determine, specific company resolutions receive votes at the Company's Annual General Meeting (AGM) where fractional ownership of the firm allows shareholders differential voting power in what are called "proxy[24] battles". Management have the upper hand in this game, they often determine the resolutions themselves and the order in which they are tabled and so in practice it can prove very difficult for a minority of shareholders to get their resolution before the AGM for a vote. Moreover, if a group of shareholders do manage to push through a resolution that is

[24]Voting while removed from the AGM, or allocating your vote to another, is known as voting by proxy.

beneficial to all shareholders (say at the expense of managers) the fact that the initiating shareholders will gain only a fraction of the benefit while incurring most of the costs will lead to a free rider problem where each small shareholder has little incentive to exert effort to improve company prospects because the marginal benefit to him may not adequately compensate for his personal cost.

This is also the stark reality of environmental issues, it is obvious that we all have a latent interest in the world's environment but we all ignore this and acting selfishly when it suits us.

8.2.10 Executive payment schemes

Just as debt can be used to solve agency problems, other corporate finance features have principal agent explanations. Executive stock options help align the interests of the manager with those of the shareholders without the need for him to own a high fraction of the firm. The more the option is set out of the money, the greater the incentive for him to work to improve its value since an out of the money option represents a higher exposure to underlying firm value. Note that stock options have a cost to the firm which is equivalent to the initial value of the stock option, how firms should account for this cost is a good question, is it compensation for past of future effort? Valuing these options is also tricky because they are not tradeable and often have call features and other provisions.

Note that firms also try and align their workers incentives with the shareholders by issuing Employee Stock Ownership Plans (ESOPs) which are like a collective stock option plan.

8.2.11 Summary

In summary, firms are continually struggling to minimise all their costs including their contracting costs not just their explicit costs. The corporate forms that we observe are the result of the struggle and currently the joint stock corporation with public or private debt and managers who often have stock options seems to be the most efficient vehicle. Whether or not this remains the case or whether a yet more efficient form emerges remains to be seen, the introduction of the internet poses one of the most severe challenges to corporate form yet encountered.

8.3 Information

Information plays a special role in Agency because what can and cannot be included in a contract is dependent on the extent to which information can be verified (i.e. upheld in a court of law). Furthermore different parties can have different amounts of information which can lead to different incentives. Finally, information can play a role through *signalling*, where companies can increase the credibility of their communications by associating them with financial transactions ("Actions speak louder than words").

8.3.1 Inside information

Managers typically have more information about the affairs of the company than the shareholders, it would clearly be inefficient for the shareholders to be involved in each and every of the day to day decisions that the managers must face, indeed this would not be delegated management at all it would be micro–management! What consequences does this information asymmetry have? It will create conflicts between the competing interests of the managers. The managers wealth is based upon many aspects:

- His salary
- His stock holdings
- His other compensation (stock options)
- His reputation (will affect future salaries)
- The information itself is valuable, it could be exploited as insider information

Suppose the manager was in possession of inside information that the firm had found a great $NPV > 0$ project that the market knew nothing of. He could trade using the inside information and make money but probably lose his reputation and maybe his job, he could announce the information hoping that the market would believe him, he could offer to take a pay cut in return for more share options or he could get the firm to take some action (such as purchasing its own shares) that could only be interpreted as being positive if the firm had good news. The shareholders hope that they have got the balance of his compensation package just about right so that most of the time, his interest is aligned with theirs and he acts to maximise firm value.

This raises the secondary issue of which actions would be believable. If would be highly credible if the CEO of a firm were to deal in the shares of his own firm in the open market. It is recognised that managers should not be forbidden to trade their own firms' stock (they may have liquidity needs of their own unrelated to the fortunes of the firm) so dealing is allowed, it is however monitored closely and usually kept within a pre–specified time window when other company announcements are unlikely. Other dealings by company managers are monitored as credible signals of company fortunes. By staking company money on say a share repurchase, it is seen as a credible *signal* because the manager stands to harm his reputation if it were to fail.

8.3.2 Exploiting inside information

Although insiders may have profitable information it may be difficult for them to exploit this fully:–

- Attempting to trade in large volume may arouse suspicion (say with the market specialist)
- Supply/demand elasticity means that increased insider volume will force up the price in order to persuade the marginal investor to sell
- At some point the profit will be zero indicating that the insider has a tricky profit/volume trade-off to make

Should insider trading be banned or permitted? If banned:-

- Prices can never be strong form efficient
- Some trades will go through at a price which might subsequently be deemed unfair (when info known) although it will be difficult to say who the losers are
- Regulators have to spend public money trying to enforce the ban
- Information may be a private issue between the firm and its employees

If permitted:-

- Price might be more efficient
- Counterparties subsequently known to have traded with insiders may feel aggrieved

- People may trust the markets less and the bid ask spread may be larger
- Firms may have problems contracting with their employees not to release information

Think of our penny game.

8.3.3 Dividend policy and information signalling

Remember that investors who do not agree with the firm's policy can undo any dividend policy through the buying and selling of their own shares, thus there is no *pure valuation* difference in a high or low dividend policy. However as well as share repurchases, firms can pay special dividends as a very credible way to *signal information* that they believe is not in the market. By making a special dividend payment or increasing the current dividend, managers seek to reinforce their message that their stewardship of the firm is bearing fruit.

Dividend policy is also seen as an important signal because of the reputations of the senior managers of the firm. Managers dislike having to be the bringer of bad news and being the one to have to cut dividends so it is seen to be very bad news if a dividend cut occurs. It is because cutting dividends is such bad news that increasing them in the first place is seen as such a credible signal of success..

8.3.4 Adverse selection

Akerlof (1970) [1] was the first to document adverse selection in a market setting. In the car market a car that is poor is known in America as a *lemon*, he asked the question, why is it that cars seemingly lose such a high proportion of their value in their first year of use? The answer lay not in the fact that all cars lost value initially, but that owners who happened to have got a faulty one (in the days when quality control was much worse) would try to sell it after discovering this fact while non–faulty cars would be treasured for a longer period of time and thus the only cars in the nearly new market would be "lemons". This he argued, would lead to the collapse of the car market because increasingly worse cars would always drive out the better until no–one traded anymore. Apart from the signalling implication that selling your car after a year meant it was bad (how could the negative price impact be avoided if you genuinely needed to sell and your car was not a lemon?), what he described was adverse selection, a situation where information asymmetry (who knows if the car is good or bad) means that only poorer quality goods are traded. In practice, car firms realised that it was in their interests (people would pay more) to improve quality control and thus reduce the number of lemons and the resultant negative first year price impact.

Adverse selection happens in other markets such as that for Labour too, if a job applicant is seen to be out currently of employment, it is assumed that he was pushed from his position rather than an applicant who is jumping from one firm to another. The quality of labour on offer in a labour exchange is thus unfairly viewed.

Adverse selection can operate in the market for shares, consider Microsoft a firm set up by the entrepreneur Bill Gates in the 1980's. Bill Gates still owns a large fraction of the firm (~20%). Suppose there were a news announcement that he was to offer his shares for sale to any willing buyers[25], what would the consequences be? In buying from Bill Gates there are two possibilities (see also Inside Information), i) that you have encountered a CEO who thinks the shares are set to fall in value or ii) this CEO has his own private reasons for wanting to sell. Because any attempt to portray himself as in the second category may not be fully credible, you infer that he must be in the first and that the stock price is set to fall. Without considering his offer for sale any further, you immediately decide to sell the stock short yourself and the price starts to plummet as others come to the same conclusion! This is another example of adverse selection.

8.3.5 Event studies

Clearly as information is revealed, the share price of a firm will react, the study of these reactions to infer the significance of a particular piece is called an *event study*. Many events and many pieces of information and the associated price changes are required to form a statistical test with any power, but aggregating across many firms and many different years allows inference of the average reaction to various pieces of common news such as debt issuance, share repurchases, replacement of CEO, etc.

8.3.6 Summary

Information is continually being generated within the firm and this has economic value. One of the managers jobs is to capture this value for shareholders rather than see it go to other (outsiders). The economic impact of the information is linked to the means of communication and the financial strategy.

[25] This situation and the similar decision at an initial floatation or IPO are described in Leyland & Pyle (1977) [30].

8.4 Control: Mergers & Acquisitions

This next Agency topic is so widely discusses within Corporate Finance, it is very often given its own Chapter, so ..

9 Mergers & Acquisitions

9.1 Control: Mergers & Acquisitions

9.1.1 Mechanisms

Whether or not the deal is called a merger or an acquisition is a fine point of difference, indeed sometimes a smaller firm is encouraged by a larger firm to reverse take-over the larger firm, ultimately it comes down to how control of the joint enterprise is determined.

In a merger, shareholders pool their equity by giving up their shares in return for a fraction of the combined firm and the negotiation over ownership revolves around the fraction of the joint that each firm claims (normally around their market value). In a takeover however, one firms equity will continues to exist and the others will disappear. This can be accomplished either by the purchase of all the target firms equity for cash (cash offer), where the sellers will have no involvement in the joint company or by the issues of new equity (paper offer) and the exchange of shares in the target company for equity in the now enlarged firm.

Under mergers both firms must decide what the joint firm is worth and what their share is, in a cash takeover offer the tendering firm must decide what the target's cash value is, while in a paper takeover offer the tendering firm must still decide what the target is worth but the value of the tendering firm is also relevant because its share are being offered as payment. Thus the last situation is more alike a merger than the cash offer, (firms can also tender part cash and part paper!).

Before gains and losses from mergers and acquisitions are discussed, we should assume that they should neither create nor destroy value

$$NPV(c) = NPV(a+b) = NPV(a) + NPV(b)$$

since net present valuation techniques are distributive over separation of cash flows, for perpetual flows a, b and their sum c and their values A, B, C

$$\begin{aligned} a + b &= c \\ R_a A + R_b B &= R_c C \\ A + B &= C \end{aligned}$$

Firm	**Debt**	**Equity**	**Assets**	**% A**	**% B**
Initial firm A	ADebt	AEquity	AAssets	100	0
Initial firm B	BDebt	BEquity	BAssets	0	100
A and B merge → C	AD+BD	AE+BE=CE	CA	$\frac{AE}{CE}$	$\frac{BE}{CE}$
A cash acquires B	AD+BD+BE	AE	CA	100	0
A paper acquires B → C	AD+BD	AE(1+$\frac{BE}{AE}$)	CA	$\frac{AE}{CE}$	$\frac{BE}{CE}$

Table 18: Three forms of mergers and acquisition

so long as the joint discounting cost R_c is determined by a WACC

$$R_a\frac{A}{C} + R_b\frac{B}{C} = R_c$$

The three possibilities of merger, acquisition by cash and acquisition by paper can be summarized by Table 18.

Under the merger and the paper offer, both firms end holding dome fraction of the joint company however under the former both forms of shares are converted to Company C shares while under the latter Company B shares cease to exist having been replaced with a now enlarged base of A shares. Under the cash offer Company A must find Cash to a value of BE (the equity value of B), either by borrowing this amount or by reducing cash. Thus here the B equity holders take the cash and retain no interest in C.

Under a cash offer B shareholders must judge the cash offer vis a vis the current market price and the likelihood of other or increased bids. Under a merger and paper acquisition offers they must judge their current market price compared to the value and fraction of the joint firm they will partly inherit. Part cash and part paper offers must be judged as a combination of the two.

9.1.2 Exchange offers

Exchange offers are where one firm bids for another by offering equity rather than cash. The Margrabe (1978) [36] formula *may* be useful for evaluating exchange offers but the correlation of acquiring and acquirer stock may change pre and post announcement[26] and other dilution and endogeneity issues may also need careful treatment.

Before the merger or announcement, consider two firms of market values (no debt) E_1, E_2 and share prices $S_1 = E_1/n_1$, $S_2 = E_2/n_2$. Suppose firm

[26] due to changes in managerial policy and or competitive issues

1 offers the firm 2 shareholders n_1' new shares in return for their n_2 shares. What sort of option exchange does this represent? The joint firms have value E_1+E_2 but if joined will incur dilution to n_1+n_1' total new shares in the combined firm (n_2 shares in firm two are retired). Thus firm 2 shareholders have to decide if they should give up n_2S_2 in return for $(E_1+E_2)\,n_1'/\,(n_1+n_1')$. Thus the exchange option has a payoff condition which depends on

$$\begin{aligned} n_2S_2 &\lesseqgtr (n_1S_1+n_2S_2)\,n_1'/\,(n_1+n_1') \\ (n_1+n_1')\,n_2S_2 &\lesseqgtr (n_1S_1+n_2S_2)\,n_1' \\ n_2S_2 &\lesseqgtr n_1'S_1 \end{aligned}$$

On exchange offer expiry, they will accept if $n_1'S_1 > n_2S_2$ and reject if not. If the number of new shares is equal to the number of shares to be rescinded (retired) $n_2 = n'_1$, the exercise condition will equate to matching current share prices $S_2 \lesseqgtr S_1$. For a ratio of new to old that is not one, the effective exercise ratio of share prices will not be one either.

We can evaluate the option to exchange S_2 for S_1n_1'/n_2 if we know the relative volatility of the two (as if they were combined) and if we know the dilution, it is the dilution factor multiplied by a Margrabe exchange option with no dilution. A no dilution (or naked, i.e. stand alone) Margrabe call on $X_1 = \frac{n_1'}{n_2}S_1$ with a strike of $X_2 = S_2$ is given by

$$\begin{aligned} \frac{dX_{1,2}}{X_{1,2}} &= \mu_{1,2}dt + v_{1,2}dZ_{1,2} \qquad dZ_1dZ_2 = \rho dt \\ \text{Margrabe} &= X_1N(d_1) - X_2N(d_2) \\ d_1, d_2 &= \frac{\ln X_1 - \ln X_2 \pm 0.5v^2T}{v\sqrt{T}} \qquad v^2 = v_1^2 - 2\rho v_1v_2 + v_2^2 > 0 \end{aligned}$$

where $d_{1,2}$ and ν are defined.

However a naked Margrabe exchange option (sold by an investment bank and not part of the merger deal i.e. not issued as a merger offer) generates no dilution whereas the merger exchange offer does generate dilution. The payoff to the exchange offer is a diluted fraction of the combined firm

$$\begin{aligned} \text{Exchange offer} &= \max\left(\frac{n_1'}{n_1+n_1'}\left(S_1\frac{n_1}{n_2}+S_2\right) - S_2, 0\right) \\ &= \frac{n_1}{n_1+n_1'}\max\left(\frac{n_1'}{n_2}S_1 - S_2, 0\right) \\ &= \frac{1}{1+q}\text{Margrabe} \qquad q = \frac{n_1'}{n_1} \end{aligned}$$

9.1.3 Exchange offer example

Suppose we have the following information on two firms who merge (where value is preserved–gains on merging are discussed at the end).

		n	S	E
pre	Firm 1	$n_1 = 50$	$S_1 = \$100$	$\$5,000$
pre	Firm 2	$n_2 = 50$	$S_2 = \$100$	$\$5,000$
new	offer	$n_1' = 50$	$S_1 = \$100$	
post	Merged	$n_1' + n_1 = 100$	$S_1 = \frac{\$10,000}{100} = \$100$	$\$10,000$

The exchange offer is worth $\$100 \times 50$ new shares to the firm 2 holders, but they must give up 50 old shares at $100, thus the value per old share is still $100. If they have say a **three month** ($T = 0.25$) option time in which to exercise, then we can use the Margrabe exchange option to value the option to give up a share worth $100 for one worth $100 under future uncertainty. The dilution factor $q = \frac{n_1'}{n_1} = 1$ so the exchange option is worth $\frac{1}{2}$ of a naked Margrabe.

First evaluate the naked Margrabe option, if both firms have the same volatility $\nu_{1,2} = 25\%$ and the correlation of returns once the exchange offer is announced is say 75% then the Margrabe volatility ν is 17.68%[27]. Since the offer is a one for one, the ratio of $\frac{n_1'}{n_2}$ is one and $X_1 = S_1$ and $X_2 = S_2$. Note that the risk free rate r is not required because we are in effect taking the PV of one asset using another asset as the numeraire (or PV factor).

$$
\begin{aligned}
(X_1, X_2, \nu, T) &= (100, 100, 0.1768, 0.25) \\
d_{1,2} &= \pm 0.0442 \\
N(d_{1,2}) &= 0.518, 0.482 \\
\text{Margrabe (call on } X_1 \text{ for } X_2\text{) } M &= \$3.525
\end{aligned}
$$

In fact the exchange offer will involve dilution, if it goes ahead (but no dilution if it does not, i.e. conditional dilution!), the full dilution factor is $\frac{1}{1+q} = \frac{1}{2}$ so the diluting Margrabe is worth a half of the non–diluting one.

$$
\text{Diluting Margrabe (call on } X_1 \text{ for } X_2\text{) } \frac{1}{1+q} M = \$3.525 \frac{1}{1+q} = \$1.763
$$

So we can identify an extra $1.763 units of value that are transferred from the acquiring firm to the target firm. This value is happily provided by the buying firm, presumably because the value of the merged firm to them exceeds this loss.

---

[27] $25\% * (1 - 2 * .75 + 1)^{\frac{1}{2}}$

There are some extra caveats that have to be added when moving from a market traded (investment bank provided) Margrabe exchange option to a merger exchange offer. Firstly the correlation between the two firms will change, this is because their future histories will be linked if the merger goes ahead (increased correlation) but conversely the merger negotiation may still redistribute wealth between the two firms (lower correlation due to fighting over the merger gains). In practice it may be difficult to say if the correlation coefficient will go up or down, but it may not stay the same as the pre–merger announcement figure.

Secondly, the option value transferred from offerer to offeree must also be included in the option valuation model itself! This is to say that the addition of \$1.763 to the \$100 share price for the target firm is too simplistic a representation. In fact the diluted exchange offer must satisfy some sort of implicit equation[28] where the exchange option price augments the target and diminishes the offering firm values.

In practice, the holders of $X_1' = S_1'$ may not lose (M) at all because of perceived gains to merging, i.e. post news announcement the sum of equity values may exceed pre–existing values

$$X_1' + X_2' > X_1 + X_2$$

9.1.4 Gains and losses to M&A; good reasons to merge

Having looked at the market and (book valuation) of M&A activity that is value neutral, we can now examine possible sources of gains to M&A activity. We shall leave the issue of who gets the gains aside for the moment.

$$NPV(c) = NPV(a + b) > NPV(a) + NPV(b)$$

Efficiency If the management of A is more efficient than B then the two run by A's management will be worth more than the parts and there will be firm gains to merging (and also gains to society in the sense that overall resources will be better employed). Synergies (what is meant by this?) and economies of scale and scope are often cited in this category as reasons for merging but in some way the *cash flows must improve* if value is to be created.

Information theories The negotiation or tendering strategy may involve the dissemination of new information or lead the market to believe that the bidders have superior information. The market may then revalue previously undervalued shares as a result of M&A activity.

[28]Note that strictly $X_{1,2} \mp M$ will not follow the same process (with associated parameters) as $X_{1,2}$ so that there is an approximation present.

Agency problems An agency problem arises when the managers of the firm only own a fraction of the firm when they might be tempted to work less vigourously and consume more resource than they would if they bore the whole cost (they will only bear some fraction). Firms with managers prone to such value destroying activity will be subject to higher likelihood of takeover because there will be gains to replacing them with harder working, more honest or more committed (owning a higher fraction) managers. In practice firms put incentive schemes in place to obviate some of the agency problems.

Market power Monopolistic and oligopolistic firms enjoy higher profits than firms in competitive environments because price is no longer driven down to long run marginal cost but "rents" exist (excess profits due to quantity restriction for example). Firms have incentives to form cartels and oligopolistic structures even without merging which is why there are commissions & regulatory bodies[29] to oversee and prevent abuse of market power. If rival firms are allowed to merge they will of course be able to pursue a joint strategy and will potentially gain from the increased market power they enjoy. This is a strong motivation for merging and unchecked will create gains for the their shareholders at the expense of the customers (who will pay more for the product).

Tax (& costs of financial distress) If there is an internal optimum to a firms capital structure, but firms face transactions costs in returning to the optimum, then it may be cheaper for firms to return to their optimum leverage by acquiring or merging with another firm. For example, consider one firm currently very profitable and a long way from financial distress but paying a lot of tax on its earnings and equity distributions and another in loss and financial distress and paying no tax as a result (it may even have accumulated tax losses - an asset that it cannot realise!).

It would pay one firm to acquire the other (either way although the firm in loss is less able to raise funds or issue equity to effect a takeover) or for the firms to merge, since the mutual gains are twofold. The firm in default can offer tax losses to shelter profits elsewhere while the firm in profit can offer equity and cashflow to relieve the financial distress of the other. The joint firm will have higher cashflows than the parts because both payments to third parties (lawyers etc.) and taxes (govt.) will reduce. These gains can form a powerful incentive to merge or acquire. How the gains are distributed is a function of the negotiation process and is not easy to predict, it could

[29] the Mergers & Monopolies Commission is one

be that the bankrupt firms shareholders get the majority of the gains or the profitable firm.

9.1.5 Neutral/Bad reasons to merge

Diversification Diversification in itself is not a reasonable motivation for merging since shareholders can do this themselves and although it does reduce specific risk, this is a risk which is not priced anyway so shareholders will not reduce their expected return as a result. The reduction in specific risk will alter the debt and equity values but not firm value if the firm is optimally leveraged. If it is not optimally leveraged, presumably there are other costs preventing them from being at the optimum which if avoided by merger acquisition could be valuable. However the traditional conglomerate theory of the 70's and 80's should not contribute to firm value through a reduction in specific risk itself.

Hubris (meaning overexuberance or excess pride). Firms want to demonstrate that they are busy trying to create shareholder value. If the best way to do this was actually to sit tight and wait some time for the opportunity to come along, the firm would have problems distinguishing itself from a lazy firm and so all firms tend to generate some level of activity irrespective of fundamental need.

It is not just firms that are subject to "Hubris" but Investment Banks, Lawyers and other advisers, whose revenues rely on doing deals have incentives to recommend M&A activity *irrespective of its benefit.* There is a natural tendency toward action over inaction, "Don't just sit there, do something!"; beware of being talked into questionable deals.

Costs Finally, M&A activity must have some gross benefit if it is to cover the costs and return some net gain above zero. Explicit costs include bankers fees, management time etc.

9.2 Determinants of firm size and scope

Ronald Coase (Nobel prize for Economics 1991) in his 1937 paper [11] discusses what affects firm size and scope.

9.2.1 Sharing and redistributing the gains and losses

It is difficult to predict which firm will grab the largest share of the perceived gains to M&A activity since this ultimately will depend on negotiating power and skill. However because M&A activity changes the specific risk of both firms once they are united, it also redistributes wealth between the stock and bondholders of the firms because of the effect the merger has on the default options embedded in each debt value pre merger. If the separate and joint capital structures are

$$\begin{aligned} V_a &= D_a + E_a \\ V_b &= D_b + E_b \\ V_c &= D'_a + D'_b + E_c \end{aligned}$$

where the debts have face values X_a, X_b Galai & Masulis and Shastri formulate the consequences of the specific risks of two firms and their joint risk

$$\begin{aligned} \sigma_c^2 &= w_a^2\sigma_a^2 + 2w_a w_b \rho_{ab}\sigma_a\sigma_b + w_b^2\sigma_b^2 \\ w_a &= \frac{V_a}{V_c} \qquad w_b = \frac{V_b}{V_c} \end{aligned}$$

Typically ρ is less than one so that $\sigma_a, \sigma_b > \sigma_c$ and the effect on the equity of firms A&B is to reduce their value since under a BS option pricing model of debt and equity

$$\begin{aligned} E_a &= Call(V_a, X_a, \sigma_a, T, R_f) \\ E_b &= Call(V_b, X_b, \sigma_b, T, R_f) \\ E_c &= Call(V_c, X_a + X_b, \sigma_c, T, R_f) \\ E_c &< E_a + E_b \end{aligned}$$

The stock holders are hurt because their limited liability is weakened. Typically both debt values D'_a, D'_b will appreciate from the merger because they are both now co-insuring each other and are being backed by each others equity holders

$$\begin{aligned} D'_a &> D_a \\ D'_b &> D_b \end{aligned}$$

To make sure that the bondholders are not the major beneficiaries of M&A activity, they are often asked to increase their commitment at the time of bidding by contributing new capital for the cash part of the bid.

Mixed effects are also possible if the risk are $\sigma_a > \sigma_c > \sigma_b$ or vice versa and depending on whose default option is closest to the money can also change the outcome for the two sets of bond and shareholders.

10 Real Options

10.1 Introducing flexibility value

If NPV analysis is sound, then once a project has been initiated, there is no further role for the manager since the firm has "bought a bond" (although with a variable coupon), but the bond cannot be sold or terminated if its cashflows were to deteriorate. Variability of cashflows was envisaged and factored into a risk premium by increasing the required rate of return to compensate for the systematic risk present in the cashflows but they were treated symmetrically, large losses as well as large gains were included and no dynamic action was foreseen. However we know that managers have the ability to affect the cashflows over time, indeed this is their role! If a project were to become severely loss making, the manager would attempt to terminate it.

Under the standard NPV this flexibility has not been taken into account; the manager has nothing to manage! We know this to be an incomplete description of managerial action since if unlikely but possible negative flows occur, the manager will not sit back and do nothing he will attempt to do something about them. This is a long standing criticism of straight NPV analysis (see Hertz (1964) [24] & Magee[33]), NPVs do not incorporate the strategic value that is associated with the flexibility that project cashflows may contain. The introduction of options technology allows many of the issues of flexibility to be addressed, we can now calculate NPVs of cashflows were some of the future cashflows contingent on the state variables of the problem (e.g. oil profit depends on oil price). In an options context, this means that the required rates of return may vary with the degree of "moneyness"[30] and that NPVs may allow for the return and risk of the project to vary over its life.

This we have seen in BS pricing, valuing expected cashflows through the use of a CAPM cost of capital is not enough unless we allow for the cost of capital to change (as it does in BS) as the likelihood of exercising the option comes in to the money.

How can this managerial flexibility be incorporated into NPV analysis before assets are put in place? How should the flexibility associated with a termination option be exercised? How should the manager actually manage ongoing cashflows? The answers to many of these questions are to be found in the topic of Real Options.

[30] moneyness is often calclated as $Se^{-\delta t}/Xe^{-rt}$ for BS options, for Merton's Perpetual Options and Real Options we mean the proximity of S to the critical exercise thresholds $\underline{S}, \overline{S}$

10.2 Merton's Perpetual Call

Before dealing with specific examples, it is worthwhile treating the main model that underlies most Real Options work, that of Merton's Perpetual Calls and Puts (Merton (1973) [41]).

The BS formula satisfied the following partial differential equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - \delta) S \frac{\partial C}{\partial S} - rC = 0 \tag{6}$$

(go on, try it, it can't bite! differentiate the call option expression once and twice with respect to S, once with respect to t and plug the values into Equation 6). Now for very long time to maturity options, the problem becomes time independent and the term $\frac{\partial C}{\partial t} \rightarrow 0$ so that we can actually value infinitely lived, continuously exercisable (American) options. Merton solved this problem (1973) [41]. The following *ordinary differential equation* (in one variable, S only)

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - \delta) S \frac{\partial C}{\partial S} - rC = 0$$

has two general solutions as functions of the form (A, B are arbitrary constants to be determined)

$$\text{General Option Solution} = AS^a + BS^b$$

where a, b ($a > 1, b < 0$) solve the fundamental equation

$$\begin{aligned} 0 &= \frac{1}{2}\sigma^2 x (x - 1) + (r - \delta) x - r \\ a, b &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} \pm \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\ &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} \pm \sqrt{\frac{(r-\delta)^2}{\sigma^4} + \frac{(r+\delta)}{\sigma^2} + \frac{1}{4}} \end{aligned}$$

However since BS^b diverges at zero (N.B. $b < 0$ and so S^b does not conform to the boundary condition Call $\rightarrow 0$ as $S \rightarrow 0$) B must be set to zero for the call. Other boundary conditions state that at exercise the price of the exercised option must equal its value and that this transition must occur smoothly, namely the value matching condition:

$$C\left(S = \overline{S}\right) = \overline{S} - X = A\overline{S}^a$$

and the smooth pasting condition:

$$\left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}} = 1 = aA\overline{S}^{a-1}$$

These allow determination of the arbitrary constant A and the critical point of exercise $\overline{S}$

$$\begin{aligned} A &= \tfrac{\overline{S}-X}{\overline{S}^a} \\ \overline{S} &= \tfrac{a}{a-1}X \quad > X \\ a &= \tfrac{\overline{S}}{\overline{S}-X} \quad > 1 \\ a-1 &= \tfrac{X}{\overline{S}-X} \quad > 0 \end{aligned}$$

yielding for the call price itself

$$\begin{aligned} \text{Call} &= \begin{matrix} \left(\overline{S}-X\right)\left(\frac{S}{\overline{S}}\right)^a & \text{for } S < \overline{S} \\ S-X & \text{for } S > \overline{S} \end{matrix} \\ a &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \end{aligned}$$

Alternatively the call price can be expressed without explicit reference to the critical stock price

$$\text{Call} = \frac{X}{a-1}\left(\frac{S\,(a-1)}{Xa}\right)^a = \left(\frac{X}{a-1}\right)^{1-a}\left(\frac{S}{a}\right)^a$$

or without explicit reference to X

$$\text{Call} = \frac{\overline{S}}{a}\left(\frac{S}{\overline{S}}\right)^a = \frac{1}{a}\overline{S}^{1-a}S^a$$

The way to think of the Merton Perpetual Call exercise time (time when $S = \overline{S}$) is the first time that the opportunity cost of waiting exceeds the value of the option (see rates of return section below). There is an opportunity cost because the underlying stock has a dividend yield δ that the option itself does not enjoy, thus waiting for ever is not optimal! At some point the stock S is so valuable that the dividend on it δS is large enough to compensate for the opportunity cost of capital r on the exercise price X as well as the lost future capital growth on the option itself. At this point the option should be exercised and the stock taken in its place.

This formula can also be derived using expected time to exercise arguments

$$\text{Call} = \max_{\overline{S}} E\left[e^{-rt}\left(\overline{S}-X\right)\right] = \left(\overline{S}-X\right)\left(\frac{S}{\overline{S}}\right)^a$$

See Shackleton and Wojakowski (2002) [59].

10.3 The delta of a Merton Call

Before exercise, we can evaluate the sensitivity of a Merton call to the underlying stock price

$$\Delta = \frac{\partial C}{\partial S} = \frac{\left(\overline{S} - X\right)}{\overline{S}^a} aS^{a-1} = \left(\frac{S}{\overline{\overline{S}}}\right)^{a-1} = \frac{C}{S}a = \frac{C}{S}\frac{\overline{S}}{\overline{S} - X}$$

This tends to zero as $S, C \to 0$ and when evaluated at $S = \overline{S}$ (using the definition of $\overline{S} = \frac{a}{a-1}X$) yields

$$\Delta\left(S = \overline{S}\right) = \frac{\overline{S} - X}{\overline{S}} a = 1$$

Thus, unsurprisingly, the option is exercised as soon as its delta becomes one. The second derivative or gamma is given by

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{C}{S^2} a\left(a - 1\right)$$

which is positive at $S = \overline{S}$.

10.4 Sensitivity to Exercise Price X

The partial κ with respect to exercise price X can also be calculated, using the version with the eliminated critical threshold

$$\begin{aligned}
C &= \left(\frac{X}{a-1}\right)^{1-a}\left(\frac{S}{a}\right)^a \\
\kappa &= \frac{\partial C}{\partial X} = (1 - a)\left(\frac{X}{a-1}\right)^{-a}\left(\frac{S}{a}\right)^a \\
&= -\left(\frac{S\left(a-1\right)}{Xa}\right)^a = -\left(\frac{S}{\overline{\overline{S}}}\right)^a \\
&= -\frac{a-1}{X}C \qquad \text{or} \qquad -\frac{C}{\overline{S} - X}
\end{aligned}$$

Now the option can be expressed using

$$\begin{aligned}
C &= \Delta S + \kappa X = S\frac{\partial C}{\partial S} + X\frac{\partial C}{\partial X} \\
\frac{\partial C}{\partial S} &= \frac{\left(\overline{S} - X\right)}{\overline{S}^a} aS^{a-1} = \frac{C}{S}a = \frac{C}{S}\frac{\overline{S}}{\overline{S} - X} \\
\frac{\partial C}{\partial X} &= (1-a)\left(\frac{X}{a-1}\right)^{-a}\left(\frac{S}{a}\right)^a = -\frac{a-1}{X}C = -\frac{C}{\overline{S} - X} \\
C &= Ca - C\left(a-1\right) = \frac{C\overline{S}}{\overline{S} - X} - \frac{CX}{\overline{S} - X} \text{ i.e. fractions in } S, X
\end{aligned}$$

Thus ownership of a Merton call is equivalent to an instantaneous hedging strategy owning $\Delta = \frac{\partial C}{\partial S}$ stocks through borrowing of $\kappa = \frac{\partial C}{\partial X}$, a constant fraction of the call value $\frac{X}{\overline{S}-X} = a - 1$ is borrowed and a constant fraction of the call value $\frac{\overline{S}}{\overline{S}-X} = a$ is invested in the stock. The two hedge ratios both yield results that are proportional to the call price itself and thus have a common factor, that of the call price itself.

$$\text{Holdings as a \% of } C = S\frac{1}{C}\frac{\partial C}{\partial S} + X\frac{1}{C}\frac{\partial C}{\partial X} = S\frac{a}{S} - X\frac{a-1}{X}$$

Whereas BS hedging involves increasing stock holdings as S increases and gradually approaching $\Delta = 1$ in the limit as S becomes very large (or time to maturity small and in the money), Merton hedging increases the stock holding proportional to the call value itself until the critical point is reached where the $\Delta = 1$ condition is satisfied. This means increasing the Δ at an increasing rate as S increases, something that stops happening in the BS case once the option moneyness exceeds one!

If for example if $a = 2$, then the hedge strategy would be to maintain twice the \$ call value in the \$ stock value through borrowing one unit of \$ call value at all times until exercise. $\Delta S = 2C \; \kappa X = -C$.

Again (like BS) this is a following strategy where the amount of stock has to increase (decrease) as the stock price increases (decreases) however, no satiation occurs and unlike BS the delta continues to increase at an increasing rate until exercise is triggered.

10.5 Instantaneous return on a Merton call

Now the instantaneous rate of return on a Merton call can be evaluated using the delta and gamma, using Itô's Lemma (detailed in Shackleton and Sødal 2005 [57])

$$\begin{aligned} dS &= (\mu - \delta)\, Sdt + \sigma SdZ \\ (dS)^2 &= S^2\sigma^2 dt \end{aligned}$$

the rate of return on the call R_C can be determined[31]. See Figure 13 again.

$$\begin{aligned} dC &= \frac{\partial C}{\partial S} dS + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (dS)^2 \\ E(dC) &= \frac{C}{S} a (\mu - \delta) S dt + \frac{1}{2} \frac{C}{S^2} a (a-1) S^2 \sigma^2 dt \\ R_C &= \frac{1}{C} \frac{E(dC)}{dt} \\ &= a (\mu - \delta) + \frac{1}{2} a (a-1) \sigma^2 \\ &= r + a (\mu - r) \qquad \text{or} \qquad \mu a + r (1-a) \end{aligned}$$

Since a is at least 1, the rate of return on the call is at least μ. Unlike with a BS option, the rate of return of the call does not vary with the degree of moneyness S/X, instead it is purely a function of the parameters of the underlying process, μ, r, δ, σ. However these parameters themselves determine a and the relative critical exercise price

$$\begin{aligned} a &= \frac{\overline{S}}{\overline{S} - X} \\ R_C &= r + \frac{\overline{S}}{\overline{S} - X} (\mu - r) \\ &= \frac{\mu \overline{S}}{\overline{S} - X} - \frac{rX}{\overline{S} - X} \\ &= \mu a - r (a-1) \\ &= r + a (\mu - r) \end{aligned}$$

Thus for all values of the call (up to the exercise point), the % rate of return is given by the stock return on the leveraged position of $\frac{\overline{S}}{\overline{S}-X} = a$ stocks through borrowing of $\frac{-X}{\overline{S}-X} = 1 - a$ (the weights sum to one).

10.6 Matching rates of return at exercise

In terms of dollar flows, the exercise condition is an equivalence between the dollar return on holding the call and the dollar return on borrowing the

[31] using the condition from the fundamental equation $\frac{1}{2}\sigma^2 \beta (\beta - 1) = -(r - \delta) \beta + r$. The rate of return R_C can also be derived from the call's $\Delta S = C\overline{\beta}$, its elasticity wrt S and the CAPM as was done for BS.

exercise price to finance the stock holding.

$$\begin{aligned} C\left(\overline{S}\right)R_C &= \left(\overline{S}-X\right)\left(a\left(\mu-r\right)+r\right) \\ &= \left(\overline{S}-X\right)\left(\frac{\overline{S}}{\overline{S}-X}\left(\mu-r\right)+r\right) \\ &= \mu\overline{S}-rX \end{aligned}$$

This is because the smooth pasting condition is driven by the same partial as the rate of return

$$\left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}} = 1$$

10.7 Comparison to Black Scholes

The Merton rate of return could be compared to the BS rate of return

$$R_C = N\left(d_1\right)\frac{Se^{-\delta T}}{C}R_S - N\left(d_2\right)\frac{Xe^{-R_f T}}{C}R_f$$

i.e. the Merton and BS call returns are in some sense equivalent if

$$\begin{aligned} N\left(d_1\right)\frac{Se^{-\delta T}}{C} &\equiv \frac{\overline{S}}{\overline{S}-X} = a \\ N\left(d_2\right)\frac{Xe^{-R_f T}}{C} &\equiv \frac{X}{\overline{S}-X} = a-1 \end{aligned}$$

i.e. if the Normal weighted ratios of $Se^{-\delta T}/C$ and $Xe^{-R_f T}/C$ were constant. The Merton call should be thought of as being similar to the BS if it is written thus

$$\begin{array}{ccccc} C_{\text{Merton}} & = & S\left(\frac{S}{\overline{S}}\right)^{a-1} & - & X\left(\frac{S}{\overline{S}}\right)^{a} \\ C_{\text{Black Scholes}} & = & Se^{-\delta T}N\left(d_1\left(S,X\right)\right) & - & Xe^{-rT}N\left(d_2\left(S,X\right)\right) \end{array}$$

$$\begin{array}{cccccc} 0 & \leq & e^{-\delta T}N\left(d_1\left(S,X\right)\right) & , & \left(\frac{S}{\overline{S}}\right)^{a-1} & \leq 1 \\ 0 & \leq & e^{-rT}N\left(d_2\left(S,X\right)\right) & , & \left(\frac{S}{\overline{S}}\right)^{a} & \leq 1 \end{array}$$

In the Merton model, the "probabilities" equivalent to $N\left(d_{1,2}\right)$ are to be thought of as "discounted" proximities to the exercise threshold $\frac{S}{\overline{S}}$, i.e. at the threshold $\overline{S}$ it is just optimal to finance exercise.

10.8 Perpetual Call as an option to defer investment

Question. Your firm (and your firm alone) has a real (non–financial) option to commence a new business activity by launching a product. It can postpone launching the new product indefinitely. If launched now, the revenue from the product would be $R = \$1,000$; future revenues are subject to geometric uncertainty ($\sigma = 20\%$) although they grow at an expected compound rate of 5% in perpetuity. The annual costs, if the product is launched now would be $C = \$500$p.a. for certain forever without any growth and a one time capital investment in (very durable and non–depreciating assets!) land, machinery etc of $I = \$10,000$ is required. The required rate of return is 10% for revenues and costs.

What is the value of the project including its deferment option?

Answer. The present value of the costs and investment are given by a perpetuity with required rate of return 10% and current flow of \$1,500 growing at 0% and I

$$\overline{X} = \frac{C}{r} + I = \frac{\$500}{0.10} + \$10,000 = \$15,000$$

The present value of revenues *if launched* are given by a growing perpetuity with required rate of return 10% and current flow of \$1,000 growing at 5%; the current present value of revenues if the product were launched is S

$$\begin{aligned} S &= \frac{R}{r-g} = \frac{\$1,000}{0.10 - .05} = \$20,000 \\ \delta &= r - g = \frac{R}{S} = 0.05 \end{aligned}$$

The dividend yield on the project is given by the required rate of return less the (capital) growth in dividends. On the face of things, the product should be launched now since its straight NPV is positive, $S > \overline{X}$, however what about the possibility that S in the future may fall below $\overline{X}$ i.e. the cashflow associated with the product may fall? What is the option (but not the obligation to invest at a given time) worth and how does this affect our NPV decision? **Merton's perpetual call option,** trading $\overline{X}$ for S, must be evaluated since there is a chance that the **call option** C is currently worth more than the **payoff** $S - \overline{X}$ in which case it is not NPV maximising to trade $\overline{X} + C$ for S!

$$\begin{aligned}
\sigma &= 0.20, \qquad r = 0.10 \qquad \delta = 0.05 \\
a &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\
a &= 0.5 - \frac{0.05}{0.04} + \sqrt{\left(\frac{0.05}{0.04} - \frac{1}{2}\right)^2 + \frac{0.2}{0.04}} = 1.6085 \\
\overline{S} &= \frac{a}{a-1}\overline{X} = 2.6434\overline{X} = \$39,650 \\
\overline{R} &= \delta\overline{S} = \$1,982
\end{aligned}$$

$\overline{S}$ is higher than $\overline{X} = \$15,000$ because this perpetual option must generate sufficient immediate profit ($\overline{S} - \overline{X}$) \$24,650 to compensate for its demise on exercise. The project should not be launched until revenues R reach $\overline{R} = \$1,982$ (i.e. when S reaches $\overline{S} = \$39,650$). Note that at this point, we would expect the immediate gain on exercise to just compensate us for the loss of the option itself (at $S = \overline{S}$ the call is worth $S - \overline{X}$).

However we are not near the boundary! We are stuck in the $S \ll \overline{S}$ region, the option to invest is worth the call value

$$\begin{aligned}
\text{Call} &= \begin{array}{ll} \left(\overline{S} - \overline{X}\right)\left(\frac{S}{\overline{S}}\right)^a & \text{for } S < \overline{S} \\ S - \overline{X} & \text{for } S > \overline{S} \end{array} \\
&= \$\left(39,650 - 15,000\right)\left(\frac{20,000}{39,650}\right)^{1.6085} \\
&= \$24,650 * 0.3326 = \$8,191
\end{aligned}$$

Thus we have established the following facts

- Straight NPV of the project (decision to launch now) is \$5,000
- Seemingly we should go ahead now!
- But the option to invest (at some later time) is worth \$8,191
- So if launched we would gain the \$5,000 (\$20,000 − \$15,000) but we would lose the option value of \$8,191 giving a net loss of −\$3,191.
- Thus immediate exercise is not Option NPV maximising!
- Exercise and investment timing should be given by an Option NPV=0 criterion, not a straight NPV=0 criterion.

- We can't say exactly when this will happen in time (in fact we don't need to know), but we can say at what project value level S we should go ahead $S = \overline{S} = \$39,650$.
- Because the project can be deferred indefinitely (while incurring the opportunity cost δS), the Option NPV must compete with itself *deferred* if launch now is to be optimal.
- Only when the launch now PV is equal to (or just exceeds) the delay launch PV will we go ahead with the project.

Further examples of options to defer entry are included in Pindyck (1988) [52], McDonald & Siegel (1986) [39].

10.9 Merton's Perpetual Put

The analysis can be repeated for a put, where the positive solution is rejected (this time A must be set to zero as S^a does not converge to 0 as $S \to \infty$)

$$b = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

Thus the general solution reduces to

$$\text{Put} = BS^b$$

and value matching and smooth pasting can be used to determine both the constant B and the lower level $\underline{S}$ at which the Put should be exercised

$$\begin{aligned} B\underline{S}^b &= X - \underline{S} \\ bB\underline{S}^{b-1} &= -1 \end{aligned}$$

and the Merton Perpetual Put price is given by

$$\begin{aligned} \text{Put} &= \begin{array}{ll} (X-\underline{S})\left(\frac{S}{\underline{S}}\right)^b & \text{for } S > \underline{S} \\ X - S & \text{for } S < \underline{S} \end{array} \\ b &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\ \underline{S} &= \frac{b}{b-1}X \qquad < \qquad X \end{aligned}$$

The way to think of a Merton perpetual put is converse to the call, if the stock price ever falls to some low value $\underline{S}$ it may be worthwhile putting the stock (foregoing its now reduced dividend δS as well as any further capital appreciation on the put due to potential increased dividends) in return for receiving the exercise price as cash (which has a yield of rX). This is the case even if the stock price although depressed is expected to rise in the future! Note that the actual rate of return on the stock does not appear in either of Merton's formulae (for the same reasons as it drops out of BS) because a hedging strategy wrt to S can make us indifferent to changes in S.

10.10 The role of dividend yield for the call

Note that for calls a positive dividend yield is required or the model collapses. The call tends to the stock price as $\delta \to 0$ ($\overline{S} \to \infty$) because for $\delta = 0$ the betas in the fundamental equation are given by and we require $a > 1$

$$\begin{aligned} a, b\,(\delta = 0) &= \frac{1}{2} - \frac{r}{\sigma^2} \pm \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\ &= \frac{1}{2} - \frac{r}{\sigma^2} \pm \sqrt{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2} \\ &= 1, -\frac{2r}{\sigma^2} \end{aligned}$$

The general solution does not have terms that can conform to all (value matching and smooth pasting) the boundary conditions. It is only the presence of a non–zero cost of waiting δ that makes these infinite horizon problems sensible. If waiting were costless, perpetual options would never be exercised.

10.11 The role of interest rate in the put

For the put problem it is the interest rate that must be positive else the put tends to zero ($\underline{S} \to 0$), for $r = 0$

$$\begin{aligned} a, b\,(r = 0) &= \frac{1}{2} + \frac{\delta}{\sigma^2} \pm \sqrt{\left(\frac{-\delta}{\sigma^2} - \frac{1}{2}\right)^2} \\ &= \frac{1}{2} + \frac{\delta}{\sigma^2} \pm \left(\frac{-\delta}{\sigma^2} - \frac{1}{2}\right) \\ &= 1 + \frac{2\delta}{\sigma^2}, 0 \end{aligned}$$

whereas we require $b < 0$ indicating that the put problem is not well defined for $r = 0$.

10.12 Perpetual put is a divestment or shutting option

Your firm has now commenced operating in the line of business described above (S exceeded $\overline{S}$ and investment was sunk) but now S has collapsed because revenues have fallen back to $R = \$400$ p.a.; future revenues are still subject to geometric uncertainty ($\sigma = 20\%$) and growth at a compound rate of 5% in perpetuity. The costs are the same $C = \$500$ for certain forever without any growth and the required rate of return is 10% for revenues and costs.

However since the product became less profitable, it has been discovered that a termination option that was not previously anticipated (before investment occurred) is actually present and that if operations are terminated, half ($\$5,000$) of the initial investment in land, machinery etc I can be recovered as well as operating under–performance (losses) terminated. Thus on termination, the firm can save PV costs of \$5,000 and also recoup $\$5,000$ from the sale of the specific assets that it was using, i.e. $\underline{X}= \$10,000$.

Question. What is the installed project worth once the termination option has been discovered?

Answer. This firm has discovered that it has a perpetual option to terminate its activities, selling its remaining assets if a time comes when it's profitability is sufficiently low. The present value of revenues is given by a growing perpetuity with required rate of return 10% and current flow of \$400 growing at 5%;

$$
\begin{aligned}
S &= \frac{R}{r-g} = \frac{\$400}{0.10 - .05} = \$8,000 \\
\delta &= r - g = \frac{R}{S} = 0.05
\end{aligned}
$$

On the face of things, the product should be terminated now since its straight NPV is negative (it is running at a loss of $\$100 = \$500 - \$400$ pa), $S < \underline{X}$, and termination will liberate \$10,000. In fact an argument could have been made to terminate at \$500 since at this revenue level the NPV was equal to zero. However what about the possibility that S in the future may increase above $\underline{X}$ i.e. the cashflow associated with the product may rise? If dis–investment occurs now and this happens, we may regret our decision. What therefore is the option to dis–invest (but not the obligation to dis–invest at a given time) worth and how does this affect our NPV decision?

Merton's perpetual put, trading S for $\underline{X}$, must be evaluated since there is a chance that the **put option** P is currently worth more than the

payoff $\underline{X} - S$ in which case it is not NPV maximising to trade $S + P$ for $\underline{X}$!

$$
\begin{aligned}
\sigma &= 0.20, \qquad r = 0.10 \qquad \delta = 0.05 \\
b &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\
b &= 0.5 - \frac{0.05}{0.04} - \sqrt{\left(\frac{0.05}{0.04} - \frac{1}{2}\right)^2 + \frac{0.2}{0.04}} = -3.1085 \\
\underline{S} &= \frac{b}{b-1}\underline{X} = 0.7566\underline{X} = \$7,566 \\
\underline{R} &= \delta\underline{S} = \$378.3
\end{aligned}
$$

$\underline{S}$ is lower than $\underline{X}$= $10,000 because this perpetual option must generate sufficient immediate profit ($\underline{X} - \underline{S}$) $2,433 to compensate for its demise on exercise. The project should not be launched until revenues R fall to $\underline{R}$ = $378.3 (i.e. when S reaches $\underline{S}$ = $378.3). Note that at this point, we would expect the immediate gain on exercise to just compensate us for the loss of the option itself (at $S = S$ the call is worth $\underline{X} - S$).

However we are only close to the boundary and not at it! We are still in the $S > \underline{S}$ region, the option to invest is worth the put value

$$
\begin{aligned}
\text{Put} &= \begin{matrix} (\underline{X} - \underline{S})\left(\frac{S}{\underline{S}}\right)^b & \text{for } S > \underline{S} \\ \underline{X} - S & \text{for } S < \underline{S} \end{matrix} \\
&= \$\,(10,000 - 7,566)\left(\frac{8,000}{7,566}\right)^{-3.1085} \\
&= \$2,433 * 0.8408 = \$2,047
\end{aligned}
$$

- Straight NPV to kill the project (now) is $2,000
- Seemingly we should go ahead now and kill it!
- But the option to divest (at some later time) is worth $2,047
- So if killed the project we would gain the $2,000 ($10,000 − $8,000) but we would lose the option value of $2,047 giving a net loss of −$47.
- Thus immediate exercise is not Option NPV maximising!
- Exercise and investment timing should be given by an Option NPV=0 criterion, not a straight NPV=0 criterion.

- We can't say exactly when this will happen in time (in fact we don't need to know), but we can say at what project value level S we should go ahead $S = \underline{S} = \$7,566$.
- Because divestiture can be deferred indefinitely the Option NPV must compete with itself *deferred* if divestiture now is to be optimal.
- Only when the kill now PV is equal to (or just exceeds) the delayed killing PV will we stop the project.

Further examples of options to terminate a business are included in Myers & Majd (1990) [48], McDonald & Siegel (1985) [38] and D&P.

10.13 Switching options

Suppose that the option to terminate includes a subsequent reopen option and the option to open includes a subsequent reclosure option, bot valued simultaneously? How can this be analysed? How would this have affected the initial decision (if at all)? When divesting (putting the put) suppose we want to anticipate the possibility of recommencing if the project becomes profitable again, is it possible to encompass these alternatives? Yes it is.

We can combine the put and call analysis to value the option to invest, when investing (calling) gives us the option to disinvest (a put which itself is valued as a function of the call). Basically it involves solving the two sets of equations from above *simultaneously* rather than independently.

For an underlying project value price process S that follows a Geometric Brownian Motion [32]

$$\frac{dS}{S} = (\mu - \delta)\, dt + \sigma dZ$$

no arbitrage or risk–neutral valuation implies that the price of a time homogeneous option claim $O(S_0)$ as a function of the current project value S_0 must satisfy a Hamilton–Bellmann–Jacobi ordinary differential equation (we assume $r > 0, \delta > 0$)

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 O}{\partial S^2} + (r - \delta)\, S \frac{\partial O}{\partial S} - rO = 0$$

The general solution to this is of the form $AS_0^a + BS_0^b$ (constants A, B are determined using boundary conditions) where a, b $(a > 1, b < 0)$ solve the

[32] This process has expected capital gain $g = \mu - \delta$ and dividend yield δ under the real world process dZ. Under the risk neutral process, its drift and yield are $r - \delta, \delta$. We assume that both r and δ are positive.

fundamental equation[33]

$$0 = \frac{1}{2}\sigma^2 x (x-1) + (r-\delta) x - r$$

$$a, b = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} \pm \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$O(S_0, T, \alpha)$$

10.14 Two value matching, smooth pasting conditions

Now suppose the project S can be activated or opened at any time by paying $\overline{X}$ (i.e. the "stock" S can be called for $\overline{X}$) with further rights to deactivate or shut the project at any time and retrieve $\underline{X}$ (i.e. the "stock" can be put back for $\underline{X}$). When the former happens the activation option together with the activation exercise amount are exchanged for the project value plus the deactivation option, when the latter happens the deactivation option and the current project value are exchanged for the deactivation proceeds plus the activation option (a bond plus call are switched for the stock plus put and vice versa). We label the perpetual option to activate or open $RO(S_0)$ and the option to deactivate, or shut the project $RS(S_0)$ both a function of the current underlying project value with no optionality S but not time

$$\begin{aligned} \overline{X} + RO(S) &\rightarrow S + RS(S) \\ \underline{X} + RO(S) &\leftarrow S + RS(S) \end{aligned}$$

Two boundary conditions are immediately determined because the option to open goes to zero as S tends to zero and the option to shut goes to zero as S becomes large. Therefore these open and shut options conform to the two general solutions AS_0^a, BS_0^b respectively. The remaining boundary conditions must be determined endogenously through optimality conditions.

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$$\begin{aligned} a &> 1, b < 0 \\ ab &= \frac{-r}{\frac{1}{2}\sigma^2} \\ a + b &= -\frac{r - \delta - \frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2} \\ a + b - ab - 1 &= \frac{\delta}{\frac{1}{2}\sigma^2} \end{aligned}$$

The activation costs and deactivation proceeds (forward and reverse exercise) may not be equal $\left(\underline{X} \neq \overline{X}\right)$ so forward and reverse exercise may be separated in underlying project value S and therefore in time. This investment hysteresis is similar to Brennan and Schwartz (1985) [8] and also described by Dixit (1989) [13].

The optimal policy is determined by two value matching and two smooth pasting conditions (see Dumas (1991) [14] for a treatment of these conditions and Dumas and Luciano (1991) [15] for another two sided transaction cost control problem) at an upper $\overline{S}$ **and** a lower threshold $\underline{S}$ (the optimal underlying project values at which to switch). This yields a system of four equations in four variables $\left(A, B, \overline{S}, \underline{S}\right)$ representing the remaining boundary conditions. The four equations co–determine the intervention points $\overline{S}, \underline{S}$ and the option constants A, B. All four are a function of the inputs to the system, the (de)activation amounts $\left(\overline{X}, \underline{X}\right)$, although we will see that it is actually easier to evaluate $\left(A, B, \overline{X}, \underline{X}\right)$ as a function of $\left(\overline{S}, \underline{S}\right)$. Writing the two value matching conditions and the smooth pasting conditions (made homogenous by a $\overline{S}, \underline{S}$ multiplication) out in matrix form allows inversion for $\left(A, B, \overline{X}, \underline{X}\right)$

$$\begin{array}{rcl} A\overline{S}^a + \overline{X} & = & B\overline{S}^b + \overline{S} \\ A\underline{S}^a + \underline{X} & = & B\underline{S}^b + \underline{S} \\ Aa\overline{S}^a & = & Bb\overline{S}^b + \overline{S} \\ Aa\underline{S}^a & = & Bb\underline{S}^b + \underline{S} \end{array} \Longleftrightarrow \begin{bmatrix} 1 & 0 & \overline{S}^a & -\overline{S}^b \\ 0 & 1 & \underline{S}^a & -\underline{S}^b \\ 0 & 0 & a\overline{S}^a & -b\overline{S}^b \\ 0 & 0 & a\underline{S}^a & -b\underline{S}^b \end{bmatrix} \begin{bmatrix} \overline{X} \\ \underline{X} \\ A \\ B \end{bmatrix} = \begin{bmatrix} \overline{S} \\ \underline{S} \\ \overline{S} \\ \underline{S} \end{bmatrix}$$

Inverting (see the Appendix) to recover $\left(A, B, \overline{X}, \underline{X}\right)$ as a function of $\left(\overline{S}, \underline{S}\right)$, the matrix product is most easily evaluated as a function of the fraction $\gamma = \underline{S}/\overline{S}$ (a ratio of the lower to the upper intervention thresholds)

$$\begin{bmatrix} \overline{X} \\ \underline{X} \\ A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & \overline{S}^a & -\overline{S}^b \\ 0 & 1 & \underline{S}^a & -\underline{S}^b \\ 0 & 0 & a\overline{S}^a & -b\overline{S}^b \\ 0 & 0 & a\underline{S}^a & -b\underline{S}^b \end{bmatrix}^{-1} \begin{bmatrix} \overline{S} \\ \underline{S} \\ \overline{S} \\ \underline{S} \end{bmatrix}$$

$$= \frac{\overline{S}}{ab\left(\gamma^b - \gamma^a\right)} \begin{bmatrix} ab\left(\gamma^b - \gamma^a\right) - b\gamma^b + a\gamma^a - (a-b)\gamma \\ ab\left(\gamma^{b+1} - \gamma^{a+1}\right) + b\gamma^{a+1} - a\gamma^{b+1} + (a-b)\gamma^{a+b} \\ \frac{b\left(\gamma^b - \gamma\right)}{\overline{S}^a} \\ \frac{a\left(\gamma^a - \gamma\right)}{\overline{S}^b} \end{bmatrix}$$

Thus if $\underline{S}, \overline{S}$ were known a priori it would be a simple matter to determine $\left(A, B, \overline{X}, \underline{X}\right)$. Since $\overline{S} \neq 0$ and $\gamma^b \neq \gamma^a$, a new variable $\alpha = \underline{X}/\overline{X}$ (a ratio of the deactivation and activation amounts) can be expressed as a function of γ by dividing the first two lines[34].

$$\alpha \equiv \underline{X}/\overline{X} = \frac{(ab-a)\,\gamma^{b+1} + (b-ab)\,\gamma^{a+1} + (a-b)\,\gamma^{a+b}}{(ab-b)\,\gamma^{b} + (a-ab)\,\gamma^{a} - (a-b)\,\gamma} = PY\,(\gamma) \qquad (7)$$

Obviously we would like to be able determine the optimal intervention threshold ratio $\gamma = PY^{-1}\,(\alpha)$ that solves Equation 7 as a function of the amounts $\underline{X}, \overline{X}$ as oppose to determining the ratio of the amounts as a function of the thresholds $\alpha = PY\,(\gamma)$ but a numerical solution for the inverse is always easy to obtain for any particular values. This polynomial PY that represents α as a function of γ is monotonic and increasing in γ and therefore there is a unique α for every γ and vice versa. For each α, the optimal γ can be retrieved numerically and then the thresholds and option constants recovered.

$$\begin{bmatrix} \overline{S} \\ \underline{S} \end{bmatrix} = ab\left(\gamma^{b} - \gamma^{a}\right) \begin{bmatrix} \frac{\overline{X}}{ab\left(\gamma^{b}-\gamma^{a}\right)-b\gamma^{b}+a\gamma^{a}-(a-b)\gamma} \\ \frac{\underline{X}}{ab\left(\gamma^{b}-\gamma^{a}\right)+b\gamma^{a}-a\gamma^{b}+(a-b)\gamma^{a+b-1}} \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{\overline{S}}{ab\left(\gamma^{b} - \gamma^{a}\right)} \begin{bmatrix} \frac{b\left(\gamma^{b}-\gamma\right)}{\overline{S}^{a}} \\ \frac{a(\gamma^{a}-\gamma)}{\overline{S}^{b}} \end{bmatrix}$$

We now examine the properties of Equation 7 for special cases.

10.15 Asymptotes of γ

This function $\alpha = PY\,(\gamma)$ has three asymptotes which correspond to degenerate cases when the matrix determinant is zero. For values of γ that are large, small or close to one the limiting behaviour of α/γ is given respectively by

$$\begin{aligned} \lim\,(\gamma \to \infty)\,\frac{\alpha}{\gamma} &= \frac{b\,(a-1)}{a\,(b-1)} \\ \lim\,(\gamma \to 0)\,\frac{\alpha}{\gamma} &= \frac{a\,(b-1)}{b\,(a-1)} \\ \lim\,(\gamma \to 1)\,\frac{\alpha}{\gamma} &= 1 \end{aligned}$$

The first two of these correspond to the Merton perpetual American irreversible calls and puts that are widely used in the real options literature.

[34] $= \frac{ab\left(\gamma^{b+1}-\gamma^{a+1}\right)+b\gamma^{a+1}-a\gamma^{b+1}+(a-b)\gamma^{a+b}}{ab(\gamma^{b}-\gamma^{a})+a\gamma^{a}-b\gamma^{b}-(a-b)\gamma}$

This can be seen by evaluating the intervention thresholds for large (small) α and γ corresponding to large $\overline{X}$ (small $\underline{X}$)[35]. For large (small) γ, taking the highest (lowest) powers of γ in denominator and numerator yields a form for the other threshold $\underline{S}$ as a function of $\underline{X}$ ($\overline{S}$ as a function of $\overline{X}$)

$$\text{Critical } S \text{ threshold} = X \begin{cases} \frac{a}{a-1} & \text{call} \\ \frac{b}{b-1} & \text{put} \end{cases}$$

For extreme α, γ the two variable reversible problem is reduced to a one variable, one way problem since one threshold becomes unattainable as it either goes to zero or to infinity. Thus these conform to the Merton perpetual American options that are not reversible.

However the third asymptote $\gamma = 1$ preserves the ability to reverse and is indeed perfectly reversible in the sense that forward and reverse exercise does not consume value. As $\gamma, \alpha \to 1$ both the critical thresholds collapse to a common value as the exercise proceeds become equal. In this case there is no longer any hysteresis and switching occurs infinitely often at the common activation and deactivation threshold. Using l'Hopital's rule to evaluate this limiting case and labelling this common threshold $K\left(= \overline{S} = \underline{S}\right)$ and the common (de)activation amount $X\left(= \underline{X} = \overline{X}\right)$ this implies

$$\begin{bmatrix} \overline{S} \\ \underline{S} \end{bmatrix} = \begin{bmatrix} \frac{ab(b-a)\overline{X}}{ab(b-a)-b^2+a^2-(a-b)} \\ \frac{ab(b-a)\underline{X}}{ab(b-a)+(a-b)(a+b-1)} \end{bmatrix} = \begin{bmatrix} K \\ K \end{bmatrix}$$

$$K = \frac{ab}{ab-(a+b)+1} X = \frac{rX}{\delta}$$

This means that the optimal exercise strategy is now to activate the project when the opportunity cost on the project δS exceeds the opportunity cost on the required investment rX and to deactivate it when $rX > \delta S$. The change happens at $S = K$ when the so called Jorgenson (1963) [27] user costs of capital are equal. Thus a current yield criteria $\delta S \lesseqgtr rX$ is employed, not a naive or myopic present value condition $S \lesseqgtr X$. Because there is no penalty for early exercise, waiting does not have to be deferred to the Merton threshold.

Although not obvious at the outset this yield argument is not surprising.

10.16 Costless reversion

$$\begin{aligned} \text{Reversible Open} &= RO\left(S_0, T, \alpha\right) \\ \text{Reversible Shut} &= RS\left(S_0, T, \alpha\right) \end{aligned}$$

[35] i.e. for a fixed exercise amount, the reverse exercise amount could be arbitrarily large or small corresponding to the irreversible Merton put and call cases respectively.

We can now define values for perpetual opening and shutting options with the option to reverse when no loss of exercise proceeds occurs[36] $RO\left(S, T=\infty, \alpha=1\right)$ and $RS\left(S, T=\infty, \alpha=1\right)$ where opening and shutting occurs at the new joint level of $K = rX/\delta$

$$
\begin{aligned}
RO\left(S_0, \infty, 1\right) &= AS_0^a = K\frac{b-1}{a\left(b-a\right)}\left(\frac{S_0}{K}\right)^a \\
RS\left(S_0, \infty, 1\right) &= BS_0^b = K\frac{a-1}{b\left(b-a\right)}\left(\frac{S_0}{K}\right)^b
\end{aligned}
$$

These are proportional to the irreversible Merton option call (open with no reverse option) and put (shut with no reverse option) values, $IO\left(S, T=\infty, \alpha=0\right)$ and $IS\left(S, T=\infty, \alpha=0\right)$

$$
\begin{aligned}
IO\left(S, \infty, 0\right) &= \frac{X}{a-1}\left(\frac{S_0\left(a-1\right)}{Xa}\right)^a = \left(\overline{S}-X\right)\left(\frac{S_0}{\overline{S}}\right)^a \\
IS\left(S, \infty, 0\right) &= \frac{X}{1-b}\left(\frac{S_0\left(1-b\right)}{Xb}\right)^b = \left(X-\underline{S}\right)\left(\frac{S_0}{\underline{S}}\right)^b
\end{aligned}
$$

The perpetual reversible options are always worth more than their Merton counterparts since they include the set of investment strategies of the irreversibles, indeed they are always worth a constant fraction more

$$
\begin{aligned}
\frac{RO\left(S, \infty, 1\right)}{IO\left(S, \infty, 0\right)} &= K\frac{b-1}{a\left(b-a\right)}\left(\frac{S}{K}\right)^a\left(\frac{X}{a-1}\left(\frac{S\left(a-1\right)}{Xa}\right)^a\right)^{-1} = \frac{r}{\delta}\frac{\left(a-1\right)\left(b-1\right)}{a\left(b-a\right)}\left(\frac{a}{a-1}\right)^a > \\
\frac{RS\left(S, \infty, 1\right)}{IS\left(S, \infty, 0\right)} &= K\frac{a-1}{b\left(b-a\right)}\left(\frac{S}{K}\right)^b\left(\frac{X}{1-b}\left(\frac{S\left(1-b\right)}{Xb}\right)^b\right)^{-1} = \frac{r}{\delta}\frac{\left(a-1\right)\left(b-1\right)}{b\left(b-a\right)}\left(\frac{b}{1-b}\right)^b > 1
\end{aligned}
$$

(Add comparative statics of relative value) and the thresholds are higher (lower) for the call (put)

$$
\begin{aligned}
\frac{K\left(\infty, 1\right)}{\overline{S}\left(\infty, 1\right)} &= \frac{K\left(a-1\right)}{Xa} = \frac{r\left(a-1\right)}{\delta a} < 1 \\
\frac{K\left(\infty, 0\right)}{\underline{S}\left(\infty, 0\right)} &= \frac{K\left(b-1\right)}{Xb} = \frac{r\left(b-1\right)}{\delta b} > 1
\end{aligned}
$$

[36] The option constants are given by

$$
\begin{aligned}
\begin{bmatrix} A \\ B \end{bmatrix} &= \lim\left(\gamma \to 1\right)\frac{1}{ab\left(\gamma^b-\gamma^a\right)}\begin{bmatrix} b\left(\gamma^b-\gamma\right)\overline{V}^{1-a} \\ a\left(\gamma^{a+b-1}-\gamma^b\right)\underline{V}^{1-b} \end{bmatrix} \\
&= \frac{1}{ab\left(b-a\right)}\begin{bmatrix} b\left(b-1\right)\left(\frac{rX}{\delta}\right)^{1-a} \\ a\left(a-1\right)\left(\frac{rX}{\delta}\right)^{1-b} \end{bmatrix}
\end{aligned}
$$

i.e. the reversible opening (closing) option is exercised at a lower (higher) threshold than the irreversible opening (closing). Since the opening (closing) boundary is only ever activated from below (above) it can be said that the reversible opening and shutting options are always activated earlier than the irreversible. (Add comparative statics of relative threshold)

10.17 Rates of return

The reversible opening option has the same constant percentage rate of return as the Merton call irreversible opening option, although its dollar return is higher because it is worth more

$$E\left[\frac{dRO(\infty)}{RO(\infty)}\right] = (r + a(\mu - r))\, dt > \mu dt$$

(where the true rate of return on S is μ). The reversible shutting option has the same constant percentage rate of return as the Merton put

$$E\left[\frac{dRS(\infty)}{RS(\infty)}\right] = (r + b(\mu - r))\, dt < rdt$$

10.18 Switching example

$\overline{X}, \underline{X}$ are the same as the first sections (\$15,000 and \$10,000 respectively)

$$\begin{aligned} \overline{X} &= 15,000 \\ \underline{X} &= 10,000 \\ a &= 1.6085 \\ b &= -3.1085 \end{aligned}$$

however the critical thresholds $\underline{S}, \overline{S}$ will be different since exercise either way (putting or calling the project) generates more value than was previously the case. This means that exercise will occur *earlier* (in S) because of the increased flexibility, i.e. that the band between $\underline{S}, \overline{S}$ will narrow. These threshold (and the two new unknown constants in the general solution A, B) are uniquely determined by the two value matching and two smooth pasting equations.

$$\begin{aligned} A\overline{S}^a + \overline{X} &= \overline{S} + B\overline{S}^b \\ aA\overline{S}^{a-1} &= 1 + bB\overline{S}^{b-1} \\ A\underline{S}^a + \underline{X} &= \underline{S} + B\underline{S}^b \\ aA\underline{S}^{a-1} &= 1 + bB\underline{S}^{b-1} \end{aligned}$$

Although non linear, there is nothing in practice to prevent these four equation from being solved for $A, B, \underline{S}, \overline{S}$.

$$\begin{aligned} \overline{S} &= 39,333 \\ \underline{S} &= 11,364 \end{aligned}$$

The switching points are now both within the previous Merton points 39,650 and 7,566.

Straight NPV is limited because it:-

- Incorporates risk through covariance with the market (CAPM β), but not contingent cashflows
- Does not incorporate the "strategic" aspects of planning, i.e. the option to change the plan
- Allows no role for managers, once the project is undertaken, it cannot be changed
- Does not incorporate the option value embedded in a project

Managers can respond to changes by:-

- Adjusting capacity
- Redesigning or developing new products
- Redeploying assets or changing markets
- Making decisions sequentially instead of all at once
- Liquidating or abandoning the firm!

Conclusions

- Don't throw away negative NPV projects, try and defer them to preserve the option to exploit them later.
- Try and maximise the life of your option to invest.
- Don't invest when NPV just exceeds zero, wait until the option NPV to invest is sufficiently in the money (e.g. $\frac{S}{X} = 2.64$).

Financial Variable	**Straight NPV**	**Flexible, Real Option PV**
Present value of revenues	S	S
Present value of costs	X	X
Risk free rate		r
Dividend yield		δ
Project risk		σ
Time to option expiry		T (must exercise within this time)
Project value	$S - X$	$C(S, X, r, \delta, \sigma, T)$
Decision rule	Invest if $S - X > 0$	Invest if project call $C = S - X$

Table 19: Standard NPV and Real Option Valuation inputs

- Increasing uncertainty can raise the value of your Real Options.
- There is real option value in the future disinvestment decision, Keswani and Shackleton (2006) [28].

10.19 Summary of Examples

- Defer market entry (call on underlying project value)
- Liquidate/abandon the firm (put on underlying project)
- Compound and switching options (call exercise buys you the put, put exercise buys the call Brennan & Schwartz (1985) [8], Margrabe [36])
- Make decisions incrementally and sequentially (call buys another call on a larger project, etc Carr (1988) [9])
- Time to build (Majd & Pindyck (1987) [34])
- Option to adjust capacity if demand changes (incrementally adjust total investment level)
- Redesign or redevelop new products (switch assets from one market to another)
- Spend money on a test market; Paddock Siegel & Smith [51]
- Abandonment Myers & Majd [48], Leland & Toft [31]
- Pricing when sudden death may occur (example).

References

[1] G. A. Akerlof. The market for lemons. *Quarterly Journal of Economics*, 84, 1970.

[2] L. Bachelier. *Theorie de la speculation*. PhD thesis, Annales de l'Ecole Normale Superieure, 1900.

[3] L. Bachelier. Theorie de la speculation. In P. H. Cootner, editor, *On the Random Character of Stock Prices*, pages 21–86. MIT Press, Cambridge, Mass., 1964.

[4] F. Black. The pricing of commodity contracts. *Journal of Financial Economics*, 3(1):167–179, 1976.

[5] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(May–June):637–659, 1973.

[6] A. J. Boness. *A theory and measurement of stock option value*. PhD thesis, University of Chicago, 1962.

[7] A. J. Boness. Elements of a theory of stock option value. *Journal of Political Economy*, 72(2):163–175, 1964.

[8] M. J. Brennan and E. S. Schwartz. Evaluating natural resource investments. *Journal of Business*, 58(2):135–157, 1985.

[9] P. Carr. The valuation of sequential exchange opportunities. *Journal of Finance*, 43(5):1235–1256, 1988.

[10] P. Carr and M. Chesney. American Put Call Symmetry. Working Paper, 1996.

[11] R. Coase. The nature of the firm. *Economica*, November:386–405, 1937.

[12] J. C. Cox and S. A. Ross. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166, 1976.

[13] A. K. Dixit. Entry and exit decisions under uncertainty. *Journal of Political Economy*, 97(3):620–638, 1989.

[14] B. Dumas. Super contact and related optimality conditions. *Journal of Economic Dynamics and Control*, 15(4):675–685, 1991.

[15] B. Dumas and E. Luciano. An exact solution to a dynamic portfolio choice problem under transaction costs. *Journal of Finance*, 46(2):577–595, 1991.

[16] E. F. Fama. Efficient capital markets: A review of theory and empirical work. *Journal of Finance*, 25:383–417, 1970.

[17] E. F. Fama. Efficient capital markets II. *Journal of Finance*, 46(5):1575–1617, 1991.

[18] E. F. Fama and J. D. MacBeth. Risk, return and equilibrium: empirical tests. *Journal of Political Economy*, 81(3):607–636, 1973.

[19] K. A. Froot, D. S. Scharfstein, and J. C. Stein. Risk management: Coordinating corporate investment and financing policies. *Journal of Finance*, 48(5):1629–1657, 1993.

[20] D. Galai. On the Boness and Black Scholes models for valuation of call options. *Journal of Financial and Quantitative Analysis*, 13(March):15–27, 1978.

[21] M. B. Garman and S. W. Kohlhagen. Foreign currency option values. *Journal of International Money and Finance*, 2(3):231–237, 1983.

[22] M. Gordon. Security and a financial theory of investment. *Quarterly Journal of Economics*, 74(3):472–492, 1960.

[23] J. O. Grabbe. The pricing of call and put options on foreign exchange. *Journal of International Money and Finance*, 2(3):239–253, 1983.

[24] D. B. Hertz. Risk analysis in capital investment. *Harvard Business Review*, 42(Jan):95–106, 1964.

[25] M. C. Jensen. Agency costs of free cash flow, corporate finance and takeovers. *American Economic Association Papers and Proceedings*, 76(2):323–329, 1986.

[26] M. C. Jensen and W. H. Meckling. Theory of the firm: Managerial behaviour, agency costs and ownership structure. *Journal of Financial Economics*, 3:305–360, 1976.

[27] D. W. Jorgensen. Capital theory and investment behavior. *American Economic Review*, 53(2):247–259, 1963.

[28] A. Keswani and M. B. Shackleton. How real option disinvestment flexibility augments project NPV. *European Journal of Operational Research*, 168(1):240–252, 2006.

[29] H. E. Leland. Corporate debt value, bond covenants and optimal capital structure. *Journal of Finance*, 49(4):1213–1252, 1994.

[30] H. E. Leland and D. H. Pyle. Informational asymmetries, financial structure and financial intermediation. *Journal of Finance*, 32(2):255–271, 1977.

[31] H. E. Leland and K. B. Toft. Optimal capital structure, endogenous bankruptcy, and the term stucture of credit risk. *Journal of Finance*, 51(3):987–1019, 1996.

[32] J. Lintner. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, 47:13–37, 1965.

[33] J. F. Magee. How to use decision trees in capital investment. *Harvard Business Review*, 42(Sept):79–96, 1964.

[34] S. Majd and R. S. Pindyck. Time to build, option value and investment decisions. *Journal of Financial Economics*, 18:7–27, 1987.

[35] B. B. Mandelbrot. The variation of certain speculative prices. *Journal of Business*, 36(4):394–419, 1963.

[36] W. Margrabe. The value of an option to exchange one asset for another. *Journal of Finance*, 33(1):177–186, 1978.

[37] H. M. Markowitz. Portfolio selection. *Journal of Finance*, 7, 1952.

[38] R. L. McDonald and D. R. Siegel. Investment and the valuation of firms when there is an option to shut down. *International Economic Review*, 26(2):331–349, 1985.

[39] R. L. McDonald and D. R. Siegel. The value of waiting to invest. *Quarterly Journal of Economics*, 101:707–727, 1986.

[40] R. C. Merton. An intertemporal capital asset pricing model. *Econometrica*, 41, 1973.

[41] R. C. Merton. The theory of rational option pricing. *Bell Journal of Economics*, 4(1):141–183, 1973.

[42] R. C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29(May):449–470, 1974.

[43] M. H. Miller. Debt and taxes. *Journal of Finance*, 32, 1977.

[44] M. H. Miller and F. Modigliani. Dividend policy, growth and the valuation of shares. *Journal of Business*, 34(4):411–433, 1961.

[45] F. Modigliani and M. H. Miller. The cost of capital, corporation finance and the theory of investment. *American Economic Review*, 48(3):261–297, 1958.

[46] F. Modligliani and M. H. Miller. Corporate income taxes and the cost of capital: A correction. *American Economic Review*, 53(June):433–443, 1963.

[47] S. C. Myers. Determinants of corporate borrowing. *Journal of Financial Economics*, 5:147–176, 1977.

[48] S. C. Myers and S. Majd. Abandonment value and project life. *Advances in Futures and Options Research*, 4:1–21, 1990.

[49] S. C. Myers and N. S. Majluf. Corporate financing and investment decisions when firms have information that investors do not have. *Journal of Financial Economics*, 13:187–221, 1984.

[50] F. O'Brien and M. B. Shackleton. An empirical investigation of option returns: Overpricing and the role of higher systematic moments. *Derivatives Use, Trading and Regulation*, 10(4), 2005.

[51] J. L. Paddock, D. R. Siegel, and J. L. Smith. Option valuation of claims on physical assets: the case of offshore petroleum leases. *Quarterly Journal of Economics*, 103(3):479–508, 1988.

[52] R. S. Pindyck. Irreversible investment, capacity choice and the value of the firm. *American Economic Review*, 78:969–985, 1988.

[53] R. Roll. A critique of the asset pricing theory's tests' part i: On past and potential testability of the theory. *Journal of Financial Economics*, 4:129–176, 1977.

[54] S. A. Ross. The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13:341–360, 1976.

[55] M. Rubinstein. A simple formula for the expected rate of return of an option over a finite holding period. *Journal of Finance*, 39(5):1503–1509, 1984.

[56] P. A. Samuelson. Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6(Spring):41–49, 1965.

[57] M. B. Shackleton and S. Sødal. Smooth pasting as rate of return equalization. *Economics Letters*, 89(2 November):200–206, 2005.

[58] M. B. Shackleton and R. Wojakowski. On the expected payoff and true probability of exercise of European options. *Applied Economic Letters*, April, 2001.

[59] M. B. Shackleton and R. Wojakowski. The expected return and time to exercise of Merton style Real Options. *Journal of Business Finance and Accounting*, 29(3–4):541–555, 2002.

[60] M. B. Shackleton and R. Wojakowski. Finite maturity caps and floors on continuous flows. *Journal of Economic Dynamics and Control*, 31(12):3843–3859, 2007.

[61] W. F. Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19(3):425–442, 1964.

[62] B. H. Solnik. An equilibrium model of the International Capital Market. *Journal of Economic Theory*, 8:500–524, 1974.

11 Appendices

11.1 Financial Economics Nobel Laureates

See www.nobel.se/economics/

- 1970 Paul Samuelson (MIT). Partial and General Equilibrium Theory.
- 1972 Kenneth Arrow (Harvard) (joint with Sir John Hicks, Oxford). General Equilibrium Theory.
- 1981 James Tobin (Yale). Macroeconomics.
- 1983 Paul Debreu (Berkeley). General Equilibrium Theory.
- 1985 Franco Modigliani (MIT). Macroeconomics.
- 1990 Harry Markowitz (CUNY), Merton Miller (Chicago) and William Sharpe (Stanford). CAPM.
- 1991 Ronald Coase (Chicago). Theory of Institutions.
- 1994 John Nash (Princeton). Game Theory.
- 1997 Robert Merton (Harvard) and Myron Scholes (Stanford). Options.
- 2001 George Akerlof (Berkeley) and Joe Stiglitz (Columbia) (also joint with Michael Spence, Stanford). Asymmetric Information.
- 2003 Robert Engle (NYU) and Clive Granger (SanDiego). Econometrics.

11.2 Glossary of Symbols

11.2.1 Greek symbols

11.2.2 Lower case

11.2.3 Operators

11.2.4 Upper case

11.3 Formula Sheet

Type	Symbol	Use
	α	CAPM intercept, Ratio of investment amounts
	β	CAPM slope
	γ	Ratio of investment thresholds
	δ	Dividend yield
	ϵ	Regression error
	ε	
	ζ	
	η	(Mean reversion coefficient)
	θ	(Partial) Sensitivity of option w.r.t. time T
	ϑ	
	ι	
	κ	(Partial) Sensitivity of option w.r.t. exercise price X
	λ	Market price of risk
	μ	True stock return or market return
	ν	Margrabe volatility, Option return
	ξ	
	π	Pi = 3.141592654...
	ϖ	
	ρ	Correlation coefficient
	σ, σ^2	Standard deviation, variance (volatility)
	ς	
	τ	Tax rate
	υ	
	ϕ	Probability density
	Δ	Delta (Partial) Sensitivity of Option to Underlying S
	Γ	Gamma (2nd Partial) Convexity of Option to Underlying S

Table 20: Glossary of Greek Symbolic Notation

Type	Symbol	Use
	a	Merton call exponent
	b	Merton put exponent
	c	Coupon
	$d_{1,2,3,4}$ d	Black Scholes parameters, size of down jump
	e^1	Napier's continuous compounding variable, 2.71828...
	f	
	g	Growth rate
	h	
	i	Interest payment
	j	Index variable
	k	Index variable
	l	
	m	Number of bonds, binomial hedge ratio
	n, n'	Number of shares, $n(.)$ is a normal density
	o	
	p	A probability (real world)
	q	A probability (risk neutral)
	r	An interest rate, the risk free interest rate
	s	Variable of integration
	t	A general time
	u	Size of up jump
	v	
	w	Portfolio weight
	x	a variable or return
	y	
	z	

Table 21: Glossary of Uncapitalised Symbolic Notation

Type	Symbol	Use
	$E[\cdot]$	(General) Expectation or mean value
	$E^P[\cdot]$	Expectation or mean value in Real World
	$E^Q[\cdot]$	Expectation or mean value in Risk Neutral World
	$Var[\cdot]$	Variance or squared standard deviation
	$Cov[\cdot,\cdot]$	Covariance or two random variables
	$n[\cdot]$	Normal Density
	$N[\cdot]$	Cumulated Normal Density
	$d[\cdot]$ e.g. $dY = Y_t - Y_{t-dt}$	The change in a variable Y in infinitessimal time dt
	$\Delta[\cdot]$ e.g. $\Delta Y = Y_t - Y_{t-\Delta t}$	The change in a variable Y in finite time Δt
	$\frac{\partial[\cdot]}{\partial X}$	The partial derivative w.r.t. variable X

Table 22: Glossary of Operators

Type	**Symbol**	**Use**
Level,	A	Asset value
Stock	B	Riskless bond value
or	C	Call option price
Value	D	Debt value
(capitals)	E	Equity value
	F	
	G	
	H	
	I	Interest payment
	J	
	K	Convertible value
	L	
	M	
	N	Cumulative normal
	O	Opening Option
	P	Put option price, Real World Measure
	Q	Risk Neutral Measure
	$R_{(i)}$	Rate of return (on asset i)
	S	Stock price, Shutting Option
	T	Time to maturity
	U	
	V	Firm value
	W	Warrant value
	$X, X', X_1, X_2, \overline{X}, \underline{X}$	Strike prices
	Y	
	Z	Brownian motion variable

Table 23: Glossary of Capitalised Symbolic Notation

Type	**Call Values**	**Put Values**
GBM	$\frac{dS}{S} = (\mu - \delta)\, dt + \sigma dZ$	
Black Scholes European	$Se^{-\delta T} N(d_1) - Xe^{-rT} N(d_2)$ AssetNothingCall–BinaryCall	$Xe^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$ BinaryPut–AssetNothingPut
	$d_{1,2} = \frac{1}{\sigma\sqrt{T}} \left(\ln S - \ln X + (r - \delta \pm 0.5\sigma^2)\, T\right)$	
Boness Expected **Payoff** at T	$Se^{(\mu-\delta)T} N(d_3) - XN(d_4)$	$XN(-d_4) - Se^{(\mu-\delta)T} N(-d_3)$
	$d_{3,4} = \frac{1}{\sigma\sqrt{T}} \left(\ln S - \ln X + (\mu - \delta \pm 0.5\sigma^2)\, T\right)$	
Merton Perpetual American	$\left(\overline{S} - X\right) \left(S/\overline{S}\right)^a$ $\quad S < \overline{S}$ $S - X$ $\quad S > \overline{S}$ $\overline{S} = \frac{a}{a-1} X$	$X - S$ $\quad S < \underline{S}$ $(X - \underline{S})\,(S/\underline{S})^b$ $\quad S > \underline{S}$ $\underline{S} = \frac{b}{b-1} X$
Perpetual Reversible	$K \frac{b-1}{a(b-a)} (S/K)^a$ $\quad S < K$ $K = \frac{r}{\delta} X$	$K \frac{a-1}{b(b-a)} (S/K)^b$ $\quad S > K$ $K = \frac{r}{\delta} X$
	$a, b = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \quad b < 0, 1 < a$	
Local option rate of return	$r + a(\mu - r)$	$r + b(\mu - r)$
GBMs $1, 2$	$\frac{dX_{1,2}}{X_{1,2}} = \mu_{1,2} dt + \nu_{1,2} dZ_{1,2} \quad dZ_1 dZ_2 = \rho dt$	
Margrabe Exchange Option	$X_1 N(d_1) - X_2 N(d_2)$	$X_2 N(-d_2) - X_1 N(-d_1)$
	$d_{1,2} = \frac{1}{\nu\sqrt{T}} \left(\ln X_1 - \ln X_2 \pm 0.5\nu^2 T\right) \quad \nu^2 = \nu_1^2 - 2\rho\nu_1\nu_2 + \nu_2^2$	
Exchange rate	$\frac{dS}{S} = (r - r^*)\, dt + \sigma dZ$	
Garman Kohlhagen Grabbe	$Se^{-r^* T} N(d_1) - Xe^{-rT} N(d_2)$	$Xe^{-rT} N(-d_2) - Se^{-r^* T} N(-d_1)$
	$d_{1,2} = \frac{1}{\sigma\sqrt{T}} \left(\ln S - \ln X + (r - r^* \pm 0.5\sigma^2)\, T\right)$	

Table 24: ACF Formula Sheet